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Stochastic Dominance Rules for Truncated Normal Distributions: A Note

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THE PURPOSE OF THIS paper is to analyze stochastic dominance (SD) rules for truncated normal distributions, symmetrically truncated at least in the lower tail. This allows us to use distributions which conform to the limited liability condition of real life situations. We show that the SD rule reduces to an intuitively appealing rule that also provides a theoretical rationale for Baumol's mean-confidence limit criterion [2].

We analyze the preference between two risky options having truncated normal distributions. We contrast the results with the preference that would obtain without truncation, employing in the analysis stochastic dominance (hereafter SD) decision rules well known in the finance and economic literature. We note that if two distributions F and G intersect only once and F crosses G from below, then second order SD (SSD) is equivalent to the condition $E_F(x) \geq E_G(x)$. Also, if F and G are normally distributed, then SSD and M–V are equivalent. (For proof and review of all these rules, see Kroll and Levy [4]).

We turn now to the analysis of the relationship between preference by the expected utility criterion and by the M–V rule when the normal distributions are truncated. The truncation is defined in such a way that the density functions $f(x)$ and $g(x)$ are zero for all values x which deviate more than some fixed number of standard deviations from the mean. Thus, we have

$$\Phi\left(\frac{A_1 - \mu_1}{\sigma_1}\right) = \Phi\left(\frac{A_2 - \mu_2}{\sigma_2}\right) = \frac{\alpha}{2}$$

and

$$1 - \Phi\left(\frac{B_1 - \mu_1}{\sigma_1}\right) = 1 - \Phi\left(\frac{B_2 - \mu_2}{\sigma_2}\right) = \frac{\alpha}{2}$$

where Φ stands for the cumulative area under the standardized normal distribution and A_i and B_i are the truncation points. For each truncated normal distribution, an area of α is eliminated from the two tails and is shifted to the center. Such symmetric truncation is not essential for our analysis and the results of this paper can be generalized to any kind of truncation.

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Define[1]

$$\frac{A_1 - \mu_1}{\sigma_1} = \frac{A_2 - \mu_2}{\sigma_2} = -\delta \quad \text{and} \quad \frac{B_1 - \mu_1}{\sigma_1} = \frac{B_2 - \mu_2}{\sigma_2} = \delta \quad \text{where} \quad \delta > 0,$$

to obtain for the two truncated distributions (see Johnson and Kotz [3], p. 81)

$$E_F(x) = \mu_1, \qquad E_G(x) = \mu_2$$

$$\mathrm{Var}_F(x) = \left[1 - \frac{2\delta Z(\delta)}{2\Phi(\delta) - 1}\right]\sigma_1^2, \qquad \mathrm{Var}_G(x) = \left[1 - \frac{2\delta Z(\delta)}{2\Phi(\delta) - 1}\right]\sigma_2^2$$

where

$$Z(\delta) \overset{\text{def}}{\equiv} \frac{1}{\sqrt{2\pi}} e^{-1/2\delta^2}$$

Thus, the truncated distributions $F(x)$ and $G(x)$ are given by

$$F(x) = \begin{cases} 0 & x < A_1 \\ \dfrac{F^*(x) - \alpha/2}{1 - \alpha} & A_1 \le x \le B_1 \\ 1 & x > B_1 \end{cases}$$

and

$$G(x) = \begin{cases} 0 & x < A_2 \\ \dfrac{G^*(x) - \alpha/2}{1 - \alpha} & A_2 \le x \le B_2 \\ 1 & x > B_2 \end{cases}$$

Like the normal distribution, the truncated normal distribution also belongs to location-scale family of distributions. For more details see Bawa [1].

