The Arbitrage Pricing Model and Returns on Assets Under Uncertain Inflation
Author(s): Edwin Elton, Martin Gruber, Joel Rentzler
Source: *The Journal of Finance*, Vol. 38, No. 2, Papers and Proceedings Forty-First Annual Meeting American Finance Association New York, N.Y. December 28-30, 1982 (May, 1983), pp. 525-537
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327988
Accessed: 27/11/2009 07:59

THE JOURNAL OF FINANCE • VOL. XXXVIII, NO. 2 • MAY 1983

INFLATION AND CAPITAL ASSET PRICES

Presiding: JAMES BICKSLER†

The Arbitrage Pricing Model and Returns on Assets Under Uncertain Inflation*

EDWIN ELTON**, MARTIN GRUBER** and JOEL RENTZLER***

THE IMPACT OF INFLATION on equilibrium returns in capital markets has been the subject of several articles in recent years. Chen and Boness [1], Friend, Landskroner and Losq [4], Long [7] and Roll [13] have all constructed general equilibrium models which incorporate the impact of uncertain inflation on returns.[1] All of these authors have derived models in a mean-variance world utilizing methodologies similar to that first utilized by Mossin [9] or Merton [8].

The basic underlying tenet of mean-variance analysis is that for any given level of expected return investors wish to minimize the variance of return. In mean-variance equilibrium models the only risk that cannot be diversified away is the covariance of an asset's return with the market return. Assets which have the same covariance with the market have the same equilibrium required return. Thus in a mean-variance world if each member of the following pairs of assets had the same covariance with the market each member would have the same expected return:

(a) a stock which was a perfect hedge against inflation versus one which was not,
(b) a stock which was a perfect hedge against oil price changes versus one which was not,
(c) a security that hedged against interest rate changes versus one which did not.

In this article we break out of the limitations of the mean variance assumptions by utilizing the Arbitrage Pricing Theory (APT) of Ross [15], [16], [17]. This

† Rutgers University.

* The authors would like to thank The Dexter Corporation, Windsor Locks, Connecticut for funding this research.

** New York University.

*** Baruch College, CUNY.

[1] Long's [7] paper is different from the other three. Long analyzes a multiperiod framework to derive a model that incorporates an intertemporally changing opportunity set and intertemporally changing interest rate. In his model equilibrium returns are affected by varying future interest rates and prices on each good. While it is an interesting model it differs from the model derived in this paper both by its complexity and its assumption of multivariate normality.

allows us to examine equilibrium returns when returns on alternative securities are impacted differentially by inflation and where investors care about (price) inflation sensitivity.

This paper will proceed as follows. In the first section we show that if the process generating real returns is affected by both the return on the market and the rate of inflation then a model for equilibrium real returns can be derived. Equilibrium real returns will be expressed in terms of the expected return on the market, the expected real return on a zero beta portfolio and the expected real return on an asset which is riskless in nominal terms. In the second section of this paper, we show that if either:

(1) the return generating process depends only on the market return or
(2) the return generating process depends on both the market return and inflation but the inflation factor is not priced,

then the arbitrage pricing model leads to results identical to the mean-variance inflation models which have been presented in the literature. Of special interest is the meaning and derivation of conditions under which a factor is not priced. In particular, we show that multiple factors can be priced using the Roll and Ross [14] testing procedure and the model still reduces to a mean-variance model. The third section expresses the equilibrium model in nominal terms. In the final section we compare our model with prior models and examine the implications for equilibrium prices and empirical testing.

The major purpose of this paper is the construction of richer descriptions of equilibrium under uncertain inflation. However, there is a second purpose which is at least as important. This concerns the role of the APT pricing model as a central model in the economics of uncertainty. The methodology of the APT involves an unspecified return generating process. Tests of the model [3], [10], [12] and [14] have attempted to identify the return generating process through empirical methods. As of this point in time, no economic rationale has been associated with the empirical processes which have been identified. While the empirical approach is one meaningful way to proceed with research on the APT model, we believe that the acceptance of this theory as the central equilibrium theory in finance must await the development of APT models defined in terms of factors which are not only statistically identifiable but which have economic meaning. This paper starts with an hypothesis which sets forth factors that might affect return in a world of uncertain inflation and shows how APT can be used to develop an equilibrium model in terms of observable variables. We do not claim to have developed the "best" description of the return generating process under inflation or to have captured all of the influences which might affect the return generating process. But we do illustrate how a model of the return generating process of the APT based on economic variables is converted to an equilibrium model. We believe this is a necessary complement to the empirical APT studies if the potential importance of APT as an equilibrium model is to be realized. In addition, we take comfort in our description of the return generating process from the fact that under simplifying assumptions about either the process itself or the pricing of the factors which generate the return, our equilibrium model reduces to the equilibrium models of Chen and Boness

