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Constant Absolute Risk Aversion Preferences and Constant Equilibrium Interest Rates

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ABSTRACT

This paper constructs a general equilibrium model with endogenous stochastic production and establishes that the equilibrium interest rate can be constant in a closed production economy when the preferences are represented by constant absolute risk aversion utility functions. The results in this paper and their limitations are compared and contrasted with related contributions in the financial economics literature.

IN THIS PAPER AN intertemporal allocation problem with endogenous stochastic production is solved for consumption function and the interest rate. Preferences are assumed to be of constant absolute risk aversion type. There are a number of stochastic constant returns to scale production techniques with independently and identically distributed shocks. It is shown that the wealth dynamics of agent can be then approximated by a "square root" diffusion process which is a special case of constant elasticity of variance diffusion. With this as the starting point the consumption function and interest rates are derived and discussed. At equilibrium the interest rate is shown to be a constant and consumption is shown to be drawn from a distribution which belongs to the constant elasticity of variance diffusion class. Restrictions are placed on the form of the problem so that results make economic sense.

The assumption of a constant interest rate has been made frequently in the asset pricing literature. We provide a plausible equilibrium model in which this assumption is valid. The findings are compared with those of Bhattacharya [1], Constantinides [2], Rubinstein [8], and Stapleton and Subrahmanyam [10] in subsequent sections.

This paper is organized as follows. In the next section the characteristics of the economy are described. Section II develops the dynamic optimization problem facing the consumer in the economy. In Section III, the optimal consumption function is derived and discussed. Section IV contains a result on the equilibrium interest rate. In this section it is shown that the equilibrium interest rate is a constant, and its properties are analyzed. In Section V a discussion of the results is provided.

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I. Description of the Economy

In our economy there is one individual. The objective of this individual is to maximize his or her expected lifetime utility of consumption of the only good in the economy. It is assumed that the preference of the agent can be represented as[1]

$$u(c(t), t) = -\exp(-(\rho t + \eta c)) \frac{1}{\eta} \tag{1}$$

where ρ is the rate of impatience of the agent and $c(t)$ is the stochastic process representing the consumption at time t. The Arrow-Pratt measure of constant absolute risk aversion is denoted by η. It is assumed that $\eta > 0$. The agent is infinitely lived and the objective functional to be maximized is given as

$$\text{Max } E_0\left[\int_0^\infty u(c(t), t)\, dt\right] \tag{2}$$

The assumption that there is only one consumer in the economy is not as restrictive as it may seem at a first glance. The model can be easily extended to incorporate a large number of identical consumers. The assumption of a single consumer is made only for expositional ease. It may aid one's intuition to regard the consumer as a composite individual.

In the economy there are a number of instantaneous production technologies whose returns are independently and identically distributed and have constant returns to scale with a mean rate of return α and variance σ^2. We further assume that each method of production has a minimum input requirement of χ.[2] In other words

$$\frac{dq}{q} = \alpha\, dt + \sigma\, dz_i \quad \text{when} \quad q \geq \chi (i = 1, 2, 3, \cdots) \tag{3}$$

The consumer is assumed to have a wealth of $Q(t)$ available for investment. At this stage it is useful to note that $Q(t)$ is the total wealth in the economy. It is also important to note that the optimal investment policy is an equally weighted one as the technologies have shocks that are independently and identically distributed. That the optimal investment policy is an equally weighted one in such a production scheme has been established by Samuelson [9].

As pointed out, the agent in the economy at any time t has $Q(t)$ units to invest. This problem may be viewed as one of investing $Q(t)$ units in Q/χ technologies. Then approximately we can represent the stochastic evolution of the wealth gross

[1] Negative exponential utility function has been employed by Merton [6], Stapleton and Subrahmanyam [10], and Hakansson [5] in studying portfolio and consumption behavior. With negative exponential utility function as $c \to 0$ $u'(c)$ will not approach infinity. So generally nonpositive consumption cannot be ruled out.

[2] I am grateful to Jon Ingersoll for suggesting this approach.

of consumption as[3]

$$dQ = \alpha Q\, dt + \hat{\sigma}\sqrt{Q}\, dz \tag{4}$$

where $\hat{\sigma}$ is a scalar parameter equal to $\sigma\sqrt{\chi}$. This approximation is an accurate one as long as $Q/\chi \gg 1$. This approximation has been made in our subsequent discussions. Hence the results of the paper must be viewed within the context of this assumption. We can handle an identical consumer economy as well in this model by suitably redefining units, in particular $\hat{\sigma}$.

