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Additive Insurance Premiums: A Note

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IF A COMPETITIVE MARKET is in equilibrium, values must be additive in the sense that the value of a basket of goods must be equal to the sum of the values of the commodities it contains. Similarly, the value of a portfolio of securities must be equal to the sum of the values of the constituent securities. Applied to insurance, this principle means that the total premium should be the same whether a complex risk is insured under one single or several separate insurance contracts, provided, of course, that administrative expenses and transaction costs are ignored. One implication is that there must be some restrictions on the methods used to compute premiums if additivity and market equilibrium are to be preserved. With this starting point, we shall derive some well known results and indicate how they can be generalized.

I. Additive Premiums

Let the stochastic variable x represent claim payments under an insurance contract, and let $P\{x\}$ be the market premium for this contract.

In insurance, it is often natural to assume that x is stochastically independent of claim payments under all other insurance contracts. If one assumes further, as a first approximation, that the market premium depends only on the two first moments of x, it follows from the Capital Asset Pricing Model that

$$P\{x\} = \gamma_1 E\{x\} + \gamma_2 \operatorname{var} x \tag{1}$$

Traditionally, one will here take $\gamma_1 \geq 1$, so that the first term in (1) is sufficient to cover expected claim payments and the insurer's administrative expenses. It is then natural to interpret the second term, with $\gamma_2 \geq 0$, as a risk premium, determined by market forces. Formula (1) is additive, in the sense that if x_1 and x_2 are stochastically independent, we have

$$P\{x_1 + x_2\} = P\{x_1\} + P\{x_2\}$$

Many other formulae, based for instance on the standard deviation, have been used to compute insurance premiums. These will, however, not in general have the additivity property which seems essential in equilibrium.

The probability distributions of claim payments under insurance contracts can be very skew, and of very different shapes. A risk premium proportional to the variance may therefore be unsatisfactory. It is, however, easy to generalize (1) by adding a term $\gamma_3 E\{(x - \bar{x})^3\}$. By writing the expectation $\bar{x} = 0$ it is easy to see that $E\{(x_1 + x_2)^3\} = E\{x_1^3\} + E\{x_2^3\}$ if x_1 and x_2 are stochastically independent.

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Things do not work quite so well for higher moments, since we have $E\{(x_1 + x_2)^4\} = E\{x_1^4\} + 6\,E\{x_1^2\}\,E\{x_2^2\}$.

It is, however, easy to verify that the expression

$$\kappa_4 = E\{(x - \bar{x})^4\} - 3(E\{(x - \bar{x})^2\})^2$$

has the desired additivity property.

In general, we can define κ_n as the coefficient of $(it)^n/n!$ in the expansion

$$\psi(t) = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \kappa_n$$

where

$$\psi(t) = \log \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

The coefficients $\kappa_1, \kappa_2, \cdots$ are the cumulants of the stochastic variable x, introduced by Thiele [2] and [3], just because they are additive for independent variables. Hence, the generalization of (1) is

$$P\{x\} = \sum_{n=1}^{\infty} \gamma_n \kappa_n \tag{2}$$

The first cumulants are

$$\begin{aligned} &\kappa_1 = \bar{x}, \qquad \kappa_2 = \text{var } x, \qquad \kappa_3 = E\{(x - \bar{x})^3\} \\ &\kappa_4 = E\{(x - \bar{x})^4\} - 3(\text{var } x)^2 \\ &\kappa_5 = E\{(x - \bar{x})^5\} - 10E\{(x - \bar{x})^3\}(\text{var } x)^2 \text{ etc} \end{aligned}$$

It is remarkable that any additive premium formula, for independent variables, can be written in Form (2). This follows from a theorem by Lukacs [1] which says essentially:

If for an arbitrary bounded kernel $K(t, x)$ the transform

$$\varphi(t) = \int_{-\infty}^{+\infty} K(t, x)\, dF(x)$$

has the properties

(i) $\varphi_1(t) = \varphi_2(t)$ implies $F_1(x) = F_2(x)$

(ii) $F(x) = \int_{-\infty}^{+\infty} F_1(x - y)\, dF_2(y)$ implies $\varphi(t) = \varphi_1(t)\varphi_2(t)$

then

$$K(t, x) = e^{ixA(t)}$$

II. Stochastic Dependence

In order to generalize Formula (1) to the case of stochastic dependence, we shall write $x = x_1 + x_2$. We then have

$$\begin{aligned}\text{var } x &= \text{var } x_1 + 2 \text{ cov } x_1x_2 + \text{var } x_2 \\ &= \text{cov } xx_1 + \text{cov } xx_2\end{aligned}$$

