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Author(s): Itzhak Venezia
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A Bayesian Approach to the Optimal Growth Period Problem: A Note

ITZHAK VENEZIA*

A CLASSICAL PROBLEM IN finance is the following: a firm has a forest that grows with time and will be worth x_t if cut at time t. When should the forest be cut down and what is its present value? The solution of this problem is straightforward when the function x_t is known with certainty (see, e.g., Bierman [2], Brealey and Myers [4], Hartman [9], Hirshleifer [10]): the forest should be cut down when the proportionate increase in its sale value equals the interest rate. If the optimal cutting time is T^* then the present value of the forest is $\rho^{T^*}x_{T^*}$, where ρ is the appropriate discount factor.

This problem is more difficult when the proceeds x_t from cutting and selling the forest are unknown (i.e., random). The optimal strategy for selling the asset (henceforth, we shall refer to the forest as "the asset") is then defined by a set of reservation prices which are functions of the age of the asset. At each period t, the firm compares x_t with the appropriate reservation price. If x_t is higher, the firm sells the asset; otherwise it holds it for one more period.[1] The reservation prices must be determined by using dynamic programming techniques since the decision whether or not to sell the asset affects the firm's opportunity set (if the firm sells, it will not have the asset next period). Venezia and Brenner [14], and Brock, Rothschild, and Stiglitz [6] have analyzed the firm's optimal strategy, and investigated the effect of increased dispersion of the proceeds $\tilde{x}_t$ on the reservation prices and on the value of the asset. They show that increased dispersion increases the reservation prices and the value of the asset. In these studies, however, it has been assumed that the distributions of future proceeds from selling the asset are known at the start and will not be reassessed. This is a strong assumption when the duration of growth investments is long. The firm may then obtain information which changes its prior beliefs about the future prospects of the asset; it may observe a streak of extremely high or low offers for the asset.

The main objectives of this article are first, to determine the optimal selling strategy of the firm, assuming that it reassesses its beliefs about future prospects, using Bayes's rule; and second to compare the results obtained in the present model with those of the non-Bayesian case.[2]

* The Hebrew University and The University of California, Berkeley. The author wishes to thank Carliss Baldwin, Professor Michael Brennan, and Professor Sasson Bar-Yosef for helpful comments and suggestions.

[1] By "selling the asset," we mean terminating the physical investment and, in the forest's case, cutting and selling the forest for lumber.

[2] The strategy of selling an asset when the parameters of the distribution are unknown has also been investigated by DeGroot [7] and Ross [13]. They do not consider, however, the case where there is a growth trend in the offers for the asset, do not compare the Bayesian solution to the non-Bayesian one, and do not investigate the effect of increased dispersion on the results. Furthermore, DeGroot assumes no discounting, and Ross only considers inference between two different distributions.

In the Bayesian case, as in the non-Bayesian case, in each period the firm compares the current offer with the value of the asset if it is kept. If the former is higher, the firm sells the asset; otherwise, it keeps it. In the Bayesian case, however, the value of the asset if it is kept depends on the current offer because the current offer affects the firm's beliefs about the future prospects of the asset. We demonstrate how the optimal selling strategy is determined, and prove that as in the non-Bayesian case, the reservation prices and value of the asset increase with increased dispersion. We show that in the Bayesian case the reservation prices may be infinite (this cannot happen in the non-Bayesian case), and we present and interpret conditions which allow this. It is also shown that the lower the discount rate, the higher the growth rate, and the higher the variance, the more likely that the Bayesian firm has higher reservation prices than the non-Bayesian firm.

The paper is organized as follows. In Section I we present the basic model and the statement of the objective function of the firm. An application of our model is also discussed. In Section II, as a preliminary to the analysis of subsequent sections, we analyze the case of stationary investments (by "stationary" we mean that time per se does not affect the expected proceeds from selling the asset). In Section III we consider the case of growth investments, and in Section IV we compare the Bayesian and non-Bayesian decision policies.