[1], Friend, Landskroner and Losq [4] and Roll [13] which were developed under uncertain inflation in a mean-variance framework.[2]

I. Derivation of the Model

The simplest return generating process investigated in the literature (see Ross [15]) assumed a single factor return generated process where the single factor was the market. As noted in Roll and Ross [14], pages 1079–1080, although the market portfolio plays no special role whatsoever in the APT, "as a well diversified portfolio, indeed a convex combination of diversified portfolios, the market portfolio ... might serve as a substitute for one of the factors." Earlier Ross [15] suggested that it might be more appropriate to decompose the return process into real returns and a term or terms associated with inflation. In a world of uncertain inflation, investors should be concerned with real return and riskless arbitrage should be defined in terms of real returns.[3] Furthermore, under uncertain inflation we believe that returns on different assets would be affected differentially by inflation. Reasons advanced in support of this contention include the existence of differences in creditor-debtor positions, differences in contractual obligations at fixed prices, and differences in tax obligations.[4] The simplest model under which real returns are differentially related to the real returns on the market and uncertain inflation is a linear relationship of the form

$$R_i = \alpha_i + \beta_{1i} R'_M + \beta_{2i}\pi + \epsilon_i$$

where subtracting $\bar{R}_i$ from this equation yields

$$R_i = \bar{R}_i + \beta_{1i}(R'_M - \bar{R}'_M) + \beta_{2i}(\pi - \bar{\pi}) + \epsilon_i \tag{1}$$

Here: 1. R_i is the real return on a security,
2. R'_M is the real return on a "market" portfolio of all risky assets. This portfolio is composed of a fraction α of assets M which are risky in nominal returns and a fraction $(1 - \alpha)$ of an asset F riskless in nominal returns. Thus we are allowing for the possibility of a nominally riskless asset with a positive net supply,
3. π is the rate of inflation,
4. ϵ_i is a random error term,
5. β_{1i} is a measure of the sensitivity of the security's real return to a change in real return of the market index,
6. β_{2i} is a measure of the sensitivity of the security's real return to a change in the inflation rates,
7. Bars over random variables indicate expected values.

The only assumption made concerning the relationship between the market

[2] We are referring to that section of Roll's [13] paper in which he derives a CAPM. He clearly has other models that analyze the hedge characteristics of assets.

[3] In particular, when we discuss arbitrage pricing and formulate zero risk strategies, these should be zero risk strategies in terms of real purchasing power.

[4] See Lintner [6] and Reilly [11] for a fuller statement and some empirical evidence.

factor and the inflation factor is the absence of perfect correlation. Thus the market factor can be, and probably is, affected by inflation.

Equation (1) is a special case of the general equation used by Ross [15] to develop the arbitrage pricing model. To get our arbitrage pricing model we follow Ross in setting up an arbitrage portfolio. This portfolio involves zero net investment, zero systematic risk with both of the factors, and is sufficiently diversified that there is a negligible effect of ϵ_i on the portfolio risk. Such a riskless portfolio should have an expected return of zero. Since the vector of portfolio proportions is orthogonal to expected return and the betas, expected return can be expressed as a linear combination of the betas. If a's denote coefficients, this results in

$$\bar{R}_i = a_0 + \beta_{1i}a_1 + \beta_{2i}a_2. \tag{2}$$

There is nothing new in equation (2): it is identical to Ross' [15] equation (9.21) which was derived for the general case. The remainder of this section is a new extension of Ross' work in that the special form of equation (1) will be used to eliminate the a's and replace them with observable economic variables. If we define R_Z as the return on a portfolio that is orthogonal to both R'_M and π then for this portfolio $\beta_{1Z} = \beta_{2Z} = 0$ and $\bar{R}_Z = a_0$.

