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The Optimal Pricing Policy of a Monopolistic Marketmaker in the Equity Market

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ABSTRACT

This paper presents a stochastic optimization model for marketmaking in security markets with a single dealer. Buy and sell orders are assumed to arrive at rates that are functions of the ask and bid prices. The dealer incurs both proportional and fixed transaction costs as well as portfolio costs. Methods of dynamic programming and semi-Markov Decision Processes are used to characterize optimal pricing policies and to perform sensitivity analysis. Both bid and ask prices are nonincreasing functions of the dealer's inventory. Spread is unrelated to inventory position but positively related to order size. Computational examples demonstrate various results.

MARKETMAKERS (SOMETIMES CALLED DEALERS, specialists, jobbers) in the equity markets set bid and ask prices at which they are willing to buy and sell, respectively, an explicitly or implicitly specified number of shares of one of their specialty stocks. By transacting at these prices, marketmakers provide marketability, referred to by Demsetz [5] as the service of the immediacy.

The crucial role of the marketmaker in U.S. stock markets has stimulated a wide range of research on marketmaking, consisting mainly of empirical studies dealing with factors influencing the size of the marketmaker's spread, i.e., the difference between the ask and bid prices at a given point of time.[1] The dynamics of the marketmaker's pricing decision has either been ignored, treated in an intuitive manner, or addressed without taking the relevant internal and external factors of the marketmaker into account. For example, West [16] assumes, in his simulation of a continuous security market, that the marketmaker does not change his bid and ask prices unless demand or supply shifts occur.

Other authors, including Granger and Morgenstern [7], Smidt [13], and Barnea [2], state that marketmakers decrease (increase) their bid or ask prices in response to increases (decreases) in their inventory. Thus, marketmakers are assumed to change their quotes simply on the basis of the stochastic nature of order arrivals. The primary explanation given for this reaction of the marketmaker is related to the cost of holding inventory. An additional reason for the marketmaker to adjust his prices in response to inventory changes is advanced by Garman [6] who demonstrates that a marketmaker must adjust prices in order to avoid an extreme

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[1] A review of the more recent work dealing with the marketmaker's spread is given by Cohen, Maier, Schwartz, and Whitcomb [4].

inventory position at which he will only quote either a bid price or an ask price. More recent research into the dynamics of marketmaking has begun to address some of the aforementioned problems. The reader is referred to the work of Bradfield [3], Ho and Stoll [8], and Stoll [14].

The purpose of this paper is to develop a stochastic model for marketmaker pricing. We characterize the behavior of optimal bid and ask prices obtained from the functional equations of a semi-Markov Decision Process model. Our model attempts to incorporate the internal and external factors of the marketmaker's firm that are relevant to the pricing decision. These factors include stochastic arrival rate functions, transaction costs, and holding costs. Although we recognize that other complexities of marketmaking exist, our model does not incorporate such factors as competition, the effect of making a market in more than one stock, and shifts in the stochastic arrival rate functions.

The paper is organized as follows. In Section I we discuss the assumptions of the model and develop the relevant functional equations of dynamic programming. Section II demonstrates various model properties by way of an example. In Section III the structure of the optimal pricing policy is discussed. In addition, we examine the sensitivity of the optimum prices and resulting spreads to various model parameters. In Section IV, the model is extended to include multiple order sizes, and other potential modifications are discussed.

I. Model Development

The security market is characterized by the following assumptions:[2]

M.1: Public sell orders are transacted at the marketmaker's bid prices and public buy orders at the ask prices.

M.2: There exists a single marketmaker for the security under consideration.

M.3: The arrival processes for buy and sell orders are Poisson processes. The arrival rate for sell orders, $\lambda(b)$, which are executed at the marketmaker's bid price, b, is an increasing function of the bid price. The arrival rate for buy orders, $\mu(a)$, which are executed at the marketmaker's price, a, is a decreasing function of the ask price. Each bid price, b, and ask price, a, is selected from a discrete set of prices,[3] i.e.,

$$b \epsilon \{b^1, b^2, \cdots, b^k\} \quad \text{where} \quad b^1 < b^2 < \cdots < b^k$$

and

$$a \epsilon \{a^1, a^2, \cdots, a^m\} \quad \text{where} \quad a^1 < a^2 < \cdots < a^m$$

If the arrival rate functions for buy and for sell orders intersect, the intersection point will be called the equilibrium price. Even though the Poisson arrival rates, $\lambda(b)$ and $\mu(a)$, may change as prices change, we assume that the arrival rate functions are stationary over time.