I. The Model

Consider a firm whose shares are traded in a perfect securities market and which holds an asset (say, a forest) whose value increases stochastically with time. Suppose that the firm is small enough that at any period t, its actions do not affect the market price of lumber π_t. If the volume of lumber contained in the forest at t is k_t, then by cutting the forest and selling its lumber, the firm may receive the cash flow $x_t = \pi_t k_t$, and we say that the firm obtained for its asset at period t an offer of x_t.[3] At any period, the future offers the firm will obtain for its asset are uncertain because future prices of lumber are unknown, and the physical growth pattern of the asset is uncertain. These two sources of uncertainty are represented by saying that prior to t, $\tilde{x}_t$ is some random variable. It is assumed that the firm does not know with certainty the mean of $\tilde{x}_t$ and its mean growth rate. The firm, however, can learn about these parameters from previous observations of offers. We address the questions: when should the firm terminate its investment and what is its value?[4]

If the firm must terminate the investment at some specified period in the future

[3] The cash flow x_t is net of transaction costs and costs of cutting the forest, and hence may be negative.

[4] One may wonder why the firm is not always indifferent between selling or keeping the asset. i.e. why the value of the asset at t is not just x_t. The answer is that whereas lumber (of a given quality) is a homogeneous commodity, forests are not since they may differ in age, and thus in growth potential. It may be profitable to cut and sell for lumber an older forest, at the going price of lumber but not a younger one. Actually, owning a forest is the same as having an option to sell it for lumber at the market price for lumber. If there were a perfect market for *uncut* forests, the firm would be indifferent between selling the *uncut* forest and keeping it. This does not eliminate the problem of determining when to cut the forest.

T, and if it does not revise its estimates of the relevant parameters, then the value of the asset is

$$W_0 = E_0(\tilde{x}_T) \prod_{t=1}^{T} 1/[1 + E(\tilde{R}_t)] \tag{1}$$

where $E_0(\tilde{x}_T)$ is the expected cash flow, as seen at the start, from selling the asset at T, and $E(\tilde{R}_t)$ is an appropriate risk adjusted discount rate.[5]

In our case, the parameters of the distribution of $\tilde{x}_T$ will be revised during the interval $(0, T)$ since the firm learns about the expected offers from experience. However, since Fama [8] has shown that (1) is the proper measure of the asset's value if the only reassessments of parameters are reassessments of means, and since it is shown below that only such reassessments are needed in our model, (1) is indeed the appropriate measure of the asset's value.

In general, $E(\tilde{R}_t)$ depends on the correlation between $\tilde{x}_t$ and the market rate of return, $\mathrm{Cov}(\tilde{x}_t, \tilde{R}_{M,t})$.[6] In our case, however, it seems reasonable to assume that $\mathrm{Cov}(\tilde{x}_t, \tilde{R}_{M,t})$ is zero or close enough to zero (the correlation between prices of lumber and the market rate of return on securities seems to us to be small); hence, the value of the asset is

$$W_0 = E_0(\tilde{x}_T) \prod_{t=1}^{T} \rho_t \tag{2}$$

where $\rho_t \equiv 1/(1 + r_{Ft})$, and r_{Ft} is the risk free interest rate.

Suppose now that the firm does not have to sell the asset at T but can sell it at any period prior to T. In this case the firm will base its decision whether or not to sell the asset at any period $t(t < T)$, on the current offer it has for the asset, and on its expectations about future offers. The sale time $\tilde{s}$ is not known at the start since the decision to sell depends on future offers. The firm has then to determine a sales strategy that will maximize

$$W_0 = E_0(\tilde{x}_{\tilde{s}}) \left[\prod_{t=1}^{\tilde{s}} \rho_t\right] \tag{3}$$

In the next section we analyze the learning process of the firm and its optimal sale strategy for the case in which there is no growth trend in the evolution of $\tilde{x}_t$. This is a necessary preliminary to the analysis of subsequent sections.

