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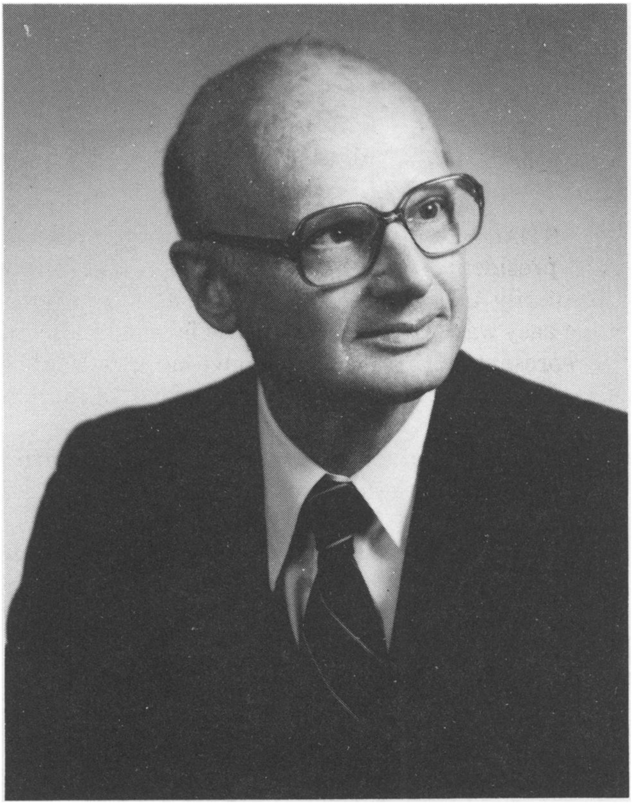

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Nonnegative or Not Nonnegative: A Question about CAPMs.

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THE AFA PRESIDENTIAL ADDRESS usually involves a report on the current research of the AFA president. Unfortunately for the present purpose, my current research is not primarily in finance. I could give a lecture on how to build a database system the easy way [8] [10], using financial examples; but this probably would not suit the present purpose. Since I have no new financial research to report, I have decided to use this occasion to defend an assumption I made previously.

No, I am not going to defend the use of mean and variance as criteria for portfolio selection. If I did, it would be a rehash of arguments I presented in chapters VI and XIII of my 1959 book [9], as reinforced by Young and Trent [18], Levy and Markowitz [6], and Kroll, Levy and Markowitz [5]. The use of mean and variance does not particularly seem to need defense at this time. I am not aware of any complaints about the Levy, Markowitz analysis of the Borch [2], Pratt [12] and Arrow [1] objections to mean-variance; and I listened with great interest to Franco Modigliani's presidential address last year [11] in which he used mean-variance analysis to resolve some problems in corporate finance which had bothered him in prior Modigliani-Miller analysis.

Rather than spring to the defense of assumptions no one seems to be attacking at the moment, I will use this opportunity to defend another assumption of mine, also made in my 1959 book. This assumption has been widely rejected. It has not been rejected by institutions who use portfolio analysis to guide investment—the results of portfolio analysis without this assumption would be obviously impractical, as I will show below. Rather, it has been rejected (or just ignored) by those who create theories about how investors act and consequently how markets work. The result, I shall argue, are Capital Asset Pricing Models whose central implications are subject to considerable doubt.

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As you may recall, in my 1959 book I proposed that an investor choose a portfolio, efficient in terms of mean and variance, subject to certain kinds of

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constraints. These constraints consist of one or more linear equations in nonnegative variables; i.e.,

$$(LP) \quad AX = b \quad X \geq 0$$

where A is an M by N matrix, b an M component vector, and X an N component vector of variables to be chosen. I will refer to this as the LP constraint set, since it is the same constraint set as is used (with different objectives) in linear programming [3]. Appendix A of [9] derives, and chapters VII and VIII illustrate, how to compute efficient sets subject to such constraints.

It is common today in CAPM theory to drop the $X \geq 0$ assumption, and use the constraint set

$$(SE) \qquad AX = b$$

where A and b are as defined above. I will refer to this as an SE constraint set, where SE stands for simultaneous equations. It is usually assumed that $M = 1$, but my remarks will apply to SE models in general. For the most part we shall consider the implications of LP and SE when investors seek mean-variance efficiency. We shall note, however, that the principal objections to SE do not depend on the mean-variance assumption.