In order to eliminate a_1 and a_2 we must first derive expressions for β_{1i} and β_{2i}. We have not made any assumption about the orthogonality of the two factors R'_M and π so that we should allow for the possibility of correlation between them. In order to determine β_{1i} and β_{2i}, first calculate $\text{cov}(R_i, R'_M)$ and $\text{cov}(R_i, \pi)$. Utilizing equation (1) for R_i we have[5]

$$\text{cov}(R_i, R'_M) = E[((\bar{R}_i + \beta_{1i}(R'_M - \bar{R}'_M) + \beta_{2i}(\pi - \bar{\pi}) + \epsilon_i) - \bar{R}_i)(R'_M - \bar{R}'_M)].$$

Without loss of generality we can assume that ϵ_i and R'_M are orthogonal. Thus

$$\text{cov}(R_i, R'_M) = \beta_{1i}\,\text{var}(R'_M) + \beta_{2i}\,\text{cov}(R'_M, \pi).$$

Likewise

$$\text{cov}(R_i, \pi) = \beta_{1i}\,\text{cov}(R'_M, \pi) + \beta_{2i}\,\text{var}(\pi).$$

Solving these equations for β_{1i} and β_{2i} results in

$$\beta_{1i} = \frac{\text{cov}(R_i, R'_M)\text{var}(\pi) - \text{cov}(R'_M, \pi)\text{cov}(R_i, \pi)}{\text{var}(R'_M)\text{var}(\pi) - (\text{cov}(R'_M, \pi))^2} \tag{3}$$

and

$$\beta_{2i} = \frac{\text{cov}(R_i, \pi)\text{var}(R'_M) - \text{cov}(R_i, R'_M)\text{cov}(R'_M, \pi)}{\text{var}(R'_M)\text{var}(\pi) - (\text{cov}(R'_M, \pi))^2}. \tag{4}$$

With these expressions we can determine the values of a_1 and a_2. Consider first the portfolio M'. Examining equation (3) shows that for this portfolio $\beta_{1i} = 1$ and examining equation (4) shows that $\beta_{2i} = 0$. Thus from equation (2) and the fact that $a_0 = \bar{R}_Z$ we have

$$\bar{R}'_M = \bar{R}_Z + a_1$$

[5] See any standard econometric text for an alternative derivation of equations (3) and (4).

or

$$a_1 = \bar{R}'_M - \bar{R}_Z.,$$

Likewise consider the nominally riskless asset. The only risk that affects the real return of this asset is due to inflation. In order to proceed further we need an expression relating real and nominal returns. We will utilize the Chen and Boness linkage. However, the same general equilibrium relationship would result from using the Friend, Landskroner and Losq linkage.[6] The Chen and Boness equation relating real and nominal returns is

$$R_i = r_i - \pi \tag{5}$$

where

r_i is the nominal return on asset i,

and π is the inflation rate.

Utilizing this equation and recognizing that $\text{cov}(r_F, \pi) = 0$ since r_F is non-stochastic

$$\begin{aligned}\text{cov}(R_F, \pi) &= E[((r_F - \bar{r}_F) - (\pi - \bar{\pi})), (\pi - \bar{\pi})]\\ &= -\text{var}(\pi).\end{aligned}$$

Likewise

$$\text{cov}(R_F, R'_M) = -\text{cov}(R'_M, \pi).$$

With these substitutions an examination of equations (3) and (4) shows that for the nominally riskless asset $\beta_{1i} = 0$ and $\beta_{2i} = -1$. Thus we have

$$\bar{R}_F = \bar{R}_Z - a_2$$

or

$$a_2 = (\bar{R}_Z - \bar{R}_F).$$

This results in an arbitrage pricing model of

$$\bar{R}_i = \bar{R}_Z + \beta_{1i}(\bar{R}'_M - \bar{R}_Z) + \beta_{2i}(\bar{R}_Z - \bar{R}_F) \tag{6}$$

which we will use in the rest of the paper.

II. Relationship to Prior Models

Up to this point we have developed one model of equilibrium returns from the Ross arbitrage pricing model. However there are two conditions under which the equilibrium model derived from arbitrage pricing can be simplified. One is that the return generating process depends only on the market return. The other is that the return generating process depends on both the market return and inflation but that investors will not (*ceteris paribus*) pay a differential return for an asset that is a superior inflation hedge.

[6] The key relationship in demonstrating this is that under either approximation $\text{cov}(r_F, \pi) = \text{cov}(r_F, r_M) = 0$.

It is easy to deal with the first of these. If the return generating process is the single factor model

$$R_i = \bar{R}_i + \beta_{1i}(R'_M - \bar{R}'_M) + \epsilon_i, \tag{7}$$

then as Ross has shown the equilibrium model is

$$\bar{R}_i = \bar{R}_Z + \beta_{1i}(\bar{R}'_M - \bar{R}_Z). \tag{8}$$

We would argue that this return generating process presented in equation (7) is less appropriate than that presented in equation (1) because both theory and empirical evidence support the fact that inflation has an independent effect on the returns on different assets beyond the market impact. As noted above, a number of theoretical hypotheses have been advanced in the literature as to why firms should have varying sensitivities to inflation.