II. The Case of Stationary Investments

Suppose that the firm obtains at each period t one offer, $\tilde{x}_t$, which is normally distributed with an unknown mean $\tilde{\mu}_t$ and a known variance σ_t^2. That is

$$\tilde{x}_t = \tilde{\mu}_t + \tilde{\epsilon}_t, \qquad t = 1, 2, \cdots \tag{4}$$

where the $\tilde{\epsilon}_t$'s are normally distributed with mean zero and variance σ_t^2. They are independent of each other and of the $\tilde{\mu}_t$'s.

Since the firm does not know $\tilde{\mu}$, it may regard this parameter as a random variable. The firm can learn about this random variable from past observations of offers. It is assumed that the firm uses Bayes's rule when revising its beliefs

[5] See Bogue and Roll [3], Brennan [5], Fama [8], Myers and Turnbull [12].

[6] Fama [8] describes the relationship between $E(\tilde{R}_t)$ and $\mathrm{Cov}(\tilde{x}_t, \tilde{R}_{M,t})$.

about $\tilde{\mu}$. The revision is done as follows. Suppose that at the beginning of period t the beliefs of the firm about $\tilde{\mu}$ are represented by the normal distribution with mean m_t and variance V_t; that is,

$$\tilde{\mu}_t = m_t + \tilde{\delta}_t \tag{5}$$

where $\tilde{\delta}_t$ is normally distributed with zero mean and variance V_t.[7] It can be shown (see, e.g., DeGroot [7], p. 167) that if an offer x has been observed at period t, then the posterior distribution of $\tilde{\mu}$ at the end of period t is normal with mean m'_t and variance V'_t given by

$$m'_t = \gamma m_t + (1 - \gamma)x, \qquad V'_t = \gamma V_t$$
$$\gamma = \sigma_t^2/(\sigma_t^2 + V_t) \tag{6}$$

The weights γ and $(1 - \gamma)$ of m and x in the future mean reflect the relative information content of prior beliefs and observations of offers.

Since we are considering the case of stationary investments ("stationary" here means that time per se does not affect the value of the asset), it follows that the prior distribution of $\tilde{\mu}$ at the beginning of period $(t + 1)$ is normal with mean $m_{t+1}(x)$ and variance V_{t+1} given by

$$m_{t+1}(x) = m'_t = \gamma m_t + (1 - \gamma)x, \qquad V_{t+1} = V'_t = \gamma V_t \tag{7}$$

where γ is given by (6).

Having described the learning process of the firm, we now turn to describe its decision process. One observes from (7) that the whole sequence of variances, V_t, is known at the start since revisions of V_t do not depend on x, so that the only parameters reassessed between the origin and period t are the means m_1, m_2, $\cdots$, m_t. Since Fama [8] has shown that the multiperiod CAPM holds in the case where the only reassessments needed are those of the m_j's, it follows that (3) is the appropriate measure of the asset's value. The objective of the firm is therefore to determine a selling strategy that will maximize (3). For this, the following definition is needed. Let $W_t(x, m, V)$ denote the maximal expected discounted proceeds from the asset (i.e. the asset's value), given that at the beginning of period t the parameters of the distribution of $\tilde{\mu}$ are m and V, and that during the period an offer of x has been made for the asset.

Suppose that the firm has received an offer x at period t. The firm can either accept it, and obtain the proceeds x, or reject it, and then move to period $(t + 1)$ where the decision process continues. In the latter case, the parameters of the distribution of $\tilde{\mu}$ at the begining of period $(t + 1)$ will be $m_{t+1}(x)$ and V_{t+1} given by (7). At period $(t + 1)$ an offer $\tilde{x}_{t+1}$ will be made, distributed according to the parameters relevant at period $(t + 1)$. It follows that the expected maximal proceeds from $(t + 1)$ on, when discounted to period t, are given by

$$\rho_t E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})]$$

[7] Although we assume no growth trend in the μ's, we index μ and m by t. By this we emphasize that the distribution of μ at two different periods t_1, t_2 will not be the same since some bids will be observed in the interval (t_1, t_2). μ_t and m_t should be interpreted as μ and m at period t and the exact dependence on time is: $\mu_t = \mu(x_1, \cdots, x_{t-1})$, $m_t = m(x_1, \cdots, x_{t-1})$. That is, the distribution of μ at period t depends on the prior distribution and on the observations of bids until that period.