Let MV/LP stand for "mean-variance model with a linear programming constraint set"; and MV/SE stand for "mean-variance model with a simultaneous equations constraint set". We start by showing that every MV/SE model can be represented by an MV/LP model; i.e., given any MV/SE model there is an MV/LP model with the "same" answer. The converse is not true. An MV/LP model cannot always be represented by an MV/SE model. Thus MV/LP theory is more general, including MV/SE theory as a special case.

Suppose that X_1 may be positive, zero or negative. (The following transformation is applicable to models in which *some* variables are thus unconstrained with respect to sign, as well as MV/SE models in which *all* variables are unconstrained with respect to sign.) Let $X_1 = Y_1 - Z_1$, where $Y_1 \geq 0$, $Z_1 \geq 0$. Substitute $Y_1 - Z_1$ for X_1 in the constraint equations and in the formulas for E and V. Do the same for all variables in an MV/SE model (or, more generally, do the same for all variables unconstrained with respect to sign). Since the X_i are homogeneous, linear functions of the Y_i and Z_i the constraints remain linear in the new variables, E remains homogeneous and linear, and V remains quadratic. Since the Y_i and Z_i are constrained to be nonnegative we now have an MV/LP model. Since the critical line algorithm of [9] does not require a positive definite covariance matrix, but permits any positive semi-definite matrix, it will not be bothered by the fact that Y_i and Z_i have a correlation coefficient of -1. In interpreting the solution Y_i may be thought of as the positive part of X_i and Z_i as its negative part. Only one of these will be nonzero along any efficient segment produced by the algorithm.

Thus for any MV/SE model there is an MV/LP model with the same answer—i.e., with the same efficient E, V combinations and with the same efficient portfolios if the old variables are suitably interpreted in terms of the new. But the converse is not true. This follows easily from a general property of SE constraint sets which we shall refer to as:

(Property P) If points (portfolios) X and Y are in an SE constraint set, then all points on the entire straight line through X and Y, extended indefinitely in both directions, are also in the particular SE constraint set.

(Proof: $AX = b$ and $AY = b$ imply that $A(aX + (1 - a)Y) = b$ for arbitrary a; but $aX + (1 - a)Y$ for any real number, a, is the locus of all points on the straight line through X and Y.) Property P implies that any SE constraint set must consist of either:

(a) no points (if $AX = b$ is inconsistent); or
(b) precisely one point; or
(c) must be unbounded, since the straight line through any two distinct feasible portfolios can be extended indefinitely in both directions without ever encountering an infeasible portfolio.

In particular an SE constraint set can never be the same as the constraint set of the "standard" three-security model:

$$X_1 + X_2 + X_3 = 1, \qquad X_i \geq 0, \qquad i = 1, 2, 3$$

since this is a triangle which is bounded (of course) and contains more than one portfolio. Furthermore, an MV/SE model cannot have the same set of efficient E, V combinations as any MV/LP model with (1) a bounded constraint set and (2) two different efficient E, V conbinations. (Proof: Appendix A of [9] implies that as we move along a critical line in the direction of increasing E, if the line is efficient it will remain efficient until we encounter a boundary (i.e., an X_i required to be nonnegative goes to zero) or the line is intersected by a "higher dimensional" line (in which another X_i is free to vary). But in an MV/SE model there are no boundaries or higher dimensional critical lines; therefore the line remains efficient without bound in the direction of increasing E. Thus E is unbounded in the efficient set, unlike the efficient set of a bounded MV/LP.)

Not only can no SE constraint set exactly equal a bounded LP constraint set, it cannot even come close. Figures 1a and 1b illustrate this, while reminding us of certain abilities of LP sets to approximate other sets.

An LP set can be used to approximate any bounded, convex set. For example, suppose we wished to approximate the disk

$$(X_1 - C_1)^2 + (X_2 - C_2)^2 \leq R^2$$

in figure 1a, where (C_1, C_2) is the center of the disk and R is the radius. We can approximate this disk as close as we please by an LP set; e.g., by a convex figure whose border is a polygon made up of chords (or tangents) of the perimeter of the disk. (Include an equality of the form $a_iX_1 + b_iX_2 + Z_i = d_i$, for each chord, where the chord satisfies $a_iX_1 + b_iX_2 = d_i$, the center of the disk satisfies $a_iC_1 + b_iC_2 < d_i$, and Z_i is a nonnegative slack variable.)