[2] A more mathematically rigorous development of the model and proofs for the optimal pricing policy are given in Schleef and Mildenstein [12].

[3] The analysis of this paper also applies when the sets of prices are continuous and bounded; however, discrete prices add realism to the model since prices are normally quoted in increments of ⅛.

M.4: Each order is for m shares, where m can be any positive integer. The assumption of a single order size will be modified in Section IV of this paper.

The security dealer or marketmaker is characterized by the following assumptions:

D.1: The marketmaker's objective is to determine bid and ask prices that maximize discounted expected cash flow.[4] The bid and ask prices must be determined for each inventory level that may occur over the relevant time interval. We ignore any fixed costs of marketmaking not affected by the choice of pricing policy. The interest rate used for discounting is denoted by i.

D.2: The marketmaker incurs two kinds of costs:

(a) a transaction cost of K per share transacted. Transaction costs for buy and sell orders are identical.

(b) a cost per time unit for holding inventory. We let $C(x)$, $C(x) \geq 0$, denote the holding cost per period when the inventory level is x shares of the trading stock.[5] We assume that $C(x) = 0$ for some value of x, e.g., $C(0) = 0$. We approximate the holding cost rate with a quadratic function of the form

$$C(x) = c \cdot x^2 \quad \text{where} \quad c > 0$$

D.3: The marketmaker is willing or able to hold inventory only within a finite range determined by

$$-\epsilon \leq x \leq \epsilon$$

D.4: The marketmaker reviews bid and ask prices at least as often as transactions occur. When appropriate, an index x to the prices a and b denotes an inventory of x units when the prices are set, while an index n denotes the number of remaining price reviews to the horizon.

Since orders arrive according to Poisson processes, the interarrival times between transactions are negative exponential random variables. The expected time between transactions varies as the bid and ask prices are changed. In order to obtain a constant expected time between events, we assume that price review events are generated by a Poisson process with constant parameter, $\Lambda = \lambda(b^k) + \mu(a^1)$ for which b^k is the largest bid price and a^1 is the smallest ask price.[6]

[4] This objective function does not necessarily imply that the marketmaker is risk neutral. His attitude towards risk is adequately incorporated in the quadratic holding cost function (Assumption D.2 b). Our approach is primarily motivated by the desire to obtain testable results in a mathematically tractable way.

[5] A theoretical foundation for this holding cost function has been provided by Stoll [14], who assumes that the marketmaker holds a portfolio of risky and riskless assets that maximizes expected utility. A deviation from the optimal portfolio due to transactions with traders to whom he supplies marketability services causes a loss in expected utility to the marketmaker. The monetary equivalent of this loss represents the marketmaker's holding costs. Return uncertainty as a marketmaker's cost factor is incorporated into the holding cost function of our model. Stoll has demonstrated that these holding costs are a convex function of the deviation from the optimal portfolio and can be approximated by a quadratic function.

[6] The use of a constant arrival rate in problems of this type has been extensive in the literature of queueing control, cf., Lippman [9]. By using the constant arrival rate mechanism, the resulting time to complete n price reviews is a random variable. Nevertheless, the expected time to complete n price reviews is a constant with a value of n/Λ. Introducing constant times between price reviews is possible, but requires an unnecessarily complex transition probability matrix.

Our approach in developing the dealer pricing model is consistent with the usual dynamic programming method of looking backward in time commencing with the last period in the time interval. First we consider the finite period problem where the number of periods represents the number of remaining price reviews. When n periods remain, the marketmaker with an inventory of x sets a bid price, $b_{x,n}$, and an ask price, $a_{x,n}$. The cost, $C(x)$, is incurred continuously until the next review occurs. The next review is caused by one of three possible events:

(1) A sell order arrives with a probability of

$$\frac{\lambda(b_{x,n})}{\Lambda}$$

(2) A buy order arrives with a probability of

$$\frac{\mu(a_{x,n})}{\Lambda}$$

(3) No arrival (null event) occurs with a probability of

$$\frac{1 - \lambda(b_{x,n}) - \mu(a_{x,n})}{\Lambda}$$

As a result, the marketmaker reviews bid and ask prices either because a transaction occurred or because the review period ended with no inventory change.