The expectation E is taken with respect to the marginal distribution of $\tilde{x}_{t+1}$ which is normal with mean $m_{t+1}(x)$ and variance $(V_{t+1} + \sigma_{t+1}^2)$.

Since $W_t(x, m, V)$ represents the outcome of a maximizing behavior:[8]

$$W_t(x, m, V) = \text{Max}\{x, \rho E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})]\} \tag{8}$$

The firm will thus sell the asset only if

$$x \geq \rho E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})] \tag{9}$$

In order to determine the optimal strategy of the firm, and the value of the asset, the functions $W_t(x, m, V)$ should be computed. This is straightforward in the case where the planning horizon is some finite T.[9] In this case $W_T(x, m, V) = x$, and hence

$$\begin{aligned} W_{T-1}(x, m, V) &= \text{Max}\{x, \rho E[\tilde{x}_T, m_T(x), V_T]\} \\ &= \text{Max}\{x, \rho[\gamma m + (1 - \gamma)x]\} \end{aligned} \tag{10}$$

Based on $W_{T-1}(x, m, V)$ one can compute $W_{T-2}(x, m, V)$, and similarly all other $W_t(x, m, V)$ can be computed.[10]

From (10) it follows that at period $(T - 1)$ the firm will sell the asset if and only if the offer x satisfies

$$x \geq x_{T-1}^* \equiv m \frac{\rho\gamma}{(1 - \rho) + \rho\gamma} \tag{11}$$

For periods $t < T - 1$, it is also true that there exist "reservation prices" x_t^* such that the asset will be sold if and only if $x \geq x_t^*$. These reservation prices can be obtained by solving (after replacing the inequality sign by equality sign) the implicit equation (9). It can be shown by induction (see Venezia [15]) that for any t, m, V, the reservation price x_t^* is unique and finite (this is not the case, as we shall later see, with growth investments).

We now investigate the effect of increased dispersion on the reservation prices. The function $W_t(x, m, V)$ represents the value of an offer x, in the sense that it denotes the maximal expected discounted proceeds, given that an offer x has been received. Since Venezia [15] has shown that the functions $W_t(x, m, V)$ are convex functions of x for all t, $E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})]$ is the expectation of a convex value function, and therefore increases as dispersion (σ_{t+1}^2) increases. It follows from (8) and (9) that so do the value of the asset, $W_t(x, m, V)$, and the reservation price x_t^*.

The result that increased dispersion increases the reservation prices and the value of the asset has also been obtained in non-Bayesian models by Venezia and Brenner [14], and Brock, Rothschild, and Stiglitz [6].[11]

[8] To simplify notation we drop the index t from ρ_t whenever this does not lead to confusion.

[9] Since the infinite horizon case does not provide any further economic insights, we shall not consider it here.

[10] If one considers just the case in which $t = T - 1$, then one actually considers a two period model. Our model is a multiperiod one but some of the results are inferred from the two-period model.

[11] A similar effect of increased dispersion has been obtained in other types of economic activities: e.g., job search models (Kohn and Shavell [11]), and liquidity models (Baldwin and Meyer [1]). All these models, however, are non-Bayesian.

In the non-Bayesian case, the economic intuition for the above result is the following. Increased dispersion increases the probability of both high and low offers. If the investor observes a low offer, he may wait until a "better" one comes along. The larger the dispersion, the larger the probability of obtaining high offers.