Thus an LP constraint set can be made to approximate, as close as you please, the circular disk—or any other bounded, convex set. As far as approximating the disk by an SE set is concerned, the only sets which satisfy property P in the two dimensional case are the point, the straight line, and the whole plane. Thus the only kinds of SE approximations to the disk are those illustrated in figure 1b, namely the point P, the line L, and the whole plane. The point P, of course, is not the visible point drawn in the figure, but the mathematical point of zero area; the line has zero thickness and extends infinitely in both directions; and the plane extends infinitely in all directions.

If we knew in advance of selecting an SE approximation, the means, variances and covariances of the securities, and the investor's preferences, we could first solve the problem, then choose as the SE constraint set that point P which is the optimal solution to the original problem. To rule out such peeking at the answer in advance, we may imagine the process of choosing an approximation as a two person game in which player A first chooses the approximation and then player B chooses means, variances, covariances and investor preferences so as to make A's solution as wrong as possible. Player A's objective in choosing the approximation is to minimize the maximum distance between the true optimum and the approximation.

If A is allowed to choose any LP set (perhaps subject to a computational budget or a limit on the number of equations and variables) then he can assure that the maximum distance between the true solution and the approximation equals the maximum distance between the arc of the circle and the chord, for whatever size chord he can afford to use. If A must use an SE approximation, then his best choice is $P = (C_1, C_2)$. For if A chooses a line as an approximation, or the whole plane, then B can make his choice so that the approximation is as far as B pleases from the feasible set, and therefore from the true optimum. With $P = (C_1, C_2)$ as A's choice then B makes the true optimum any point on the circumference. The radius of the circle, R, is then the distance between actual and approximation.

It might seem at first that the limitations on the SE approximation could be circumvented by an approach similar to the way we represented an MV/SE model by an MV/LP model. Perhaps we could

(1) express the original variables $X = (X_1, X_2)$ as homogenous, linear functions of any number of new variables, $X = BY$;
(2) substitute BY for X in the formulas for E and V; (X has to be linear and homogeneous in Y to assure that the substitution leaves E and V in the form assumed by an MV/SE model);
(3) think up clever linear constraints on the Y to arrive at a new SE constraint set;
(4) solve the new MV/SE model, but interpret the efficient portfolios by the formula $X = BY$.

Figure 1. LP Approximation to a Circular Disk.

If we use such an approach, the feasible set in the X-space is the projection onto the X-space of the feasible set in the Y-space, using the formula $X = BY$ to associate a feasible X with each feasible Y. Perhaps if we are clever enough in choosing B and the constraints on Y we can get a constraint set in the Y-space whose projection on the X-space is more satisfactory than either a point, a line or the whole plane.

But this approach won't help. No matter how clever we are in choosing B and the constraints on Y, the projection onto the X-space (of the SE constraint set in the Y-space) we still have Property P and indeed will be either a point, an entire line or the whole plane. (For if X_a and X_b are in the projection, then there must be Y_a and Y_b which are feasible points in the Y-space with X_a and X_b, respectively, as their projections. But then any point on the straight line through Y_a and Y_b is feasible in the Y-space, and its projection is the corresponding point on the straight line through X_a and X_b.)

Thus even with the creation of new variables and constraints, and then the interpretation of these in terms of the original variables, the SE model offers only the point, the line and whole plane in the 2-dimensional case (and similarly offers us any M-dimensional subspace $M \leq N$ in the N-dimensional case) but never a feasible set which contains two points and is bounded.

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An MV/SE model is a special case of MV/LP model, therefore, but not necessarily vice-versa. This would be of no concern if real investor constraint sets were SE, or if MV/SE models had the same implications for questions of concern as MV/LP models generally. We next consider the first of these matters: are real-world constraint sets at least approximately SE?

We begin with the one equation SE system

$$X_1 + X_2 + \cdots + X_N = 1 \tag{1}$$

This is sometimes referred to as a budget equation with short selling allowed; but it is very far from an accurate representation of real-world short selling opportunities. Note, for example, that this equation is satisfied by the solution

$$X_1 = -1{,}000, \quad X_2 = +1{,}001, \quad X_i = 0 \; i = 3, \ldots, N.$$

If the absurdity of the above portfolio is not apparent, try executing it in practice: place \$10,000 with a broker, instruct the broker to sell \$10,000,000 of some security short and use the proceeds of the sale plus the original \$10,000 to buy \$10,010,000 of some other security long. The broker will explain that the real world doesn't work that way.

