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Exact Pricing in Linear Factor Models with Finitely Many Assets: A Note

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THE ARBITRAGE PRICING THEORY (APT), originally formulated by Ross [6, 8] and later extended by Connor [1], Huberman [4], and Ingersoll [5], predicts an approximate, linear pricing relation among expected returns as the number of assets approaches infinity. Connor [1] has also shown this pricing relation to be asymptotically exact under additional conditions on asset supplies, returns distributions, and utility functions.

The purpose of this communication is threefold. We want to: (1) identify a sufficient characteristic of an economy from which we can infer an exact pricing equation with finitely many assets; (2) emphasize the distinction between the pricing relation and the portfolio separation results in [1] and [8]; and (3) facilitate the understanding of linear factor models without resorting to infinite asset economies and asymptotic mathematics. Some of the following ideas can be found in the above mentioned articles in slightly more complicated settings.

Assume the return generating process for each asset i, $i = 1, \cdots, N$, in the economy can be represented by

$$\tilde{r}_i = \mu_i + b_{i1}\tilde{\delta}_1 + \cdots + b_{iK}\tilde{\delta}_K + \tilde{\varepsilon}_i \tag{1}$$

where μ_i is the expected return; $\tilde{\delta}_j$ is a mean zero common factor; b_{ij} is the sensitivity of the return on asset i to the fluctuations in factor j; and $\tilde{\varepsilon}_i$ is the "nonsystematic" risk component with $E[\tilde{\varepsilon}_i \mid \delta_1, \cdots, \delta_K] = 0$.[1] We do not assume that the residuals are uncorrelated.

We now state sufficient conditions so that there exist factor premiums $\lambda_0, \lambda_1, \cdots, \lambda_K$ with the pricing relation

$$\mu_i = \lambda_0 + b_{i1}\lambda_1 + \cdots + b_{iK}\lambda_K \tag{2}$$

holding exactly for each asset.

PROPOSITION. *A sufficient condition for the pricing relation in* (2) *to hold for all assets in the above economy is that* (*i*) *there exists a portfolio whose return* $\tilde{r}_p$

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[1] The assumption of conditional mean zero residuals is used in Ross [7, 8] and Connor [1]; Ross [6], Huberman [4], and Ingersoll [5] use residuals which are uncorrelated with the factors. Connor requires this stronger assumption to get an exact result as we do here. Ross [8] proves an asymptotic separation result which goes beyond the pricing relation.

has no nonsystematic risk and (ii) some expected utility maximizer, with a continuously differentiable, increasing, and strictly concave utility function u and initial wealth w considers it his optimal portfolio.

Proof: From the investor's first order condition there exists a positive number γ such that for all i

$$E\{u'[w(1+\tilde{r}_p)]\tilde{r}_i\} = \gamma \tag{3}$$

As the investor's portfolio has no nonsystematic risk, the marginal utility function in (3) can be expressed as a function of the factors alone, $f(\delta_1, \cdots, \delta_K)$, which by the assumed monotonicity is strictly positive. Substituting (1) into (3) and solving for μ_i gives

$$\mu_i = \frac{\gamma}{E[f(\cdot)]} + \sum_j b_{ij}\left[-\frac{E[f(\cdot)\tilde{\delta}_j]}{E[f(\cdot)]}\right] - \frac{E[f(\cdot)\tilde{\varepsilon}_i]}{E[f(\cdot)]} \tag{4}$$

The final term in (4) is zero since

$$\begin{aligned} E[f(\cdot)\tilde{\varepsilon}_i] &= E[E[f(\cdot)\tilde{\varepsilon}_i \mid \delta_1, \cdots, \delta_K]] \\ &= E[f(\cdot)E[\tilde{\varepsilon}_i \mid \delta_1, \cdots \delta_K]] = 0. \end{aligned} \tag{5}$$

Therefore, (4) satisfies (2).[2] Q.E.D.

Comparing our result with that in an infinite asset economy, we see that the assumption of the existence of a portfolio with no nonsystematic risk plays the same role as the "uniform allocation" of factor responses in Connor's sequence of economies.[3] With uniform responses, the nonsystematic risk can be completely diversified away in the limit economy. The optimality of such portfolios for *all* investors is then implied by his other restrictions.