Suppose that the marketmaker enters the last review period with an inventory of x. If a sell order is transacted, the marketmaker buys m shares at the bid price, $b_{x,1}$, with a resultant cash flow $-m(b_{x,1} + K)$ and terminal inventory of $(x + m)$ shares with a value of $V_0(x + m)$. If the event at the end of the last period is a buy order, then the marketmaker realizes costs of $K \cdot m$, a positive cash flow of $100 \cdot m \cdot a_{x,1}$, and terminal inventory of $(x - m)$ shares with a value of $V_0(x - m)$. If at the end of period 1 no transaction occurs, then the terminal value is $V_0(x)$.

Let $V_1(x)$ denote the expected discounted return when one period remains and x shares of stock are in the marketmaker's inventory. We assume that costs and cash flows are discounted continuously to yield[7, 8]

[7] The discounted expected value of \$1 received at time t when t is a random variable with a negative exponential probability distribution is given by

$$E\,(e^{-it}) = \int_0^\infty e^{-it}\Lambda e^{-\Lambda}\,dt$$

$$= \frac{\Lambda}{\Lambda + i}$$

The parameter for the negative exponential distribution is Λ. The discounted expected cost that is incurred continuously at rate $C(x)$ is given by

$$\int_0^\infty C(x)te^{-t}\Lambda e^{-\Lambda}\,dt = C(x)[1 - \Lambda/(\Lambda + i)/i$$

$$= C(x)/(\Lambda + i)$$

[8] For a complete development of the theory of semi-Markov Decision Processes and the related functional equations of dynamic programming, the reader is referred to Chapter 7 in Ross [11].

$$V_1(x) = \{-C(x) + \lambda(b_{1,x})[-mb_{1,x} - Km + V_0(x+m)] + \mu(a_{1,x})[ma_{1,x} - Km + V_0(x-m)] + [\Lambda - \lambda(b_{1,x}) - \mu(a_{1,x})]V_0(x)\}/(\Lambda + i)$$

In general, the set of discounted return equations when n periods remain, denoted by $V_n(x)$, are defined as follows:

$$V_n(x) = \max_{a,b}\{-C(x) + \lambda(b)[-mb - Km + V_{n-1}(x+m)] + \mu(a)[ma - Km + V_{n-1}(x-m)] + [\Lambda - \lambda(b) - \mu(a)]V_{n-1}(x)\}/(\Lambda + i) \quad (1)$$

Observe that the functional equations in (1) require the determination of optimal bid and ask prices for each inventory level as well as for each period. Consequently, a large number of computations may be required to determine the optimal pricing policy. For large n, the optimal pricing policy tends to be nearly identical from period to period. For the above reasons, our computational results consider the infinite horizon functional equations analogous to (1).[9] Nevertheless, we shall use the finite horizon equations for the subsequent sensitivity analysis of the marketmaker's optimal pricing policy.

The infinite horizon equations are obtained by letting n tend to infinity; consequently, the subscript, n, is omitted and the corresponding set of functional equations are given by[10]

$$V(x) = \max_{b,a}\{-C(x) + \Lambda V(x) + \lambda(b)[-mb - Km + V(x+m) - V(x)] + \mu(a)[ma - Km + V(x-m) - V(x)]\}/(\Lambda + i) \quad (2)$$

II. Determining Optimal Bid and Ask Prices: An Example

The model developed in the previous section can be used for analyzing optimal pricing policies and for actually computing optimal policies. In this section, an example is used to offer initial insight into the problem and a preview to the sensitivity analysis of Section III.

The policy iteration method of successive approximations (see Wagner [15]) is used to determine optimal prices for each inventory level. The policy iteration

[9] The finite horizon functional equations in (1) converge in both value and policy as n tends to infinity. Generally, policy convergence occurs for finite n. However, the rate of policy convergence is a function of the interest rate, i, as well as certain characteristics of the optimal transition matrix, cf., Morton and Wecker [10]. Convergence at the limit can be demonstrated by standard methods for semi-Markov Decision Processes, cf., Ross [11].