There are two reasons why the above is not the way the world works. The first reason is that it is illegal. Neither the investor (individual or institutional) nor even the broker for the firm's own account can act that way. The broker is required to have the investor put up collateral to protect the short. The brokerage firm itself is required to reduce ("haircut") its net equity by a fraction of such short positions in its own account. The brokerage firm must maintain a minimum ratio of net equity to liabilities.

Besides being illegal, the other reason you cannot act according to (1) is that it would be irrational for someone to provide you with the opportunities that 1 allows. If you take the \$10,000,000 short and \$10,010,000 long position with only \$10,000 of equity, and you are lucky, then you make a lot of money. But if you are unlucky, who pays for the loss in excess of your equity: the person who loaned you the stock to establish the short? the broker? the Federal Government? Why should any of these, or anyone else for that matter, bear the risk of your loss—stand ready to lay out cash of their own, or lose the value of their stock, maybe in exchange for your I.O.U. if the bankruptcy court doesn't let you off the hook completely?

I am not asserting that individuals and institutions never foolishly bear more of some other investor's risk than is justified by the rewards they receive. In the recent Drysdale fiasco, for example, Chase Manhattan and other banks had, in effect, been middlemen in the loan to the new, relatively small, little-known firm of Drysdale Government Securities Inc. of enough government bonds for Drysdale to take a multi-billion dollar short position. This "lending of securities" took the form of a "reverse repurchase agreement", so wasn't covered by the aforementioned collateral requirements for short positions. When the ball didn't bounce as Drysdale predicted, the Fed pressured Chase Manhattan into absorbing (at least temporarily) the losses beyond Drysdale's resources, rather than have these fall upon the generally weaker institutions that had loaned the bonds.

The decision by Chase Manhattan to underwrite the Drysdale gamble was not a deliberate one made in terms of risk and return to Chase. Rather it was the result of actions of the six salesmen and three officers of the Integrated Securities Services group within Chase who "just did not know what they were doing" [15]. They could see the commissions they were getting from the repurchase (and reverse repurchase) business they were doing with Drysdale, but had no idea of the risks involved; nor did anyone else in giant Chase Manhattan stop to consider—at that time—the exposure to Chase of the actions of this small group within it.

The fact that there are cases in which an institution has underwritten a speculator's gamble without reasonable compensation for it, does not mean that this is the common practice. For example, many banks declined Drysdale's business. Also, the recent bankruptcy of Lombard-Wall was precipitated when Bankers Trust refused to continue clearing the former's government securities transactions because of Lombard-Wall's "deteriorating financial condition." [13] When we turn from more recent and esoteric forms for financing speculation, to more traditional forms, we find that institutions (by and large) understand that they should not underwrite someone else's gambles in exchange for the "risk-free rate of return."

The Drysdale case shows that lenders of securities (or intermediaries in the lending of securities) do not always act wisely; but it makes no sense to assume that such lenders are universally available to all investors, as assumed by equation (1). In a world in which all investors are constrained only by (1), who loans the securities? who suffers the loss if the investor exhausts his resources? what's in it for the broker or stock lender to make this risk worth bearing?

The above objections to equation 1 as a constraint set, also apply to SE constraint sets in general. Because of property P, if a move from portfolio X to a different portfolio Y is permitted, then the investor is allowed to move indefinitely in the same direction. If under some circumstances portfolio Y has slightly less return than portfolio X, then further movement in the same direction, beyond Y, will eventually lead to portfolios with very large losses (i.e., losses of any prescribed amount if you go far enough). Thus if we

(a) rule out "sure bets" (that is, directions in which you cannot make less under any circumstance, and can make as much as you please by moving far enough in the direction, if some events occur); and
(b) ignore, as irrelevant, situations in which no move can have any effect on portfolio return in any event

then we conclude that any SE constraint set allows the investor to place bets of arbitrarily large size with potential losses which he cannot cover himself. As in the case of (1), only a fool would underwrite this bet in return for the risk-free rate. This argument does not assume that the broker or banker necessarily act according to mean-variance, but only that they are not willing to bear risk without some reward for it.

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One can imagine a world in which it would be safe for the broker and lenders of stock to allow the investor to choose any portfolio allowed by (1). This world does not exist in fact; but is often assumed in CAPM theories, and therefore requires our attention. Suppose (i) that security prices varied as a continuous function of time (the exact function is not known in advance, but it is known that the path of prices will be continuous); and suppose (ii) that the broker could sell out the investor, completely and instantly, at any instant of time. In such a world there would be no risk, to anyone but the investor, in allowing the investor to choose any portfolio consistent with 1. The instant that the investor's net worth reached zero his account would be liquidated with no loss to anyone but the investor.