II. Dynamic Optimization Problem

The problem will be set forth as an intertemporal consumption allocation problem. We will then use the result established by Cox, Ingersoll, and Ross [4] and solve for the shadow price for the instantaneous rate of interest, at which no borrowing or lending would be demanded. This way of posing the problem involves no loss of generality.

At each instant the consumer solves for the consumption policy $c(t)$ given that his wealth dynamics follow

$$dQ = [Q\alpha - c]\, dt + \hat{\sigma}\sqrt{Q}\, dz \tag{5}$$

The state variables of this economy are (Q, r). As will be demonstrated later, the interest rate will be constant and hence will not enter into the determination of indirect value function. So we suppress this dependence and define the indirect value function as

$$V(Q, t) = E_t \int_t^{\infty} u(c(l), l)\, dl \tag{6}$$

Merton [6] has shown that a continuous time version of the Bellman fundamental equation of optimality can be explicitly written as[4]

$$0 = \text{Max}_{\{c\}}[u(c) - \rho J + J_Q\{Q\alpha - c\} + ½\, J_{QQ}\hat{\sigma}^2 Q] \tag{7}$$

where

$$u(c) = -\frac{1}{\eta}\exp\{-\eta c\} \tag{8}$$

and $J(Q) = V(Q, t)\exp(\rho t)$. The first order conditions of optimality can be written as

$$\frac{\partial u}{\partial c} - J_Q = 0 \tag{9}$$

[3] The properties of this diffusion process have been studied by Cox [3] and Cox, Ingersoll, and Ross [4]. We cannot assert that $Q/\chi \gg 1$ with probability 1.

[4] The value function inherits some properties from the direct utility function. It is strictly increasing and concave in Q. In this paper we shall assume that J is unique. We shall show the existence by solving for it explicitly. J_Q represents the partial derivative of J with respect to Q; etc.

Equation (9) is the envelope condition which equates the marginal utility of consumption to the marginal utility of wealth of the value function.

III. Optimal Consumption Policy

Under the assumptions stated in Section I(i) the optimal consumption policy is

$$c = \frac{aQ}{\eta} + \left[\frac{\rho}{a} - \frac{1}{\eta}\right] \tag{10}$$

and (ii) the optimal value function is

$$J(Q) = -\frac{1}{a}\left\{\exp\frac{a - \rho\eta}{a} - aQ\right\} \tag{11}$$

$$\text{where} \quad a \equiv -\frac{J_{QQ}}{J_Q} = \frac{2\eta\alpha}{2 + \eta\hat{\sigma}^2} \tag{12}$$

is the constant absolute risk aversion in wealth which enters into the consumption function, along with the impatience factor ρ and the coefficient η which is the Arrow-Pratt measure absolute risk aversion in consumption.

We sketch some properties of the consumption function. From (10) it is a linear increasing function in the wealth at any time. Some restrictions on the parameters are needed, however, in order for the model to yield sensible results. Substitution of (10) in (5) yields

$$dQ = \left[Q\alpha - \frac{aQ}{\eta} + \left(\frac{1}{\eta} - \frac{\rho}{a}\right)\right] dt + \hat{\sigma}\sqrt{Q}\, dz \tag{13}$$

Unless $\left(\frac{1}{\eta} - \frac{\rho}{a}\right) > 0$, the wealth Q with probability one in an infinite length of time becomes zero. We impose this restriction. This restriction requires $\rho < \frac{2\alpha}{2 + \eta\hat{\sigma}^2}$. Let $\delta \equiv \frac{2\alpha}{2 + \eta\hat{\sigma}^2} - \rho > 0$ by assumption. This restriction assures that the wealth will always remain positive. By substitution the consumption can be written as

$$c = (\rho + \delta)Q - \frac{\delta}{\eta(\rho + \delta)} \tag{14}$$

The first term in the consumption function is $(\rho + \delta)Q$ which is unambiguously positive. If δ is "sufficiently" small, the time rate of preference ρ is as high as possible without making δ negative, then the last term is negligible and the consumption is positive. Keeping α, η and $\hat{\sigma}^2$ at a fixed level, if the impatience factor ρ is sufficiently high as to make δ close to zero but positive, then the consumption will remain positive.[5]

[5] I wish to thank Jon Ingersoll for bringing this point to my attention. If α is sufficiently high and $\hat{\sigma}^2$ is sufficiently low, then we can select a high value of ρ without violating the restriction that $\delta > 0$ but making δ very close to zero. With a high value of ρ the third term in the consumption function (which is negative) becomes less important and the first term in the consumption function (which is positive) becomes more important.