[10] The constant transition device, Λ, (see footnote 6) is unnecessary in the infinite horizon case, since (2) can be written as

$$V(x) = \frac{1}{\lambda(b_x) + \mu(a_x) + i}\{-C(x) + \lambda(b_x)[-100mb_x - Km + V(x+m)] + \mu(a_x)[100ma_x - Km + V(x-m)]\}$$

where a_x and b_x are optimal prices.

method requires initial trial pairs of prices, b_x^1 and a_x^1, for each inventory level. For the first iteration, $t = 1$, the following set of simultaneous equations is solved to determine functional values, $V^1(x)$.

$$V^t(x) = \frac{1}{\Lambda + i}\{-C(x) + \Lambda V^t(x) + \lambda(b_x^t)[-mb_x^t - Km + V^t(x+m) - V^t(x)] + \mu(a_x^t)[ma_x^t - Km + V^t(x-m) - V^t(x)]\} \tag{3}$$

At each iteration, a check for an improved policy is performed by computing $W^t(x)$ values from (4) below

$$W^t(x) = \max_{b_x a_x} \frac{1}{\Lambda + i}\{-C(x) + \Lambda V^t(x) + \lambda(b_x)[-mb_x - mK + V^t(x+m) - V^t(x)] + \mu(a_x)[ma_x - mK + V^t(x-m) - V^t(x)]\} \tag{4}$$

The iteration count is then updated from t to $t + 1$ and Equation (3) is resolved using optimal prices, b_x^{t+1} and a_x^{t+1} determined in (4). Iterations continue until $V^t(x) = W^t(x)$ for each x.

Although Equation (4) suggests that b_x^{t+1} and a_x^{t+1} are determined jointly, a closer examination indicates that, for each inventory position, the bid price can be determined independently from the ask price by maximizing (5a) and (5b) separately,

$$F(a_x) = \mu(a_x)[ma_x - mK + V^t(x-m) - V^t(x)] \tag{5a}$$

$$F(b_x) = \lambda(b_x)[-mb_x - mK + V^t(x+m) - V^t(x)] \tag{5b}$$

For linear arrival rate functions

$$\lambda(b_x) = \lambda_1 + \lambda_2 b_x \tag{6a}$$

$$\mu(a_x) = \mu_1 - \mu_2 a_x \tag{6b}$$

the optimal prices are given by

$$a_x^{t+1} = \frac{1}{2}\left\{\frac{\mu_1}{\mu_2} + K + \frac{1}{m}[V^t(x) - V^t(x-m)]\right\} \tag{7a}$$

$$b_x^{t+1} = \frac{1}{2}\left\{-\frac{\lambda_1}{\lambda_2} - K + \frac{1}{m}[V^t(x+m) - V^t(x)]\right\} \tag{7b}$$

where prices determined by (7a) and (7b) may require rounding to the nearest ⅛ and may require equating to lower or upper bounds if a_x^{t+1} or b_x^{t+1} for (7a) and (7b) are outside of the permissible price ranges.

The following parameter values are used in the computational example. Arrival Rates:

$$\lambda(b) = -320 + 8b \quad \text{for} \quad 40\tfrac{3}{8} \le b \le 43\tfrac{7}{8}$$

$$\mu(a) = 354 - 8a \quad \text{for} \quad 40\tfrac{3}{8} \le a \le 43\tfrac{7}{8}$$

Price changes are permitted in increments of ⅛.

Table I

	A			B		
Inventory x	ask a_x	bid b_x	$V(x)$	ask a_x	bid b_x	$V(x)$
−700	—	41¼	106,086	—	61⅛	99,731
−600	43⅝	41	110,375	63⅝	61	105,996
−500	43½	41	114,596	63⅜	60⅞	112,224
−400	43⅜	40⅞	118,814	63¼	60¾	118,431
−300	43¼	40¾	123,017	63¼	60¾	124,620
−200	43¼	40¾	127,207	63⅛	60⅝	130,796
−100	43⅛	40¾	131,385	63⅛	60⅝	136,958
0	43⅛	40⅝	135,554	63	60½	143,109
100	43⅛	40⅝	139,713	63	60½	149,250
200	43	40½	143,862	62⅞	60½	155,380
300	43	40½	148,003	62⅞	60⅜	161,501
400	42⅞	40½	152,135	62⅞	60⅜	167,613
500	42⅞	40⅜	156,259	62¾	60⅜	173,715
600	42⅞	40⅜	160,373	62¾	60⅜	179,807
700	42¾	—	164,472	62⅝	—	185,889

The equilibrium price occurs at 42⅛, where

$$\lambda(42\tfrac{1}{8}) = \mu(42\tfrac{1}{8}) = 17$$

Cost Factors and Interest Rate:

$$C(x) = 0.0002x^2$$

$$K = 0.3$$

$$i = 0.1$$

Order Size and Inventory Limits:

$$m = 100$$

$$-700 \leq x \leq 700$$

Table I, Section A, gives the optimal solution to the example presented. Examination of this solution indicates that both a_x and b_x are nonincreasing as x increases. Also, the function, $V(x)$, is concave in x. In the appendix, it is proved by induction that $V_{n-1}(x)$, given by (1), is a concave function of x, i.e., $[V_{n-1}(x) - V_{n-1}(x-m)]$, is decreasing in x. As demonstrated in [12], price monotonicity follows directly from concavity.