Both assumptions are required for the protection of the broker and lenders of stock. If prices can change at any instant of time but when they change they can jump to a new value, as in a compound Poisson process [4] for example, or if there is a limitation on the rate at which the broker can liquidate the investor's account, then someone other than the investor has an arbitrarily large risk from portfolios allowed by 1. E.g., with a sufficiently large position, one jump in a compound Poisson process could more than wipe out the investor.

Some observable phenomena clearly contradict the implications of (i) and/or (ii). For example, if we look continuously at the path of a stock price, e.g., by watching the NYSE tape in a broker's office, we see immediately that the stock price changes by discrete amounts at discrete instants. It is not a tenable hypothesis, then, that (i) and (ii) are consistent with all observable phenomena. The only kind of hypothesis of this sort worth considering is one that asserts that (i) and (ii) predicts, at least approximately, certain classes of phenomena.

(i) and (ii) plus certain assumptions concerning investors' beliefs and preferences imply an MV/SE CAPM (i.e., a capital market model in which each investor acts according to MV/SE). Below we offer an empirical test of MV/SE CAPMs versus CAPMs based on other MV/LP models. This can also serve as a test of (i) and (ii). For if (i) and (ii) (plus assumptions about beliefs and preferences) imply a particular kind of MV/SE CAPM, and that kind of MV/SE CAPM is accepted or rejected by a Neyman-Pearson test, or is made to seem more plausible or less plausible by a Bayesian analysis, then the same applies to the hypothesis that (i) and (ii) (etc.) predicts the class of phenomena of the test.

Thus, one way at least to test the applicability to certain phenomena of the hypothesis of continuous price movement and instantaneous liquidation, is to test the applicability of MV/SE CAPMs generally. We return later to the question of testing MV/SE versus other MV/LPs. At this point we leave the world of continuous price movements and unlimited liquidity, to return to the comparison of the implications of MV/LP in general versus MV/SE in particular.

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So far we have seen that MV/LP models include MV/SE models as a special case, but not vice versa; and that an SE constraint set allows the investor opportunities which no broker in his right mind (in the real world) could permit. This would be of little consequence if SE models had the same implications for important financial questions as do LP models generally. We shall now show, to the contrary, that one of the main conclusions of MV/SE models is typically not true of MV/LP models with bounded constraint sets.

We shall assume, as is usual in CAPMs, that all investors have the same beliefs, and that the covariance matrix in nonsingular. We also assume that all constraint sets contain more than one portfolio. Constraints which are infeasible or can only be satisified by one portfolio are possible for general SE or LP specifications, but are of no interest for the present discussion.

Any MV/LP model, including MV/SE models as a special case, has a set of efficient portfolios which is either a single point or else is piecewise linear. For reasons cited in a preceding proof, in the case of an MV/SE model this efficient set may be either a single point (if all feasible portfolios have the same mean) or is a "half-line" starting at the portfolio with minimum V and extending without end in the direction of increasing E. This is a special case of a piecewise linear efficient set, namely one in which there is only one piece.

In the case of an MV/LP model with a bounded constraint set, the efficient set may consist of one or more linear pieces. Cases are shown in chapter VII of [9] in which the standard three-security analysis has in one case a single point for an efficient set, and in another case a single straight line for an efficient set. On the other hand, the efficient set usually consists of many connected straight line segments. For example, in practice a portfolio analysis involving hundreds of securities will typically have an efficient set consisting of hundreds of connected pieces (though only a few points from this set may be displayed, for reasons of cost or clarity).

If all investors (assumed to have the same beliefs) select portfolios from an efficient set consisting of one straight line (either bounded or unbounded) then a weighted average of these (finite number of) investors will also be on the same straight line. Since the market portfolio is such a weighted average, the market portfolio is an efficient portfolio in this case—i.e., when the efficient set consists of one line segment. Hence the implication of MV/SE models that the market is an efficient portfolio.