It is to be noted that the marginal increase in consumption with respect to wealth is the ratio of the absolute risk aversion in wealth to the absolute risk aversion in consumption. This is easily verified by noting that

$$\frac{\partial c}{\partial Q} = \frac{a}{\eta} = \frac{2\alpha}{2 + \eta\hat{\sigma}^2} > 0 \tag{15}$$

The marginal propensity to consume with wealth increases as the expected return α increases. It declines with increase in either η or $\hat{\sigma}^2$ or both. These are intuitively pleasing conclusions.

The marginal propensity to consume with respect to the impatience factor is positive as $\dfrac{\partial c}{\partial \rho} = \dfrac{1}{a} > 0$.

IV. Equilibrium Interest Rate

Cox, Ingersoll, and Ross [4] in their insightful paper established that the instantaneously riskfree rate

$$r = \rho - \frac{L(J_Q)}{J_Q} \tag{16}$$

where L is the differential operator.

This is Theorem 1 of Cox, Ingersoll, and Ross [4] as applied to an infinite horizon problem. Although we did not explicitly introduce any market for riskfree lending or borrowing, we can still obtain the shadow price for the interest rate using (16). The problem that has been solved earlier could just as well have been solved as an identical consumer problem with lending and borrowing at $r(Q)$. At equilibrium, there would have been no lending or borrowing as the consumers are identical and the specification (16) would have resulted as demonstrated by Cox, Ingersoll, and Ross [4]. For expositional ease the problem was set up as a single consumer problem.

The instantaneous interest rate is a *constant* and is

$$r = \alpha - \frac{a\hat{\sigma}^2}{2} \tag{17}$$

This can be easily seen by substituting J_Q in (16) from (11). Q.E.D.

Since the instantaneously riskfree rate is negative of the rate of change in the marginal utility of wealth we are able to solve for it easily. From (17) it is to be noted that the riskfree rate has a nice intuitive interpretation. The riskfree return is the expected return α from the risky technology minus a risk adjustment term which depends on the variance parameter $\hat{\sigma}^2$ and the coefficient of absolute risk aversion in wealth a.

Substituting for a from (12) into (17) we obtain

$$r = \alpha - \frac{\eta\alpha\hat{\sigma}^2}{2 + \eta\hat{\sigma}^2} = \frac{2\alpha}{2 + \eta\hat{\sigma}^2} \tag{18}$$

From (18) we can readily establish the following comparative statics properties for the riskfree rate. It is an increasing function in α and a decreasing function in both η and $\hat{\sigma}^2$. These results are fairly intuitive.

V. Discussion of the Results

A. In this paper we have presented the results for a constant absolute risk aversion, infinite horizon dynamic portfolio and consumption choice problem. First we show that there exists at least one plausible production scheme permitting random production of the commodity through time which causes the endogenously determined instantaneous market clearing interest rate to be constant. In this economy the technology employed also has intuitively appealing features. At the level of each production technique the rate of return is independent of the quantity invested. At the agents' wealth level, aggregate production can be approximated by a square root diffusion which belongs to constant elasticity of variance diffusion class. Merton [6, 7] has solved the dynamic consumption and portfolio problem when the preferences can be represented by a time additive negative exponential utility function. He, however, exogenously assumed the instantaneous riskless rate of interest to be a constant. Our model provides one general equilibrium scenario in which the constant equilibrium interest rate is consistent with negative exponential utility function. It appears that for Merton [6, 7] to get a constant equilibrium interest rate, there must have been an exogenous riskless investment.[6]

B. It has been established by many researchers such as Cox, Ingersoll, and Ross [4], and Rubinstein [8] that a default-free unit discount bond price is nothing but the expected discounted marginal rate of substitution between current consumption and future consumption. Once equilibrium default-free bond price are known the instantaneous riskless rate can then be found by suitable limiting arguments. These results are true for both exchange and closed production economies. In this model, since the instantaneous riskless rate is constant, the unit default-free discount bond prices can be found by simple present value arguments. It is easy to verify that the same result can be obtained by deriving a valuation equation on claims on goods, similar to the approach taken by Cox, Ingersoll, and Ross [4].

The individuals' consumption in our economy inherits stochastic properties from the stochastic process followed by the instantaneous output from the technology endowed in the economy. (The stochastic properties of constant elasticity of diffusion technology has been studied by Cox [3] in a different context and are reported in this paper for ready reference. Cox, Ingersoll, and Ross [4] provide distributional properties of the "square root" diffusion process).[7]

[6] Merton [6, 7] did not assume a riskless technology to exist. Opportunity for riskless borrowing and lending at a constant rate was, however, assumed to be available to agents.