The second condition under which the equilibrium model can be simplified is more subtle. The return generating process might be given by equation (1) but the market does not price (require a return for) any inflation risk above and beyond that captured by the market return. Shortly we will demonstrate that under either of these conditions the equilibrium model of equation (6) collapses to that of Chen and Boness (C&B), Friend, Landskroner and Losq (F, L&L), or Roll. Before we do so let us discuss why we believe inflation *should* be priced.

To see why inflation should be priced we must examine why the expected real return required by individuals should be a function of inflation. Let us look at a simple example of the conditions under which inflation risk would bear a market price. Consider the extreme case of an individual who consumed only goods that were extremely sensitive to changes in the price index while another individual consumed only goods which were relatively insensitive to changes in the price index. The investor whose consumption was sensitive to changes in the price index would prefer assets which gave exceptionally high returns in periods with a high rate of change in the price index while the other investor would not have such a preference. Because changes in the price index would not perfectly reflect changes in purchasing power for all individuals, inflation risk would be priced in the market.[7]

Another way to picture this difference is to note that if inflation risk were important to individuals then each investor's optimum portfolio should consist of a component that allows a modification of inflation risks according to the sensitivity of the pricing of his personal consumption basket to inflation. This is why, as we shall see, our general model of equation (6) leads to a three mutual fund theorem while the models which assume inflation risk is not priced—(C&B) and (F, L&L)—lead to a two mutual fund theorem. Let us now demonstrate that C&B, F, L&L and Roll can be derived from either of the two conditions.

[7] It might be worth pointing out that this discussion (namely the differential impact of measured inflation) is consistent with our earlier analysis. Despite the fact that alternative rates of inflation impact different investors-consumers differently, for any rate of inflation and market rate of return there is an equilibrium rate of return on each asset. At this equilibrium each investor and consumer is at a consumption-investment decision set which maximizes his utility. As Ross [17] has shown, the existence of equilibrium means the absence of arbitrage profits. Thus, an equation of the form of equation (2) must always follow from equation (1) as shown in Ross [17].

A. Demonstration that the Chen and Boness, Friend, Landskroner and Losq and Roll models can be derived from a single factor generating model.

If returns are generated by the single factor model of equation (7) then as Ross has shown

$$\bar{R}_i = \bar{R}_Z + \beta_i'(\bar{R}_M' - \bar{R}_Z). \tag{8'}$$

where the β_i' is the response of asset i to the market when a second factor is excluded. This is Roll's equation at the bottom of page 922.[8] To get the C&B and F, L&L equation in real terms note that the nominally riskless asset is risky in real terms and thus equation (8′) must hold for R_F. Thus

$$\bar{R}_F = \bar{R}_Z + \beta_F'(\bar{R}_M' - \bar{R}_Z).$$

Solving for $\bar{R}_Z$

$$\bar{R}_Z = \frac{\bar{R}_F - \beta_F'\bar{R}_M'}{1 - \beta_F'}.$$

Substituting into equation (8′) yields

$$\bar{R}_i = \frac{\bar{R}_F - \beta_F'\bar{R}_M'}{1 - \beta_F'} + \beta_i'\bar{R}_M' - \beta_i'\left(\frac{\bar{R}_F - \beta_F'\bar{R}_M'}{1 - \beta_F'}\right).$$

Putting all terms on the right hand side of the equation over a common denominator, adding and subtracting $\beta_F'\bar{R}_F$ in the numerator and simplifying yields

$$\bar{R}_i = \bar{R}_F + \frac{\beta_i' - \beta_F'}{1 - \beta_F'}(\bar{R}_M' - \bar{R}_F).$$

Substituting $\dfrac{\text{cov}(R_i, R_M')}{\text{var}(R_M')}$ for β_i' and $\dfrac{\text{cov}(R_F, R_M')}{\text{var}(R_M')}$ for β_F' yields

$$\bar{R}_i = \bar{R}_F + \frac{\text{cov}(R_i, R_M') - \text{cov}(R_F, R_M')}{\text{var}(R_M') - \text{cov}(R_F, R_M')}(\bar{R}_M' - \bar{R}_F). \tag{9}$$

This is equivalent to C&B's and F, L&L's equations when they are expressed in real terms. Both of these sets of authors express their equilibrium relationship in nominal terms. Elsewhere [2], we show that when equation (9) is expressed in nominal terms it leads directly to the final equilibrium equations presented by C&B and F, L&L.