But if the efficient set consists of two or more straight line segments the market will not necessarily be an efficient portfolio. For example, consider the three security analysis with standard constraint set in figure 2, where X_3 is given implicitly by $X_3 = 1 - X_1 - X_2$. Suppose that the efficient set consists of the two line segments *ab* and *bc*, as in the figure. Suppose further that some investors select portfolios from one (or more) points on *ab*, such as *d*, while others select portfolios from *bc*, such as portfolio *e*. The weighted average of such portfolios will be a portfolio, like *f*, which is neither on *ab* nor *bc*—i.e., is not in the efficient set. The market, then, may be in equilibrium with some investors holding portfolio *d*, some portfolio *e*, the market as a whole having proportions *f*—and *f* not efficient.

In a market with hundreds of securities, a bounded MV/LP model will typically consist of hundreds of linear segments, some efficient for very small ranges of *E*. Investors will be on one segment or another depending on risk-aversion. The particular weighted average of these portfolios which is the market will almost surely not fall on any of the critical lines, and therefore not be efficient.

It is not simple to state which subclasses of MV/LP models inevitably imply that the efficient set consists of a single line segment (assuming the efficient set is feasible and contains more than one point). I say "inevitably imply", since a constraint set (such as the standard constraint set in figure 2) can typically imply a many-piece efficient set yet, with careful choice of security parameters, may have instances of single line efficient sets. We have seen that MV/SE models

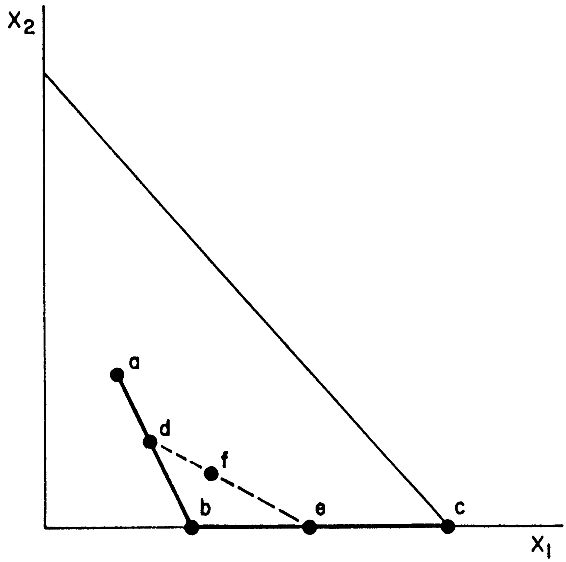

Figure 2. Three (3) SE Approximations to the Disk.

inevitably imply a single line efficient set. Other MV/LP models can do so too: see for example the original Sharpe-Lintner CAPM [17 |], [7] in which securities must be held in nonnegative amounts, but the investor may lend all he has or borrow any amount he wants, and the investor can borrow or lend at the same rate of interest. This CAPM model has an unbounded constraint set. It is not true, however, that all MV/LP models with unbounded constraint sets inevitably have single line efficient sets. Counter examples include: (a) the same model except with different interest rates for borrowing and lending, and (b) the same model without lending and with nonnegative borrowing. As noted previously, MV/LP models with bounded constraint sets typically have many-piece efficient sets; but trivial counter examples can be constructed wherein the efficient set is inevitably single line, since the entire constraint set consists of a single line segment.

Thus some MV/LP models (including the MV/SE models) inevitably have single line efficient sets; others do not. The MV/LPs which are actually computed for real investing institutions have hundreds of segments for analyses with hundreds of securities. The significance of this in the present context is that much touted "conclusion" of CAPM, that the efficient set lies on a single line and therefore the market is an efficient portfolio, is not at all robust. On the contrary, it is a fragile conclusion that depends delicately on choosing just the right assumptions. By "right" assumption I refer here to "right" in terms of getting this particular conclusion, not right in terms of plausibility of the constraint set as a description of, or approximation to, real-world investor opportunities.

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Above we showed that if we make the usual CAPM assumptions, except that we assume an LP rather than as SE constraint set, we no longer can conclude that the market is an efficient portfolio. It might be feared at this point that the substitution of the LP for the SE constraint set might generally lead only to such negative results; it might be feared, in other words, that the LP assumption is so broad that no practical or testable implications can be deduced from a CAPM based on this more general assumption.

On the contrary, it turns out that the substitution of the LP for the SE constraint set offers abundant new opportunities for theoretical deduction and empirical research. In planning this talk I had hoped to present one or two such results; but it turns out that this would increase the length of this paper beyond the reasonable bounds of a AFA presidential address. Instead, I will end with four "Exercises" for the listener or reader. These exercises are of two sorts. (1) and (4) are theoretical results which follow more or less immediately from the properties of MV/LP efficient sets as spelled out in Appendix A of [9]. Exercises (2) and (3), on the other hand, are empirical in nature. The conclusions of the latter type of exercise remain to be seen.