In the context of a finite asset economy, Ross [7] has proved that if returns are described by the linear factor model (1) and all investors hold optimal portfolios which are combinations of $K + 1$ residual risk free portfolios, then the pricing relation (2) must obtain.

Both of these results, therefore, depend on portfolio separation. Ross's does so explicitly. Connor's investors do not necessarily hold combinations of only a few mutual funds; however, since all investors hold well-diversified portfolios, everyone would be indifferent between his portfolio and one holding only positions in $K + 1$ well-diversified portfolios each constructed to be perfectly correlated with one of the factors.[4]

In this respect our assumption is weaker. While it is sufficient for exact pricing, it does not necessarily imply that all investors hold portfolios which are free of

[2] We thank Phil Dybvig for providing this shorter proof to the proposition.

[3] The existence of a linear factor model as described in (1) and a portfolio free from nonsystematic risk is always guaranteed if a sufficient number of factors is used. (It can never require more than N). To be of practical use, however, the number of factors should be small relative to the number of assets. In certain cases, such as the multivariate normal model or the diffusion model, only a single factor is required. The question of whether one such portfolio is always optimal for some investor remains open. It is known that this is true in the generic cases (including $k = N$) above.

[4] See Ingersoll [5] for description of this construction.

residual risk or that $K + 1$ fund separation obtains. The example below illustrates this point.

Consider an economy with a riskless interest rate of 0 and the following three risky assets:

$$\tilde{r}_1 = 0.5 + 0.5\tilde{2}\delta_1$$

$$\tilde{r}_2 = 0.25 + 0.25\tilde{\delta}_1 + 0.25\tilde{\delta}_2 + \tilde{\varepsilon}_2$$

$$\tilde{r}_3 = 0.25 + 0.25\tilde{\delta}_1 + 0.25\tilde{\delta}_2 + \tilde{\varepsilon}_3 \qquad (6)$$

The random variables $(\tilde{\delta}_1, \tilde{\delta}_2)$ take on the values (2, 1), (−2, 1), and (0, −1) with probabilities one-quarter, one-quarter, and one-half, respectively. They are uncorrelated but not independent. Both $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_3$ are independent of $\tilde{\delta}_1$, $\tilde{\delta}_2$, and each other. $\tilde{\varepsilon}_2$ takes on the values ±1 with probability one-half. $\tilde{\varepsilon}_3$ equals 1.5 with probability one-quarter and −0.5 with probability three-quarters.

The covariance matrix for this set of assets is

$$\underline{\Sigma} = \frac{1}{16}\begin{pmatrix} 8 & 4 & 4 \\ 4 & 19 & 3 \\ 4 & 3 & 15 \end{pmatrix} \qquad (7)$$

So the minimum-variance risky asset portfolio which is at the tangency point of the borrowing-lending line is $\underline{\Sigma}^{-1}\,\underline{\mu}/\underline{1}'\,\underline{\Sigma}^{-1}\,\underline{\mu} = (1, 0, 0)$. Quadratic utility investors, therefore hold only asset one (and the riskless asset). Since it is free of nonsystematic risk, an exact pricing result is guaranteed by our proposition. The factor premiums which provide this are $\lambda_0 = \lambda_2 = 0$, $\lambda_1 = 1.0$.[5]

Investors with constant relative risk aversion hold the risky assets in proportions (0.576, 0.049, 0.034), (1.115, 0.087, 0.055), and (1.73, 0.073, 0.035) for relative risk aversion equal to 2, 1, and 0.5 respectively. Since the ratios of investments in assets 2 and 3 are not the same for all investors, separation does not obtain in this economy, and none of these investors hold portfolios which are free from nonsystematic risk.

We have demonstrated that the usual pricing relation of linear factor models, even if exact, is not dependent on the portfolio separation result in such models. The pricing result is apparently more microeconomic in nature. Nonsystematic risk is not priced even if only a single investor chooses not to bear it.

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[5] Since asset one is an optimal portfolio for some investor with quadratic utility, it is mean-variance efficient. Thus, its excess expected return and the other assets' "betas" with respect to it could be used to predict expected returns. This, of course, does not imply that the CAPM pricing relation holds. Furthermore, if the portfolio free from nonsystematic risk were held by some investor whose utility function were not quadratic, then the factor pricing equation would not necessarily have a CAPM-like interpretation. See Dybvig [3] for a further discussion of this issue.

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