[7] The stochastic properties of the constant elasticity of variance diffusion process have been examined by Cox [3] in the option pricing context. Cox, Ingersoll, and Ross [4] have discussed the "square root" process. For the square root model, the distribution of q_T can be written [refer to Equation (3) in the paper] as

$$f(q_T, T) = k\left(\frac{x}{y}\right)^{1/2} e^{-x-y} I_1[2(xy)^{1/2}]$$

where

$$k \equiv \frac{2\alpha}{\sigma^2[\exp\{\alpha(T-t) - 1\}]}$$

$$x \equiv kq_t \exp\{\alpha(T-t)\}$$

$$y \equiv kq_T$$

I_1 = modified Bessel function of the first kind of order 1.

C. The result in our paper shares the spirit of some of the results established by Constantinides [2]. In his paper, Constantinides has established a set of conditions on the production opportunity set that would cause the market clearing interest rate to remain constant. We discuss briefly his results in the context of our paper below and place our work in a perspective. At the outset it must be pointed out that the major result established by Constantinides [2] is the sufficiency of "weak aggregation" preferences and the stationarity of the risky technologies for the instantaneously riskless rate to depend only on the aggregate wealth. He did, however, discuss, briefly the conditions under which the term structure of interest rates may evolve deterministically.

In his Proposition 1′ Constantinides shows that the term structure of interest rates evolves deterministically over time under the following set of assumptions:

(A.1) Perfect capital markets,

(A.2) Homogeneous expectations,

(A.3) State independent utility.

The production technologies conform to assumption (A.4) explained in detail in his paper.[8] He further assumed that one technology is riskless and consumers are sufficiently risk averse so that the riskless technology is active. He made, in addition, *one* of the following assumptions:

(i) Decisions are made continuously in time and the vector of returns from the competitive firms is generated by a diffusion process; and

(ii) Consumers have additive quadratic utility functions. (This assumption was made in the context of discrete time models.)

Under these sets of assumptions, Constantinides [2] has established that the interest rates evolve deterministically over time. In our model we share the Assumptions (A.1)–(A.3). However, we allow for a different production scheme and the decisions to be made continuously in time and yet show that the interest rate is constant.

D. Our results so far have some bearing on the results of Stapleton and Subrahmanyam [10] and the related aspects of the recent work done by Bhattacharya [1]. Stapleton and Subrahmanyam [10] assumed a constant riskfree rate in the context of their model where payoffs were normally distributed and the agents have time additive negative exponential utility functions. Bhattacharya [1] has argued that the *assumption* of a nonstochastic riskfree rate is not generally consistent in *closed* economies when the underlying preferences are represented through a time additive negative exponential utility function. In his paper, Bhattacharya [1] has pointed out that such an assumption made by Stapleton and Subrahmanyam [10] is generally false. It must be emphasized that Bhattacharya's comments were made in the context of normally distributed payoffs and in the case of closed production economies to stochastic constant returns to scale technologies. This paper does suggest that the production scheme described in

[8] The assumptions concerning the production sector have been articulated in detail by Constantinides [2]. In establishing proposition 1′ constant returns to scale technologies or linear technologies with the randomness generated by a diffusion process have been used. In proving his subsequent results Constantinides [2] employs

$$dk_{jt} = g_j(k_{jt}, t)[\mu_j(\phi_t, t)\, dt + \sigma_j(\phi_t, t)\, dw_t]$$

the economy is consistent with the constant interest rate with the preferences characterized by negative exponential utility functions. However, in our model the returns are locally normal and the expected return across all assets are the same. In an equilibrium model where the expected returns differ across assets such as the Stapleton and Subrahmanyam [10] model, it remains to be shown that the equilibrium interest rate can evolve in a deterministic manner.

Our model points at the difficulties in obtaining a constant equilibrium interest rate in closed economies with constant absolute risk aversion preferences.

In conclusion it must be pointed out that the results of the constancy of interest rates were predicated on a set of assumptions which are only sufficient but not necessary. For instance, if a riskless technology exists it will provide at least a lower bound for the instantaneous riskfree rate. Other restrictions on the opportunity set might also lead to a constant riskfree rate. An interesting research issue is to identify the necessary and sufficient conditions on the opportunity set that will induce a constant riskless rate.

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