(1) Roll [14] shows for an MV/SE model with equation 1 as its constraint set, that if the market Index I is an efficient portfolio, all portfolios (whether efficient or not) will have an (E,β) combination which lies on the (E,β) market line. Show

that in the case of an MV/LP model the conclusion is still valid for portfolios which use only securities found in the Index; but that a portfolio X which includes other securities may lie to the inefficient side of this line—even if X is efficient. Hint: show that the Kuhn-Tucker conditions for efficient portfolios, applied to I, may be written as

$$\beta_i V_I - \sum_{k=1}^{M} \lambda_k a_{ki} - \lambda_E \mu_i = 0 \tag{2}$$

for security i in I, and

$$\beta_i V_I - \sum_{k=1}^{M} \lambda_k a_{ki} - \lambda_E \mu_i \geq 0 \tag{3}$$

for security i out of I. (The left hand side of 3 is typically greater than zero for securities out of an efficient portfolio, even if they enter other efficient portfolios). From this deduce, for example, that

$$V_I B - \sum_{k=1}^{M} \lambda_k b_k - \lambda_E E = 0$$

for any feasible portfolio using only securities in the Index; where, here and above, V_I is the variance of the Index "portfolio", $\lambda_1 \ldots \lambda_M$ and λ_E are the values of Lagrangian multipliers for the Index portfolio, μ_i and β_i are the expected return and regression against I of the i-th security, E is the mean of any portfolio X, and B is the regression of X against the Index.

(2) Statistical tests of CAPMs necessarily involve a bundle of hypotheses: e.g., the hypothesis that all investors act according to some mean-variance model, plus the hypothesis that they all have the same beliefs, plus the hypothesis that their beliefs are exactly equal to the historic means, variances and covariances for some period (or the like). Some of these assumptions can be weakened by use of observed portfolio holdings as of some instant in time. For such a cross-section of portfolio holdings for some class of investors (e.g., investment companies), find beliefs (means, variances and covariances or betas of securities) to minimize the sum of squares of deviations in equations like 2 and inequalities like 3. By a "deviation" I mean a nonzero value of the left-hand-side of an equation like 2 or a negative (but not a positive) value of the left-hand-side of an inequality like 3. This minimum is to be taken over all portfolios and all securities in a specified set. The procedure for finding these estimates may be adopted from the procedures for finding mean-semivariance efficient sets (see chapter IX and Appendix A of [9]).

At the researcher's option, the assumed LP constraints may be a single budget equation in nonnegative variables, or can model more precisely the actual constraints of the class of investors. Another researcher option concerns the assumed model of covariance. In particular, if the one-factor model [16] is hypothesized, and its diagonal form used, the equations take on a relatively simple form, and only 3N beliefs need be estimated. Estimated beliefs concerning variances can be constrained to be nonnegative, while beliefs about means and betas can be allowed to be negative or positive.

(3) Estimates of beliefs can similarly be obtained from a cross section of portfolio holdings assuming an MV/SE model. The difference would be that the

beliefs would be chosen so as to minimize the sum of squared deviations, including both positive and negative deviations in 3 for securities not in a portfolio. Compare the likelihood of getting the observed portfolios given these beliefs and the MV/SE model with those obtained in exercise 2 for some bounded MV/LP model.

(4) [9] solves the problem of finding one efficient portfolio for each efficient (E, V) combination. This solution applies even if the covariance matrix is singular. (A singular covariance matrix will arise if historic data is used for covariances and there are more securities than years; or if slack variables are introduced as part of the constraints; or if the analysis includes one or more riskless securities, such as one or more rates at which money can be borrowed with limitations on how much can be borrowed at a given rate.) Describe the set of *all* efficient portfolios—as distinguished from the aforementioned subset consisting of one portfolio for each efficient (E, V) combination. Hint: note that constraints $AX = b$, $X \geq 0$ can always be rewritten as $DX \geq d$ for suitably chosen D and d, and vice versa. The results of Appendix A of [9] can easily be reverbalized in terms of $DX \geq d$. If necessary, rotate the axes (i.e., choose new variables) so that, in terms of the new variables, a "security" is either riskless or is part of a nonsingular risky submatrix. Such a rotation of axes is always possible, since it is always possible to rotate the axes so that the covariance matrix is diagonal. Such a rotation leaves a constraint set of the form $DX \geq d$ in the same form. Referring to new variables as securities, show that all portfolios which provide a given efficient (E, V) combination have the same combination of these (perhaps new) risky securities. Thus in the subspace of these new risky securities the efficient set is piecewise linear, as in [9]. Show that the set of all solutions for a given such piece is convex, and the sets for two successive pieces share a common interface.