In the Bayesian case, a low offer reduces the probability of obtaining high offers in the future. However, the future mean of the offers is a convex combination of the offer x and the prior m; thus, the "damage" in observing a low x is partially offset by the prior.

III. The Case of Growth Investments

We first consider the case in which the pattern of growth is known. As in Section II, it is assumed that offers $\tilde{x}_t$ are drawn from a normal distribution with uncertain mean and known variance. The mean, $\tilde{\mu}_t$, of $\tilde{x}_t$ is assumed to grow following the process

$$\tilde{\mu}_t = \tilde{\mu}_{t-1} + \alpha_t \qquad t = 1, 2, \cdots \tag{12}$$

where the α_t's are known constants.

As in the former case, the only unknown parameter is $\tilde{\mu}_t$. The learning process of the firm is similar to that described in Section II; however, (7) is replaced by

$$m_{t+1}(x) = m'_t + \alpha_t = \gamma m_t + (1 - \gamma)x + \alpha_t \tag{13}$$

where γ is given by (6).

A more interesting case is the one in which the pattern of growth is unknown and the investor learns about this pattern from past observations. In this case we assume that the increase α_t in the mean of the distribution is unkown, and (12) is replaced by

$$\tilde{\mu}_t = \tilde{\mu}_{t-1} + \tilde{\alpha}_t, \qquad t = 1, 2, \cdots \tag{14}$$

where the tilde over α_t signifies that now α_t is considered a random variable. It is also assumed that the beliefs of the firm about $\tilde{\alpha}_t$ are represented by a normal distribution with mean η_t. That is

$$\tilde{\alpha}_t = \eta_t + \tilde{u}_t \tag{15}$$

where the $\tilde{u}_t$'s are normally distributed random variables with zero mean and known variance U_t.[12]

Here the firm must update each period its beliefs both about $\tilde{\mu}_t$ and $\tilde{\alpha}_t$. The firm does so using two pieces of data: observations of past offers provide knowledge about $\tilde{\mu}_t$, and observations of past differences in offers provide information about $\tilde{\alpha}_t$. We shall first show how beliefs about $\tilde{\alpha}_t$ are revised.

[12] Alternative, more complicated, assumptions, can be analyzed using the same techniques we use here. For example, the case of a linear trend in growth can be studied by replacing (14) by $\tilde{\mu}_t = \tilde{\mu}_{t-1} + a_0 - t\tilde{\alpha}_t$, where a_0 is some known constant and $\tilde{\alpha}_t$ is defined as above. Since the results of this and the following sections are preserved also under these alternative assumptions (see Venezia [15]), we preferred using the simpler assumptions in order to facilitate the exposition.

From (4) and (14) it follows that

$$\tilde{D}_t \equiv \tilde{x}_t - \tilde{x}_{t-1} = \tilde{\mu}_t - \tilde{\mu}_{t-1} + (\tilde{\epsilon}_t - \tilde{\epsilon}_{t-1}) = \tilde{\alpha}_t + (\tilde{\epsilon}_t - \tilde{\epsilon}_{t-1}) \tag{16}$$

It hence follows from Bayes's rule that if at the beginning of period t the prior distribution of $\tilde{\alpha}_t$ has been normal with mean η_t and variance U_t, and if at period t an offer x is received yielding $D = x - x_{t-1}$, then the posterior distribution of $\tilde{\alpha}_t$ is normal with mean η'_t and variance U'_t given by

$$\eta'_t = \gamma_2\eta_t + (1 - \gamma_2)D = \gamma_2\, \eta_t - (1 - \gamma_2)x_{t-1} + (1 - \gamma_2)x$$
$$U'_t = \gamma_2 U_{t'}\, \gamma_2 = 2\sigma_t^2/(2\sigma_t^2 + U_t) \tag{17}$$