III. An Analysis of the Optimal Pricing Policy

To further investigate optimal pricing policy, we analyze the model when n price reviews remain until the end of the planning period. The functions analogous to (5a) and (5b) for determining optimum bid and ask prices are (8a) and (8b) respectively.

$$F(a_{x,n}) = \mu(a_{x,n})[ma_{x,n} - Km + V_{n-1}(x-m) - V_{n-1}(x)] \tag{8a}$$

$$F(b_{x,n}) = \lambda(b_{x,n})[-mb_{x,n} - Km + V_{n-1}(x+m) - V_{n-1}(x)] \tag{8b}$$

For linear (continuous) arrival rate functions, the optimal bid and ask prices are determined by differentiation, i.e.,

$$a^*_{x,n} = \frac{1}{2}\left\{\frac{\mu_1}{\mu_2} + K + \frac{1}{m}\left[V_{n-1}(x) - V_{n-1}(x-m)\right]\right\} \tag{9a}$$

$$b^*_{x,n} = \frac{1}{2}\left\{-\frac{\lambda_1}{\lambda_2} - K + \frac{1}{m}\left[V_{n-1}(x+m) - V_{n-1}(x)\right]\right\} \tag{9b}$$

If the marketmaker could change his prices over a continuous price range, then an increase in inventory would result in a decrease in both the bid and ask prices until the lower bound on each of the price ranges is reached. Since changes are only permitted at discrete intervals, (increments of ⅛) the optimal prices may remain unchanged when the inventory level changes.

By subtracting Equation (9b) from Equation (9a), the marketmaker's optimal spread, denoted by $s^*_{x,n}$, is determined by

$$\begin{aligned} s^*_{x,n} &= a^*_{x,n} - b^*_{x,n} \\ &= \frac{1}{2}\left\{\frac{\mu_1}{\mu_2} + \frac{\lambda_1}{\lambda_2} + 2K + \frac{1}{m}\left[V_{n-1}(x) - V_{n-1}(x-m)\right]\right. \\ &\quad \left. - \frac{1}{m}\left[V_{n-1}(x+m) - V_{n-1}(x)\right]\right\} \end{aligned} \tag{9c}$$

As the marketmaker's inventory increases (decreases), both the bid and ask prices decrease (increase) by an amount determined by the first difference of the $V_{n-1}(x)$ function. Consequently, the model indicates no systematic relationship between spread and inventory level assuming the price range is unrestricted. In our computations, we observed that the spread decreases near both the maximum and the minimum of the inventory levels. This effect is apparently due to the restrictive set of prices considered. Consequently, the ask price reaches its largest value before the minimum inventory level is reached while the bid price can still move up further. Similarly, an increase in the inventory level may cause the bid price to reach its smallest value before the maximum inventory level is reached.

We now investigate the sensitivity of the model to model parameters (λ_1, λ_2, μ_1, μ_2, m, K, c, ϵ). Our analysis focuses on first order conditions for the optimum price functions (9a), (9b), (9c). For the general case of n remaining price reviews until the end of the horizon, examination of (9a) and (9b) indicates that the behavior of $[V_{n-1}(x) - V_{n-1}(x-m)]$ determines the form and character of optimal prices.