B. Demonstration that the Chen and Boness, Friend, Landskroner and Losq and Roll models can be derived from the general model by assuming that inflation risk (beyond that captured by the market index) is not priced.

Earlier we argued that mean-variance equilibrium models made the assumption that the only risk that mattered was the covariability with the market. Whether or not an asset provided a hedge against inflation was unimportant. In this section we will assume that an asset's return is generated by the two factor model given in equation (1). From this we already have derived the general equilibrium

[8] Actually, to get Roll's equation replace R_M' with R_M.

equation (6). We will now show that if inflation (beyond that captured in the market index) is not priced, then equation (6) reduces to the results presented by C&B, F, L&L and Roll. To do so we will impose on equation (6) the constraint that two assets which have the same covariance with the market have the same required return regardless of their covariance with inflation. As discussed at the end of this section, imposing this constraint is not equivalent to assuming that a_2 in equation (6) is equal to zero (i.e., that $\bar{R}_Z = R_F$). The analysis of this section will clarify what is meant by a factor not being priced in the market. We are assuming returns are generated by two factors, the market return and inflation. However, we are assuming that investors do not care whether an asset is a good or poor hedge against inflation. *Ceteris paribus* they will not pay a differential premium for a stock with a negative β_{2i}. All they care about is the stocks' covariability with the market.

To demonstrate these assertions define an asset L with the following characteristics:

1. $\text{cov}(R_L, R'_M) = \text{cov}(R_F, R'_M)$
2. $\text{cov}(R_L, \pi) = 0$

Thus asset L has the same market related risk as the nominally riskless asset but its return is uncorrelated with inflation. Using these conditions to simplify the expressions for β_{1i} and β_{2i} presented in equations (3) and (4) the expected return for asset L derived from equation (6) becomes

$$\bar{R}_L = \bar{R}_Z + \frac{\text{cov}(R_F, R'_M)\text{var}(\pi)}{\text{var}(R'_M)\text{var}(\pi) - (\text{cov}(R'_M, \pi))^2}(\bar{R}'_M - \bar{R}_Z)$$

$$- \frac{\text{cov}(R_F, R'_M)\text{cov}(\bar{R}'_M, \pi)}{\text{var}(R'_M)\text{var}(\pi) - (\text{cov}(R'_M, \pi))^2}(\bar{R}_Z - \bar{R}_F).$$

L and F have the same market risk. In the mean-variance world when all that matters is market variability they should have the same expected return or $\bar{R}_F = \bar{R}_L$. Recalling that $\text{cov}(R_F, R'_M) = -\text{cov}(R'_M, \pi)$, substituting $\bar{R}_F$ for $\bar{R}_L$ and solving for $\bar{R}_Z$ yields

$$\bar{R}_Z = \bar{R}_F - \frac{\text{cov}(R_F, R'_M)\text{var}(\pi)}{(\text{var}(\pi))(\text{var}(R'_M) - \text{cov}(R_F, R'_M))}(\bar{R}_M - \bar{R}_F). \tag{10}$$

Examining equation (6) shows that the coefficients of β_{2i} is $(\bar{R}_Z - R_F)$. Since from equation (10) $\bar{R}_Z \neq R_F$, our assertion that the coefficient of $\beta_{21} \neq 0$ is validated. Substituting the expression for $\bar{R}_Z$ into equation (6) and rearranging yields

$$\bar{R}_i = \bar{R}_F + \left(\beta_{1i} - \frac{\text{cov}(R_F, R'_M)\text{var}(\pi)}{\text{var}(\pi)(\text{var}(R'_M) - \text{cov}(R_F, R'_M))}(1 - \beta_{1i} + \beta_{2i})\right)(\bar{R}'_M - \bar{R}_F)$$

Simplifying we again get equation (9)

$$\bar{R}_i = \bar{R}_F + \frac{\text{cov}(R_i, R'_M) - \text{cov}(R_F, R'_M)}{\text{var}(R'_M) - \text{cov}(R_F, R'_M)}(\bar{R}'_M - \bar{R}_F) \tag{9}$$

which is the C&B, F, L&L and Roll equation in real terms.