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In summary, the bad news is that the implications of many current CAPMs are based on the highly unrealistic assumption of an SE, rather than an LP constraint set. The good news is that if we substitute the more general assumption for the objectionable assumption the resulting CAPM can still be used in theoretical and empirical analyses. For example, Roll's objection to using position relative to an (E,β) market line as a measure of portfolio performance is as much supported by an MV/LP CAPM as by the original MV/SE CAPM. The possible future empirical research sketched above, using a cross section of portfolio holdings as inputs, not only permits the use of bounded MV/LP constraint sets, but also permits the weaker assumption that *some* investors have common beliefs in place of the stronger (therefore less desirable) assumption that *all* investors have the same beliefs.

I hasten to add that the use of portfolio analysis normatively does not depend on the success of any such empirical research. The use of portfolio analysis by an individual investor can be desirable even if no one else in the world uses it. Thus the success of any particular MV/LP CAPM as a positive theory is in no way a prerequisite to the desirability of MV/LP analysis for normative purposes.

REFERENCES

1. K. J. Arrow. *Aspects of the Theory of Risk Bearing.* Yrjo Johnsson Lectures, Helsinki, 1965.
2. K. Borch. "A Note on Uncertainty and Indifference Curves," *Review of Economic Studies*, January 1969.
3. G. B. Dantzig. *Linear Programming and Extensions*, Princeton University Press, Princeton, New Jersey, 1963.
4. W. Feller. *An Introduction to Probability Theory and Its Applications*, Vol 2., 2nd ed., Wiley, New York 1971.
5. Y. Kroll, H. Levy and H. M. Markowitz. "Mean Variance Versus Direct Utility Maximization," IBM Thomas J. Watson Research Center Research Report RC 9176, Yorktown Heights, New York, 1981.
6. H. Levy and H. M. Markowitz. "Approximating Expected Utility by a Function of Mean and Variance," *American Economic Review*, 69 (1979), pp. 308–317.
7. J. Lintner. "The Valuation of Risk Assets and the Selection Risky Investments in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics*, February 1965, pp. 13–37.
8. H. M. Markowitz, A. Malhotra and D. P. Pazzel. "The ER and EAS Formalisms for System Modeling, and the EAS-E Language," *Entity-Relationship Approach to Information Modeling and Analysis,* P. P. Chen (ed.), ER Institute, 1981, pp. 29–47.
9. H. M. Markowitz. *Portfolio Selection: Efficient Diversification of Investments.* New York: John Wiley and Sons, 1959; New Haven: Yale University Press, 1970.
10. A. Malhotra, H. M. Markowitz and D. P. Pazel. "The EAS-E Programming Language," IBM Thomas J. Watson Research Center Research Report RC 8935, Yorktown Heights, New York, 1981.
11. Franco, Modigliani. "Debt, Dividend Policy, Taxes, Inflation and Market Valuation," *Journal of Finance*, 37 (1982), pp. 255–273.
12. J. W. Pratt. "Risk Aversion in the Small and in the Large," *Econometrica*, 32 (1964), pp. 122–136.
13. G. Putka and G. Sanders. "Lumbard-Wall, Allied Firm File Bankruptcy Plea," *Wall Street Journal*, August 13, 1982, page 3.
14. R. Roll. "Ambiguity When Performance is Measured by the Securities Market Line", *Journal of Finance*, September 1978, pp 1051–1069.
15. J. Salamon. "Chagrined Chase: How New York Bank Got Itself Entangled In Drysdale's Dealings," *Wall Street Journal*, June 11, 1982, page 1.
16. W. F. Sharpe. "A Simplified Model For Portfolio Analysis," *Management Science*, January 1963, pp. 425–42.
17. W. F. Sharpe. "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance*, September 1964.
18. W. E. Young and R. H. Trent. "Geometric Mean Approximations of Individual and Portfolio Performance," *Journal of Financial and Quantitative Analysis*, June 1969.