To simplify the subsequent analysis, we use the optimal price function assuming two price reviews remain before the planning period ends. These optimal prices are given by,

$$a^*_{x,2} = \frac{1}{2}\left\{\frac{\mu_1}{\mu_2} + K + \frac{1}{m}\left[V_1(x) - V_1(x-m)\right]\right\} \tag{10a}$$

$$b^*_{x,2} = \frac{1}{2}\left\{-\frac{\lambda_1}{\lambda_2} - K + \frac{1}{m}\left[V_1(x+m) - V_1(x)\right]\right\} \tag{10b}$$

First differences for $V_1(x)$ are determined from the optimal prices for the last price review given by (11a) and (11b).

$$a^*_{x,1} = \frac{1}{2}\left\{\frac{\mu_1}{\mu_2} + K + \frac{1}{m}\left[V_0(x) - V_0(x-m)\right]\right\} \tag{11a}$$

$$b^*_{x,1} = \frac{1}{2}\left\{-\frac{\lambda_1}{\lambda_2} - K + \frac{1}{m}\left[V_0(x+m) - V_0(x)\right]\right\} \tag{11b}$$

We employ a liquidation price, p_0, that is independent of the inventory x; therefore,

$$V_0(x) - V_0(x-m) = p_0 \cdot m \quad \text{for all} \quad x > -\epsilon + m$$

Thus

$$V_1(x+m) - V_1(x) = \frac{1}{\Lambda + i}\{-c(x+m)^2 + cx^2 + 100\,\Lambda m p_0\}$$

and

$$V_1(x) - V_1(x-m) = \frac{1}{\Lambda + i}\{-cx^2 + c(x-m)^2 + 100\,\Lambda m p_0\}$$

We now substitute the above expressions into (10a) and (10b) to yield the equations on which our sensitivity analysis is based.

$$a^*_{x,2} = \frac{1}{2}\left\{\frac{\mu_1}{\mu_2} + K + \frac{100\Lambda p_0 + c(m-2x)}{\Lambda + i}\right\} \tag{12a}$$

$$b^*_{x,2} = \frac{1}{2}\left\{-\frac{\lambda_1}{\lambda_2} - K + \frac{100\Lambda p_0 - c(m+2x)}{\Lambda + i}\right. \tag{12b}$$

Thus, the optimal spread $s^*_{x,2}$ is given by

$$s^*_{x,2} = \frac{1}{2}\left\{\left(\frac{\mu_1}{\mu_2} + \frac{\lambda_1}{\lambda_2}\right) + 2K + \frac{2cm}{\Lambda + i}\right\} \tag{12c}$$

Note that the spread $s^*_{x,2}$ is independent of the marketmaker's inventory level x.

Differentiation of (12a), (12b), and (12c) yields relationships between prices and various parameter changes displayed in Table II.

Analysis of the cost parameters indicates that increases in the spread may result from increases in the order size, m, the transaction cost, K, or the holding cost, $C(x)$. Analysis of the slope and intercept parameters demonstrates that spread increases may result from either increases in the intercepts λ_1 and μ_1 or decreases in the slope parameters λ_2 and μ_2. From these results, the effect of changes in the equilibrium price and trading activity of the stock can be analyzed. If the equilibrium price (intersection of the arrival rate functions) of the stock increases as a result of simultaneously increasing μ_1 and decreasing λ_1 by equal amounts, then the spread will remain approximately the same. An example to demonstrate this result is given in Section B of Table I. The input for this example is obtained by adding 160 to μ_1 and subtracting 160 from λ_1 in example *A* with other parameters remaining the same. (Slight differences in the spreads

Table II

First Partial Derivatives of

w.r.t.	$a^*_{x,2}$	$b^*_{x,2}$	$s^*_{x,2}$
μ_1	$\frac{1}{2\mu_2}$	0	$\frac{1}{2\mu_2}$
λ_1	0	$-\frac{1}{2\lambda_2}$	$\frac{1}{2\lambda_2}$
μ_2	$-\frac{\mu_1}{2\mu_2^2}$	0	$-\frac{\mu_1}{2\mu_2^2}$
λ_2	0	$\frac{\lambda_1}{2\lambda_2^2}$	$-\frac{\lambda_2}{2\lambda_2^2}$
m	$\frac{c}{2(\Lambda + i)}$	$-\frac{c}{2(\Lambda + i)}$	$\frac{c}{\Lambda + i}$
K	½	− ½	1
c	$-\frac{2x - m}{2(\Lambda + i)}$	$-\frac{2x + m}{2(\Lambda + i)}$	$\frac{m}{\Lambda + i}$

between the two examples are due to roundings to discrete prices). Usually an increase in equilibrium price is accompanied by higher holding cost; consequently, the spread is positively related to the equilibrium price. Simultaneously increasing λ_1 and μ_1 effectively corresponds to higher trading activity for the stock. From the sensitivity analysis it follows that the spread should thereby increase. Several empirical studies cited in [4] indicate that there is a negative relationship between spread and trading activity. An explanation for the inconsistency between empirical findings and the result of our sensitivity analysis is related to the price elasticity of the arrival rates. As the parameters λ_1 and μ_1 increase, both arrival rate functions shift upward resulting in arrival rates which are relatively price inelastic. As in the case of static supply and demand functions, decreasing elasticity produces equilibrium points with greater profit margins.