A number of comments on the analysis in this section are in order. We simplified our asset pricing model to be consistent with mean-variance analysis by using the condition that two assets with the same covariance with the market would offer the same expected return regardless of their correlations with inflation. This is the economically meaningful definition of inflation not being priced. It is important to note that this is not equivalent to assuming that the second factor of equation (6) is not priced. If this second factor was not priced, a_2 would equal zero (i.e. $\bar{R}_Z = R_F$). However, as shown above, the conditions under which our model is equivalent to mean-variance analysis doesn't imply $\bar{R}_Z = R_F$ (see equation (10)). Thus, an arbitrage pricing model can be consistent with mean-variance analysis and still have more than one priced factor.[9] A number of recent papers have used the presence of multiple priced factors as evidence against the mean-variance CAPM. The discussion in this section should serve as a cautionary note for that conclusion.

III. The Asset Pricing Model in Nominal Terms

In section I we derived the asset pricing model when real returns are affected by the market and inflation. The asset pricing model was presented in real terms. To facilitate comparison with other models and to examine implications of our model, it is necessary to express the model of equation (6) in nominal terms. This section utilizes the inflation conversion assumed by C&B and derives the nominal asset pricing model that results in some detail. The nominal model which results from utilizing F, L&L's inflation conversion is stated in the text. This choice should not be viewed by the reader as a preference on our part of the C&B approximation to the one used by F, L&L. The reason for deriving the C&B results in the body of this paper is that, although the derivations are analogous, using C&B's approximation makes the proofs less complex.

As already stated in equation (5), C&B relate nominal to real returns with the following expression

$$R_i = r_i - \pi.$$

This expression is an approximation to a more general expression relating real and nominal returns.[10] Employing this expression and recalling that $R'_M = \alpha R_M$

[9] Another way to account for this seeming anomaly is to see that the β_{1i} that arises from the two factor returns generating process will in general be different from the β_{1i} (β_i') that will arise from the single factor returns generating process. If a_2 equals zero, inflation will still impact a_1 because of the way β_{1i} is measured.

[10] See Chen and Boness [1] for a derivation. Also see Elton and Gruber [2] for a derivation and a comparison of the approximations used by Chen and Boness and those used by Friend, Landskroner and Losq. Chen and Boness relate nominal and real returns by the expression

$$(1 + R_i) = (1 + r_i)\frac{1}{1 + \pi}.$$

They then approximate $\frac{1}{1 + \pi}$ by expanding it in a Maclaurin series dropping all terms with powers greater than zero and assuming that πr_i is sufficiently small so it can be approximated as zero.

$+ (1 - \alpha)R_F$ and that the return on asset F is non-stochastic in nominal terms we have

$$\text{cov}(R_i, \pi) = \text{cov}(r_i, \pi) - \text{var}(\pi), \tag{11a}$$

$$\text{cov}(R_i, R'_M) = \alpha\ \text{cov}(r_i, r_M) - \text{cov}(r_i, \pi) - \alpha\ \text{cov}(r_M, \pi) + \text{var}(\pi), \tag{11b}$$

$$\text{cov}(R'_M, \pi) = \alpha\ \text{cov}(r_M, \pi) - \text{var}(\pi), \tag{11c}$$

and

$$\text{var}(R'_M) = \alpha^2\ \text{var}(r_M) - 2\alpha\ \text{cov}(r_M, \pi) + \text{var}(\pi). \tag{11d}$$

Utilizing these four expressions and equations (3) and (4) for β_{1i} and β_{2i} yields

$$\beta_{1i} = \frac{[\text{cov}(r_i, r_M)\text{var}(\pi) - (\text{cov}(r_i, \pi)\text{cov}(r_M, \pi))]}{[\alpha\ \text{var}(r_M)\text{var}(\pi) - \alpha(\text{cov}(r_M, \pi))^2]} \tag{12}$$

and

$$\begin{aligned}\beta_{2i} = [&\alpha\ \text{cov}(r_i, \pi)\text{var}(r_M) - \alpha\ \text{var}(\pi)\text{var}(r_M) - \alpha\ \text{cov}(r_i, r_M)\text{cov}(r_M, \pi) \\ &+ \alpha(\text{cov}(r_M, \pi))^2 - \text{cov}(r_i, \pi)\text{cov}(r_M, \pi) + \text{cov}(r_i, r_M)\text{var}(\pi)] \\ &/[\alpha\ \text{var}(r_M)\text{var}(\pi) - \alpha(\text{cov}(r_M, \pi))^2].\end{aligned} \tag{13}$$

Recall that the asset pricing model in real terms is

$$\bar{R}_i = \bar{R}_Z + \beta_{1i}(\bar{R}'_M - \bar{R}_Z) + \beta_{2i}(\bar{R}_Z - \bar{R}_F). \tag{6}$$