Theorem: Let F^* and G^* be two normal distributions with parameters (μ_1, σ_1) and (μ_2, σ_2) respectively, and F and G the corresponding truncated distributions. Then:

(1) If $\mu_1 > \mu_2$, $\sigma_1 > \sigma_2$, F dominates G by FSD iff $(\mu_1 - \mu_2)/(\sigma_1 - \sigma_2) > \delta$;
(2) If $\mu_1 > \mu_2$, $\sigma_1 < \sigma_2$ F dominates G by FSD iff $(\mu_1 - \mu_2)/(\sigma_2 - \sigma_1) > \delta$ (δ is a function of the truncation point);
(3) If $\mu_1 > \mu_2$, $\sigma_1 < \sigma_2$ F dominates G by SSD

Proof: *Case* (1): First note that F^* and G^* intersect at most once and the intersection point x_0 is given by $(x_0 - \mu_1)/\sigma_1 = (x_0 - \mu_2)/\sigma_2$. Since $(B_1 - \mu_1)/\sigma_1 = (B_2 - \mu_2)/\sigma_2 = \delta$, it is easy to verify that $B_1 > B_2$. However, for the lower truncation points we have $(A_1 - \mu_1)/\sigma_1 = (A_2 - \mu_2)/\sigma_2 = -\delta$. Hence, $A_1 = \mu_1 -$

[1] Note that the results of this paper hold also for nonsymmetric truncation. For example, we may have $A_i = \mu_i - \delta_1\sigma_i$ and $B_i = \mu_i + \delta_2\sigma_i$. Setting $\delta_2 = \infty$ yields an analysis with only lower truncation, which does not change our basic results.

$\delta\sigma_1 \gtreqless A_2 = \mu_2 - \delta_2\sigma_2$. Nevertheless, the following always holds:

$$(\mu_1 - \mu_2)/(\sigma_1 - \sigma_2) > \delta \Leftrightarrow A_1 > A_2$$

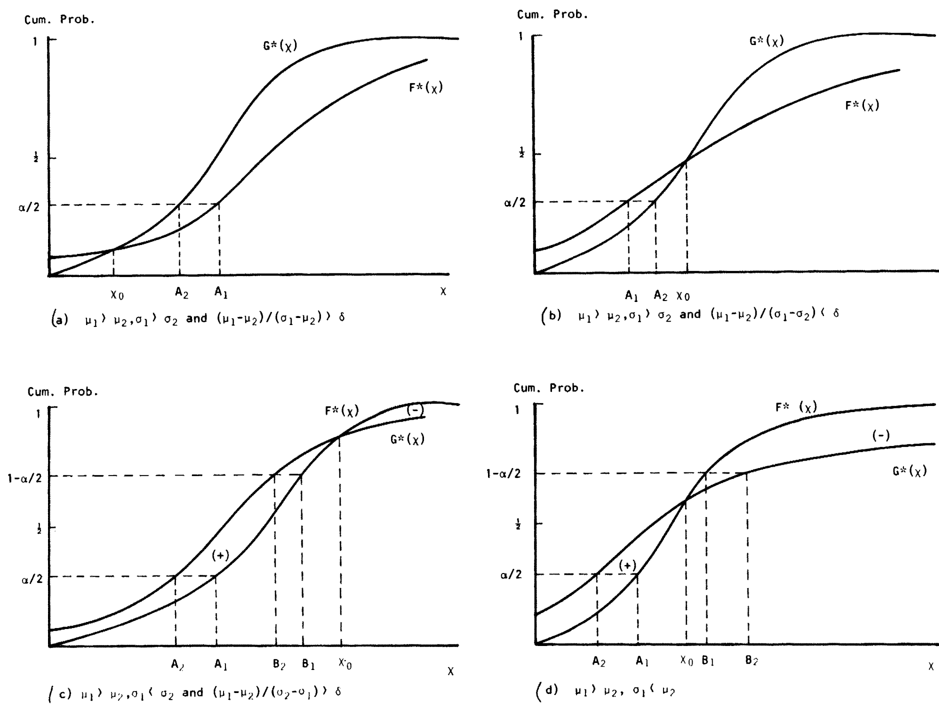
which we shall use below.