We now consider the effect of a marketmaker's decision to increase the maximum and minimum number of round lots he is willing to position. For example, suppose that the marketmaker's minimum inventory level is decreased below −700 round lots and his maximum level increased above +700 round lots of the stock in question. For the inventory level smaller than −700, the ask price increases by $\frac{\partial a^*_{x,2}}{\partial x}$ and the bid price increases by $\frac{\partial b^*_{x,2}}{\partial x}$ per share of inventory decrease. For inventory levels larger than +700, the prices are lowered accordingly. Differentiation of Equations (12a) and (12b) indicate that prices at the end points change, but that prices inside the inventory range of −700 to +700 round lots are not affected by marginal changes in the limits.

IV. Model Extensions and Conclusions

We now change Assumption M.4 (single order size m) to

M.4′: There exist k different order sizes and m is the index for order size. As in the preceding sections, m denotes the number of shares traded in a given transaction; however, we now allow multiple order sizes.

This new assumption requires that we modify Assumption M.3 to

M.3′: The arrival rate for sell orders of m shares is a function of the bid price for this order size. This arrival rate function is denoted by $\lambda_m(b_m)$. Accordingly, we change the arrival rate function for buy orders of m shares to $\mu_m(a_m)$.

In this version of the model, a pair of bid and ask prices $(b_{x,m};\ a_{x,m})$ is determined for each potential order size. The relevant functional equations are given by

$$\begin{aligned} V(x) = \frac{1}{\Lambda + i}\{&-C(x) + \Lambda V(x) \\ &+ \textstyle\sum_{m=1}^{k} \lambda_m(b_{x,m})[-mb_{x,m} - Km + V(x+m) - V(x)] \\ &+ \textstyle\sum_{m+1}^{k} \mu_m(a_{x,m})[ma_{x,m} - Km + V(x-m) - V(x)]\} \end{aligned} \tag{13}$$

(Note that (13) requires minor notational changes to accurately denote the states which are realizable when the upper and lower bounds on the marketmaker's inventory are approached).

The algorithm for determining optimal prices for each order size and inventory level is identical to the case of a single order size except that the system of linear equations has more nonzero coefficients. In other words, we observe an increase in the number of states that can be reached during a given transition. Also, k pairs of prices must be selected for each inventory level. Using the vaues, $V^t(x)$, obtained from solving these linear equations, an ask price for a given inventory level x and a given order size m can be determined independently of all other prices by maximizing:

$$F(a_{x,m}) = \mu_m(a_{x,m})[ma_{x,m} - Km + V^t(x-m) - V^t(x)] \tag{14a}$$

The respective bid price is determined by maximizing

$$F(b_{x,m}) = \lambda_m(b_{x,m})[-mb_{x,m} - Km + V^t(x+m) - V^t(x)] \tag{14b}$$

Table III gives the solution to the example of Section I, which has been modified to include three different order sizes (m = 100; 200; 300). The arrival rate functions for each of the three sizes are assumed to be identical to the rates in the previous example where only a single order size is permitted.

A central problem in any empirical study is the estimation of demand and supply curves. The fact that our model captures demand/supply curves in terms of arrival rate functions suggests a somewhat different statistical methodology for estimating the curves. One approach would be to observe inter-arrival times for different prices. Nevertheless, as in most attempts at estimating demand/supply relationships, formidable problems still remain in estimating a large number of Poisson parameters. Many of our results are not dependent upon the linearity of the arrival rate functions, but are only dependent on their general form (upward or downward sloping).