Substituting in relationship (5) relating real to nominal returns and substituting $\alpha R_M + (1 - \alpha)R_F$ for R'_M yields

$$\bar{r}_i = \bar{r}_Z + \beta_{1i}(\alpha(\bar{r}_M - r_F) - (\bar{r}_Z - r_F)) + \beta_{2i}(\bar{r}_Z - r_F).$$

Adding and subtracting r_F and combining all terms involving $(\bar{r}_Z - r_F)$ gives

$$\bar{r}_i = r_F + \beta_{1i}\alpha(\bar{r}_M - r_F) + (1 - \beta_{1i} + \beta_{2i})(\bar{r}_Z - r_F).$$

Substituting for β_{1i} and β_{2i} and simplifying yields

$$\begin{aligned}\bar{r}_i = r_F &+ \frac{\text{cov}(r_i, r_M)\text{var}(\pi) - \text{cov}(r_i, \pi)\text{cov}(r_M, \pi)}{\text{var}(r_M)\text{var}(\pi) - (\text{cov}(r_M, \pi))^2}(\bar{r}_M - r_F) \\ &+ \frac{\text{cov}(r_i, \pi)\text{var}(r_M) - \text{cov}(r_i, r_M)\text{cov}(r_M, \pi)}{\text{var}(r_M)\text{var}(\pi) - (\text{cov}(r_M, \pi))^2}(\bar{r}_Z - r_F).\end{aligned} \tag{14}$$

This is the asset pricing model which results from an assumption that returns on securities are generated by a two-factor model (a market and an inflation factor) when the C&B approximation between real and nominal returns is used. If the F, L&L approximation is used the equilibrium equation becomes

$$\begin{aligned}\bar{r}_i = r_F &- \text{cov}(r_i, \pi) \\ &+ \frac{\text{cov}(r_i, r_M)\text{var}(\pi) - \text{cov}(r_i, \pi)\text{cov}(r_M, \pi)}{\text{var}(r_M)\text{var}(\pi) - (\text{cov}(r_M, \pi))^2}(\bar{r}_M - r_F - \text{cov}(r_M, \pi)) \\ &+ \frac{\text{cov}(r_i, \pi)\text{var}(r_M) - \text{cov}(r_i, r_M)\text{cov}(r_M, \pi)}{\text{var}(r_M)\text{var}(\pi) - (\text{cov}(r_M, \pi))^2}(\bar{r}_Z - r_F - \text{cov}(r_Z, \pi)).\end{aligned} \tag{15}$$

IV. Implications

There are a number of differences between our model and earlier models which examined the impact of inflation on equilibrium returns. Our model, like those of C&B, F, L&L and Roll assume that investors use real returns to choose between assets. However ours, unlike prior models, assumes that investors are also concerned with an asset's response to inflation (its inflation hedge characteristics). Thus in the equilibrium equation expressed in real terms (equation (6)) the assets hedge characteristics explicitly affect the securities equilibrium return. This is in contrast to earlier models (equation (9)). As previously discussed equation (9) results from either assuming assets real returns are unaffected by inflation (except insofar as inflation affects the variability of the market) or that the only source of variability that affects a security's return is the covariability with a market index. While it is true that prior models expressed in nominal returns contained an inflation term, this term was present only because it entered in the conversion from real to nominal terms and not because it affected an investor's choice between assets. In these earlier models the choice between assets can unambiguously be made in terms of means and variances of real returns. The covariance of the real return of any asset with inflation was a matter of no importance.

To examine the conditions under which our model leads to a different expression for expected real returns, equation (6) should be compared to equation (9). Letting the superscript "o" stand for our model and "p" for prior inflation models, an examination of when our model leads to higher expected returns involves examining the conditions when

$$\bar{R}_i^o - \bar{R}_i^p > 0.$$

Utilizing equations (3), (4), (6) and (8) yields the condition as

$$\beta_{2i}\left[-\frac{\operatorname{cov}(R_M', \pi)}{\operatorname{var}(R_M')}(\bar{R}_M' - \bar{R}_Z) + (\bar{R}_Z - \bar{R}_F)\right] > 0. \tag{16}$$

$\bar{R}_M'$ can be shown to be larger than $\bar{R}_Z$ assuming risk aversion. Otherwise an arbitrage portfolio can be found with positive return. The sign of $(\bar{R}_Z - \bar{R}_F)$ cannot be specified. It depends on investors' feelings toward inflation as manifested by the effect of inflation on their market basket relative to the market basket used as the basis for calculating the rate of inflation in the return generating process. Thus the sign of (16) depends on the sign of β_{2i}, $\operatorname{cov}(R_M', \pi)$ and $\bar{R}_Z - \bar{R}_F$. Table 1 shows the possibilities for an asset whose real return is positively related to inflation (i.e., $\beta_{2i} > 0$). The entry "higher" means that our