Since no trend with time is assumed about the evolution of $\tilde{\alpha}_t$; the prior distribution of $\tilde{\alpha}_{t+1}$ is the same as the posterior of $\tilde{\alpha}_t$; that is

$$\eta_{t+1} = \eta'_t, \qquad U'_t = U_t \tag{18}$$

Consider now the revision of beliefs about $\tilde{\mu}$. The posterior of $\tilde{\mu}_t$ given the offer x is the same as that presented in Section II. Here, however, since the mean offers grow with time, the prior distribution of $\tilde{\mu}_{t+1}$ is not the same as the posterior of $\tilde{\mu}_t$. Actually, since $\tilde{\mu}_{t+1} = \tilde{\mu}_t + \tilde{\alpha}_t$, it follows that the mean of $\tilde{\mu}_{t+1}$ equals the mean of the posterior of $\tilde{\mu}_t$ plus the mean of the posterior $\tilde{\alpha}_t$. Thus, $\tilde{\mu}_{t+1}$ is normally distributed with mean and variance given by

$$m_{t+1}(x) = \gamma_1 m_t + (1 - \gamma_1)x + \gamma_2\eta_t + (1 - \gamma_2)D$$
$$= [\gamma_1 m_t + \gamma_2\eta_t - (1 - \gamma_2)x_{t-1}] + [(1 - \gamma_1) + (1 - \gamma_2)]x$$
$$V_{t+1} = \gamma_1 V_t + \gamma_2 U_t \tag{19}$$

where γ_2 is defined in (17) and γ_1 is the same as γ defined in (6).

In its decision process the firm must now consider four parameters: m, η, V, U. Its selling strategy can be computed as in Section II, except that now the function $W_t(\cdot)$ is defined over the above mentioned four parameters. Arguing as before, one can readily observe that $W_t(\cdot)$ satisfies

$$W_t(x, m, \eta, V, U) = \mathrm{Max}\{x, \rho E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), \eta_{t+1}(x), V_{t+1}, U_{t+1})]\} \tag{20}$$

The functions $W_t(x, \cdot)$ are computed recursively using the boundary condition $W_T(x, m, \eta, V, U) = x$.

Since $W_T(x, m, \eta, V, U)$ is a convex function of x, one can show, reasoning as in Section II, that $W_t(x, m, \eta, V, U)$ is also a convex function of x for all m, η, V, and U. From this follows the result that reservation prices (if any exist) and the value of the asset are increasing functions of the dispersion of offers in the future periods.

IV. Comparison of Bayesian and Non-Bayesian Decisions

In the present model, unlike the cases of stationary investments and non-Bayesian decision makers, the reservation prices need not be finite. This can be seen from analyzing $W_{T-1}(\cdot)$. Since $W_T(x, \cdot) = x$, from (19) and (20):

$$\begin{aligned} W_{T-1}&(x, m, \eta, V, U) \\ &= \mathrm{Max}\{x, \rho E[\tilde{x}_T m_T(x), \eta_T(x), V_T, U_T]\} \\ &= \mathrm{Max}\{x, \rho m_T(x)\} \\ &= \mathrm{Max}\{x, \rho[\gamma_1 m + \gamma_2 \eta - (1-\gamma_2)x_{T-1}] + \rho[(1-\gamma_1) + (1-\gamma_2)]x\} \end{aligned} \tag{21}$$

Now if

$$\rho[(1-\gamma_1) + (1-\gamma_2)] > 1 \quad \text{and} \quad \rho[\gamma_1 m + \gamma_2 \eta - (1-\gamma_2)x_{T-1}] > 0 \tag{22}$$

then

$$x < \rho[\gamma_1 m + \gamma_2 \eta - (1-\gamma_2)x_{T-1}] + \rho[(1-\gamma_1) + (1-\gamma_2)]x$$

for all $x \geq 0$. This means that no matter how high the offer is, the firm should not sell the asset. Constraint (22) will hold if the γ's are small, i.e. if the prior information is not reliable. In this case most of the information about $\tilde{\mu}_t$ and $\tilde{\alpha}_t$ comes from observing x, and if a high offer is made, it raises even higher the expectations of future offers. For (22) to hold, the discount rate should be small, since if the discount rate were high (e.g., $\rho < ½$), the losses from waiting would always be smaller than the gains from the increased expected growth.