In order to prove the sufficiency, recall that since $\sigma_1 > \sigma_2$, the intersection of F^* and G^* is such that the left tail of F^* is above the left tail of G^* (see Figure 1, Panels (a) and (b)). However, if the condition of case (1) holds, we must have for a $\alpha/2$ lower tail truncation that $A_1 > A_2$, as described by Panel (a) of Figure 1. Thus, if the condition of Case (1) holds, we must have $x_0 < A_2 < A_1$. For all values $x < A_2$, we have $G(x) = 0$ and $F(x) = 0$; for all values $A_2 < x < A_1$ we have $G(x) > 0$ and $F(x) = 0$; hence $F(x) \leq G(x)$. For the range $x > A_1$ we have

$$F^*(x) < G^*(x) \Rightarrow \frac{F^*(x) - \alpha/2}{1 - \alpha} < \frac{G^*(x) - \alpha/2}{1 - \alpha}$$

which implies that $F(x) \leq G(x)$ (see the definition of F and G), i.e., F dominates G by FSD. (Recall that $B_2 > B_1$, and hence $G(B_1) = 1$ while $F(B_1) < 1$, so that $F(B_1) < G(B_1)$).

To prove necessity, note that if the condition of Case (1) does not hold, then $A_1 < A_2$ as described in Panel (b) of Figure 1. Thus, in this case $x_0 > A_2 > A_1$. Hence,



The Cumulative Distributions of Two Distinct Prospects

Figure 1.

for the range $A_1 < x < A_2$, $F(x) = \dfrac{F^*(x) - \alpha/2}{1-\alpha} > 0$ but $G(x) = 0$. Since $F(x) > G(x)$ in this range, F does not dominate G by FSD. The proof of Cases (2) and (3) follows similarly, using Panels (c) and (d) of Figure 1.

Conclusions

The result of the theorem has several implications. First note that if the conditions of the theorem (Case (2)) does not hold, neither F nor G dominates the other by SSD or by TSD. For the range $(-\infty, A_2]$ we have $F(x) > 0$ and $G(x) = 0$. Thus, $\int_{-\infty}^{A_2} [G(t) - F(t)]\,dt < 0$ and $\int_{-\infty}^{A_2} [\int_{-\infty}^{V} [G(t) - F(t)]\,dt]\,dv < 0$, which implies that F does not dominate G by SSD or by TSD. Since $E_F(x) > E_G(x)$, G does not dominate F by SSD or by TSD either, since a necessary condition for dominance by SSD or by TSD is that the preferred option commands a higher expected return. Thus, if F^* and G^* are normally distributed and no option dominates the other by the M–V rule, it also implies that no option dominates the other by SSD, let alone FSD. However, the truncation produces new distributions F and G, and we may find dominance by FSD of one distribution over the other. But if the truncation condition which guarantees FSD dominance does not hold, there is no dominance by SSD or by TSD.

A second interesting point is that the condition of the theorem (Case (1)) which is based on the expected utility notion (see [8]), has a very similar structure to Baumol's intuitive criterion. Recall that according to Baumol [2], F^* dominates G^* if $E_{F^*}(x) \geq E_{G^*}(x)$ and $E_{F^*}(x) - k\sigma_{F^*} > E_{G^*}(x) - k\sigma_{G^*}$, where $k > 0$. In our framework, the distributions are truncated from below simply because the prices of the assets cannot be negative. This truncation may be considered as an objective constraint on the distribution which is identical for all investors. Another distinction is that while Baumol analyzes the original normal distributions after removing the *left* tail, we renormalize the density function in the range $A < x < B$ after removing the two tails (or one tail), namely shift to another distribution. In spite of these differences, it can easily be verified that for $k = \delta$, Baumol's rule coincides with our FSD rule (Case (1)).

Finally, note that if the condition of the theorem (Case 1) holds, F dominates G by FSD, but not by the M–V rule no matter if we look at the parameters of the original distributions F^* and G^* or at those of the truncated distributions F and G. This follows from the fact that $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$ by assumption of the theorem, and for the truncated distributions $E_F(x) > E_G(x)$ and $\sigma_F(x) > \sigma_G(x)$. Thus, with truncated normal distributions the M–V rule may fail to distinguish between two options, in spite of the fact that one dominates the other by FSD.

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