Table 1

The Relationship of Expected Return on our Model to Prior Models

	$\bar{R}_Z > \bar{R}_F$	$\bar{R}_Z = \bar{R}_F$	$\bar{R}_Z < \bar{R}_F$
$\operatorname{cov}(R_M', \pi) > 0$	indeterminate	lower	lower
$\operatorname{cov}(R_M', \pi) = 0$	higher	same	lower
$\operatorname{cov}(R_M', \pi) < 0$	higher	higher	indeterminate

model leads to an asset having a higher equilibrium expected return than that which would be predicted by the C&B or F, L&L models.

Empirical studies of the relationship between inflation and nominal returns have found that the nominal return on the market is negatively correlated (or occasionally uncorrelated) with inflation. This would imply $\text{cov}(\underline{R}'_M, \pi) < 0$. Examining the third row of Table 1 shows that in this case if $\bar{R}_Z \geq \bar{R}_F$ then assets with positive correlation of their real return with inflation would require a higher real return in our equilibrium model than in prior models.

Another implication of having an asset's expected return affected by inflation is that the usual two fund theorem is replaced by a three fund theorem. Prior CAPM inflation models allowed an investor to achieve his desired portfolio by utilizing two mutual funds. Any two minimum variance portfolios would, of course, be equally good. However, a convenient choice would be the market and a portfolio uncorrelated with it in real terms. When assets' required returns are affected by inflation a three fund theorem replaces the two fund theorem. These funds are two minimum variance portfolios and a portfolio that is perfectly correlated with inflation. Once again there are a wide variety of choices. However, a convenient set of choices would be the market, a portfolio uncorrelated in real terms with both the real market return and inflation and the nominally riskless asset. Combinations of these three funds allow the investor to select the degree of market risk he desires while adjusting the degree of inflation exposure to a level that is optimum given his consumption preferences.

REFERENCES

1. A. H. Chen and A. J. Boness. "Effects of Uncertain Inflation on the Investment and Financing Decisions of a Firm." *Journal of Finance* (May 1975), 469–83.
2. E. Elton and M. Gruber. "Non-standard C.A.P.M. and the Market Portfolio." Unpublished manuscript, NYU.
3. H. R. Fogler, J. Kise and J. Tipton. "Three Factors, Interest Rate Differentials and Stock Groups." *Journal of Finance* (May 1981), 323–35.
4. I. Friend, Y. Landskroner and E. Losq. "The Demand for Risky Assets under Uncertain Inflation." *Journal of Finance* (December 1976), 1278–97.
5. J. Lintner. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics* (February 1965), 87–99.
6. ———. "Inflation and Security Returns." *Journal of Finance* (May 1975), 259–80.
7. J. B. Long, Jr. "Stock Prices, Inflation, and the Term Structure of Interest Rates." *Journal of Financial Economics* (July 1974), 131–70.
8. R. Merton. "An Intertemporal Capital Asset Pricing Model." *Econometrica* 41, No. 5 (September 1973), 867–88.
9. J. Mossin. "Equilibrium in a Capital Asset Market." *Econometrica* (October 1966), 768–83.
10. G. S. Oldfield, Jr. and R. Rogalski. "Treasury Bill Factors and Common Stock Returns." *Journal of Finance* (May 1981), 337–50.
11. F. Reilly. "Companies and Common Stocks as Inflation Hedges." *The Bulletin*, Graduate School of Business, NYU.
12. M. R. Reinganum. "The Arbitrage Pricing Theory: Some Empirical Results." *Journal of Finance* (May 1981), 313–21.
13. R. Roll. "Assets, Money and Commodity Price Inflation Under Uncertainty." *Journal of Money, Credit and Banking* (November 1973), 903–23.

14. ——— and S. Ross. "An Empirical Investigation of the Arbitrage Pricing Theory." *Journal of Finance* (December 1980), 1073–103.
15. S. Ross. "Return, Risk, and Arbitrage." In I. Friend and J. Bicksler (eds.), *Risk and Return in Finance*. Cambridge: Ballinger, 1977, 189–218.
16. ———. "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory* 13 (December 1976), 341–60.
17. ———. "A Sample Approach to the Valuation of Risky Streams." *Journal of Business* (July 1978), 453–75.