The non-Bayesian case can be considered as a special case of the Bayesian case where $\gamma_2 = \gamma_1 = 1$; that is, the firm does not revise its beliefs when observing new information. We consider, for simplicity, the case where the α_t's are known. The function $W_t(\cdot)$ for $t = T-1$ in the Bayesian case is

$$W_{T-1}(x, m, V) = \mathrm{Max}\{x, \rho[\gamma_1 m + \eta + (1-\gamma_1)x]\} \tag{23}$$

and its non-Bayesian counterpart is given by

$$\bar{W}_{T-1}(x, m) = W_{T-1}(x, m, 0) = \mathrm{Max}\{x, \rho[m + \eta]\} \tag{24}$$

From (24) we observe that the reservation price in the non-Bayesian case is

$$\bar{x}_{T-1} = \rho(m + n) \tag{25}$$

and from (23) that its Bayesian counterpart is

$$\begin{aligned} x^*_{T-1} &= \rho[\gamma_1 m + \eta]/[1 - \rho(1-\gamma_1)] \\ &= [\bar{x}_{T-1} - \rho(1-\gamma_1)m]/[1 - \rho(1-\gamma_1)] \end{aligned} \tag{26}$$

It therefore follows that if $\bar{x}_{T-1} \gtreqless m$ then $x^*_{T-1} \gtreqless \bar{x}_{T-1}$.

For the case $\bar{x}_{T-1} > m$, by the definition $\bar{x}_{T-1}$, a non-Bayesian firm receiving an offer $\bar{x}_{T-1}$ is indifferent between selling the asset and keeping it (i.e. moving to period T possessing an asset whose mean offer is $m + \eta$). If a Bayesian firm receives an offer of $\bar{x}_{T-1}$ and rejects it, it will possess at T an asset whose mean offer is larger than $m + \eta$ (since $\bar{x}_{T-1} > m$). Thus, the Bayesian firm has a better incentive (than the non-Bayesian) to keep the asset. It hence follows that if the non-Bayesian firm is indifferent between keeping and selling the asset, the Bayesian firm strictly prefers keeping the asset. This implies that the Bayesian firm needs a higher offer in order to sell the asset, i.e. its reservation price is higher than that of the Bayesian firm. Likewise, one can argue that the opposite is true with $\bar{x}_{T-1} < m$. A similar argument (see Venezia [15]) can be used to show that for $t < T - 1$, the higher is $\bar{x}_t$ relative to m the more likely that $x_t^* > \bar{x}_t$. Since $\bar{x}_t$ is positively related to the variance of returns and the rate of growth, and negatively related to the discount rates, it follows that the higher the variances and the growth rates, and the lower the discount rates, the more likely that a Bayesian firm has higher reservation prices than a non-Bayesian firm.

V. Conclusion

We have extended and modified results obtained in previous models of growth investments, to the case in which the firm reasseses its beliefs whenever confronted with new information. We have described the learning process and the optimal policies for terminating the investment when there is uncertainty about the mean return and mean growth rate of the investment assuming that the underlying distributions are normal. A comparison of the Bayesian and non-Bayesian optimal policies has shown that (1) the Bayesian reservation prices may be infinite (this cannot happen in the non-Bayesian case) if uncertainty about the unknown parameters is considerable and if discount rates are low; (2) as in the non-Bayesian case, there is a positive correspondence between reservation prices, the value of the asset, and the variance of the proceeds from selling the asset; and (3) the higher the variances and the rate of growth, and the lower the discount rates, the more likely that a Bayesian firm has higher reservation prices than a non-Bayesian firm.

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