Substitution in (1) will then give the additive premium formula

$$P\{x_i\} = \gamma_1 E\{x_i\} + \gamma_2 \text{cov } xx_i \qquad i = 1, 2 \tag{3}$$

Elimination of γ_2 between (1) and (3) gives

$$P\{x_i\} = \gamma_1 E\{x_i\} + \frac{\text{cov } xx_i}{\text{var } x}\{P\{x\} - \gamma_1 E\{x\}\} \tag{4}$$

This formula gives the premium for an arbitrary component of an insurance contract. In practice it can be used, for instance, if industrial fire, and loss of profit, are to be reinsured under separate arrangements.

Formula (4) will clearly give the Capital Asset Pricing Model, with a different interpretation of the terms.

As another example consider:

$$x_1 = \min(x, Q) \text{ and } x_2 = \max(x - Q, 0)$$

It is then easy to verify that

$$\bar{x}_1 = \int_0^Q xf(x)\,dx + Q\int_Q^\infty f(x)\,dx = Q + \int_0^Q (x - Q)f(x)\,dx$$

$$\bar{x}_2 = \int_Q^\infty (x - Q)f(x)\,dx$$

$$\text{cov } xx_1 = \int_0^Q x(x - \bar{x})f(x)\,dx + Q\int_Q^\infty (x - \bar{x})f(x)\,dx$$

$$\text{cov } xx_2 = \int_Q^\infty x(x - \bar{x})f(x)\,dx - Q\int_Q^\infty (x - \bar{x})f(x)\,dx$$

Substitution of these expressions into (3) will give the premiums required to cover the two components of the risk. The most obvious interpretations of this example are the following:

> If $P\{x\}$ is the premium for complete cover of a risk, $P\{x_1\}$ will be the reduction which the insured will get if he accepts a deductible Q.
>
> If $P\{x\}$ stands for the total premium paid to cover a portfolio of insurance contracts, $P\{x_2\}$ will be the premium for a stop loss treaty, under which the reinsurer pays the excess, if total claims should exceed the limit Q.

In the example, one can also take x as the price of a security at some future date. $P\{x_2\}$ can then be interpreted as the price of a European option with call price Q.

As a numerical example, we take $F(x) = 1 - e^{-x}$, so that $E\{x\} = 1$ and var $x = 1$. Further, let $\gamma_1 = \gamma_2 = 1$, so that $P = 2$. For the two components we find:

Retained premium: $P_1 = 2 - 2e^{-Q} - Qe^{-Q}$
Stop loss premium: $P_2 = 2e^{-Q} + Qe^{-Q}$

The table below gives the two premiums for some selected values of Q.

Table I

Q	P_1	P_2	$\bar{x}_1$	$\bar{x}_2$	var x_1	var x_2
0.5	0.48	1.52	0.39	0.61	0.02	0.85
0.69	0.65	1.35	0.50	0.50	0.06	0.75
1.0	0.89	1.11	0.63	0.37	0.13	0.60
1.5	1.23	0.77	0.78	0.22	0.28	0.40
2.0	1.44	0.56	0.86	0.14	0.44	0.25
3.0	1.75	0.25	0.95	0.05	0.69	0.01

The last four columns in the table are really irrelevant. They do, however, indicate that one will run into difficulties if one tries to set premiums for both components with loadings proportional to expectation or variance.

These examples illustrate how Formula (1) can be generalized to hold for dependent stochastic variables. To generalize (2) in a similar manner, it is sufficient to write

$$P\{x_i\} = \sum_{n=1}^{\infty} \gamma_n \kappa_{1,n-1}^{(i)} \qquad i = 1, 2 \tag{5}$$

where $\kappa_{1,n-1}^{(i)}$ is the cumulant corresponding to the bivariate moment $E\{(x_i - \bar{x}_i)\,(x - \bar{x}_1)^{n-1}\}$. Clearly (5) is a generalization of (3).

To apply (5) to the numerical example, we consider only the three first terms in the expansion; take $\gamma_1 = 1$, $\gamma_2 = 0.4$, $\gamma_3 = 0.3$ and find

$$\text{Retained premium: } P_1 = 2 - 2e^{-Q} - Qe^{-Q} - 0.3Q^2e^{-Q}$$
$$\text{Stop loss premium: } P_2 = 2e^{-Q} + Qe^{-Q} + 0.3Q^2e^{-Q}$$

As one would expect, consideration of the skewness will lead to a higher stop loss premium.

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