To summarize our results, we demonstrate that both bid and ask prices are nonincreasing as the marketmaker's inventory increases. Even though the model uses inventory as the linking mechanism for capturing much of the dynamics of marketmaking, our analysis indicates that many characteristics of the pricing

Table III

Inventory X	Bid Prices for m shares $m = 100$	$m = 200$	$m = 300$	Ask Prices for m shares $m = 100$	$m = 200$	$m = 300$
−700	41¼	41¼	41¼	—	—	—
−600	41¼	41¼	41⅛	43⅝	—	—
−500	41¼	41⅛	41⅛	43⅝	43⅝	—
−400	41	41	41	43⅝	43⅝	43⅝
−300	41	41	41	43½	43½	43⅝
−200	41	40⅞	40⅞	43⅜	43½	43½
−100	40⅞	40⅞	40⅞	43⅜	43⅜	43⅜
0	40⅞	40⅞	40¾	43⅜	43⅜	43⅜
100	40¾	40¾	40¾	43¼	43¼	43⅜
200	40¾	40¾	40¾	43¼	43¼	43¼
300	40¾	40⅝	40⅝	43⅛	43¼	43¼
400	40⅝	40⅝	40⅝	43⅛	43⅛	43⅛
500	40⅝	40⅝	—	43	43¼	43¼
600	40⅝	—	—	43	43	43
700	—	—	—	43	43	43

policy, such as spread, are unrelated to inventory position. We demonstrate that spread is positively related to both activity level of each order type and the order size, but is negatively related to the absolute value of arrival rate slopes. The costs faced by the marketmaker are discussed and as expected the spread increases as the marketmaker's cost factors increase.

APPENDIX

A proof by induction can be used to demonstrate that $V_n(x)$ is a concave function of x. First we define

$$\begin{aligned} U_n(x, b, a) = \frac{1}{\Lambda + i} \{ & -C(x) + \Lambda V_{n-1}(x) \\ & + \lambda(b)[-mb - Km + V_{n-1}(x+m) - V_{n-1}(x)] \\ & + \mu(a)[ma - Km + V_{n-1}(x-m) - V_{n-1}(x)]\} \end{aligned}$$

Observe that

$$V_n(x) = \max_{(b,a)} U_n(x, b, a). \quad \text{Now let}$$

$$v_0(x) = p_0 m x$$

Theorem. $V_n(x)$ is a concave function of x for all n.

Proof. The hypothesis is true for $n = 1$ since

$$\begin{aligned} U_1(x, b, a) = \frac{1}{\Lambda + i} \{ & -C(x) + \Lambda m p_0 x \\ & + \lambda(b)[-mb - Km + mp_0] \\ & + \mu(a)[ma - Km - mp_0]\} \end{aligned}$$

and

$$V_1(x) = \max_{(b,a)} U_1(x, b, a)$$

are both concave functions of x. Now assume that the hypothesis is true for $n - 1$. Taking the second difference of $V_n(x)$ and letting $b_{x,n-1}$ and $a_{x,n-1}$ denote the optimal prices given x and $n - 1$, we have

$$\begin{aligned}
&V_n(x+m) - 2V_n(x) + V_n(x-m) \\
&\quad = \max_{(b,a)} U_n(x+m, b, a) \\
&\qquad - 2\max_{(b,a)} U_n(x, b, a) + \max_{(b,a)} U_n(x-m, b, a) \\
&\quad = U_n(x+m, b_{x+m,n}, a_{x+m,n}) - U_n(x, b_{x,n}, a_{x,n}) \\
&\qquad -[U_n(x, b_{x,n}, a_{x,n}) - U_n(x-m, b_{x-m,n}, a_{x-m,n})] \\
&\quad \le U_n(x+m, b_{x+m,n}, a_{x+m,n}) - U_n(x, b_{x+m,n}, a_{x+m,n}) \\
&\qquad - [U_n(x, b_{x-m,n}, a_{x-m,n}) - U_n(x-m, b_{x-m,n}, a_{x-m,n})] \\
&\quad = \frac{1}{\Lambda + i}\{-[C(x+m) - 2C(x) + C(x-m)] \\
&\qquad + [\Lambda - \lambda(b_{x-m,n}) - \mu(a_{x+m,n})][V_{n-1}(x+m) - 2V_{n-1}(x) + V_{n-1}(x-m)] \\
&\qquad + \lambda(b_{x+m,n})[V_{n-1}(x+2m) - 2V_{n-1}(x+m) + V_{n-1}(x)] \\
&\qquad + \mu(a_{x-m,n})[V_{n-1}(x) - 2V_{n-1}(x-m) + V_{n-1}(x-2m)]\} \\
&\quad \le 0 \text{ by the concavity of } -C(x) \text{ and } V_{n-1}(x) \text{ Q.E.D.}
\end{aligned}$$

As a consequence of the above theorem, $V(x)$ is also a concave function of x since the limit (as n tends to infinity) of a concave function is also concave.

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