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Optimum Distribution-Free Tests and Further Evidence of Heteroscedasticity in the Market Model

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ABSTRACT

In this paper several powerful distribution-free tests for heteroscedasticity are introduced and are used to test the hypothesis of constant variance in the market model. These tests are noted for their flexibility in specifying alternative hypotheses. It is found that the assumption of homoscedasticity is untenable for the majority of stocks analyzed. The implications of this finding for the efficient estimation of the parameters of the market model are also discussed.

IN THE PAST FEW years the amount of research devoted to the market model as developed by Markowitz [14], and Sharpe [18], has increased considerably. This model is usually defined by

$$\tilde{R}_{it} = \alpha_i + \beta_i \tilde{R}_{mt} + \tilde{\varepsilon}_{it} \qquad \begin{array}{l} i = 1, 2, \ldots, I \\ t = 1, 2, \ldots, T \end{array} \tag{1}$$

where $\tilde{R}_{it}$ is the rate of return on stock i during period t; $\tilde{R}_{mt}$ is the rate of return on a market index; $\tilde{\varepsilon}_{it}$ is a random disturbance with mean zero, constant variance σ_i^2, and serially uncorrelated; α_i is the expected return for the ith security when $E(\tilde{R}_{mt}) = 0$; and β_i is the systematic risk coefficient for the same stock. In particular, the assumption of constant variance has been investigated by severeal authors: Martin and Klemkosky [15], Brown [6], Belkaoui [1], and Bey and Pinches [2]. There are two major reasons for this interest. First, the Ordinary Least Squares (OLS) estimators of the parameters of the market model are inefficient when the error terms have heteroscedastic variances (see Section IV, and also Goldfeld and Quandt [10] for some experimental evidence on the extent of inefficiency). And second, tests for the impact of firm-specific events (e.g., Brown and Warner [7]) on stock prices implicitly assume that the variance of returns is constant.

The evidence presented in the literature is incomplete for several reasons. First, the traditional tests for heteroscedasticity, such as the Goldfeld-Quandt, Bartlett, or the Glejser tests, as used in the previous literature, are unreliable if the basic assumption of normality for the error distribution in the model is violated. Seber [17, p. 147], Brown and Forsythe [4], and Layard [13] have reported that if the error distribution, and hence that of R_{it}, is not normal, the

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probability of Type I error (rejecting a true null hypothesis) can be several times that of the prespecified nominal level of significance. Given the studies of Fama [9], Praetz [16], Clark [8], and many others that argue that the distribution of stock returns is fat tailed, it is possible that the incidence of heteroscedasticity detected by these tests may be overstated. Bey and Pinches [2] addressed this issue and found some support for such a suspicion. Second, some of the tests used in the literature require that the errors be independent. However, these tests have been applied to the estimated errors (OLS residuals) which are correlated, and thus, conclusions from these tests may not be tenable (see Section I for further elaboration of this point). Third, so far only a limited number of specifications for the alternative hypothesis to homoscedasticity have been investigated. Thus, in many of the previous investigations, implicitly or explicitly, it is hypothesized that the variance is increasing with the market rate of return, R_{mt}. In the presence of our ignorance of a specific alternative, it is important to have tests which are flexible in specifying a variety of alternatives and, to identify, we hope, the most suitable one. It may be noted that the Generalized Least Square (GLS) method of estimation is more efficient than OLS when the error variance is not constant, provided we have a proper knowledge of the behavior of the variance.

Although the nonparametric tests used in the literature avoid most of the above criticisms, they are not as powerful and as flexible in investigating a variety of alternatives as one would like them to be. Also, they are applied to OLS residuals which are not independent. In this study, we introduce two classes of distribution-free tests for heteroscedasticity which are robust to non-normality, and also asymptotically powerful. These classes of tests have comparable power and specifically the class of tests based on ranks is locally most powerful not only within the class of rank tests but also in the class of all possible tests — parametric or nonparametric [11, p. 249]. Therefore, by construction our tests are locally more powerful than the conventional tests used in the literature, such as the Goldfeld-Quandt, Bartlett, Glejser, modified Glejser as defined in [2], and the much used nonparametric tests like Kendall's rank correlation and Peak tests. Our tests are also flexible in investigating a wide range of alternatives. Since these tests, like the traditional ones, require that the errors be independent, we introduce the use of recursive residuals in this context instead of the OLS residuals commonly used to estimate the errors. The recursive residuals by construction are independently distributed.

The nonparametric tests, well known in the applied literature (especially in Finance), are briefly described in Section I. The effects of using the OLS instead of the recursive residuals on the powers of these tests are also explored by applying them to a sample of securities obtained from New York Stock Exchange. A class of distribution-free tests is introduced in Section II and is applied to the same sample. The class of rank tests is introduced in Section III and is again applied to the sample. In these applications, a variety of alternative hypotheses is considered. The analysis indicates that heteroscedasticity is a serious problem and that possibly the variance increases in proportion to the square of R_{mt} rather than R_{mt} itself. The implications of the findings for the efficient estimation of the parameters of the market model, along with concluding remarks, are given in Section IV.

I. Data, Traditional Robust Tests, and Residual Estimation

I.1 *The Data*

The data used in this study consist of monthly rates of return on 384 securities randomly selected from the Compustat Tapes. The sample period is from January 1965 through December 1977; therefore, there are a total of 156 observations per stock. The SP 425 index over the same period is used as a proxy for the market rate of return. This particular time period covers a wide range of economic activity: from the Boom of 1966, the "Credit Squeeze" of 1968, the wage and price controls of 1971–2, the recession of 1974, and finally the recovery of 1975–6. Hence, one would suspect a priori that the random shocks $\tilde{\varepsilon}_{it}$ may not be homogeneous throughout the sample period.

I.2 *Traditional Robust Tests and Residual Estimation*

Both parametric and nonparametric tests have been used previously in testing for heteroscedasticity in model (1). As discussed earlier, the parametric tests are sensitive to violations of the normality assumption regarding the errors. The nonparameteric tests are robust to non-normality but they require that the errors to which these tests are applied be independent. In practice, the errors are unknown and are estimated by the OLS residuals. Unfortunately, even when the true errors are independent and homoscedastic, the OLS residuals are neither independent nor homoscedastic. Thus, the conclusion of these tests may not be reliable. Fortunately, there exists an alternative procedure to estimate a set of independent and homoscedastic residuals provided that the errors have the same properties. These residuals are usually called "Recursive Residuals," (see [5]) and are defined by the sequence of one-step-ahead prediction errors, estimated on the basis of previous data and normalized by the standard deviation of each one step ahead prediction error.[1]

$$\hat{\varepsilon}_{it} = \frac{R_{it} - \hat{\alpha}_{i,t-1} - \hat{\beta}_{i,t-1} R_{mt}}{\left(1 + \dfrac{1}{t-1} + \dfrac{(R_{mt} - \bar{R}_{m,t-1})^2}{\sum_{j=1}^{t-1} (R_{mj} - \bar{R}_{m,t-1})^2}\right)^{1/2}} \qquad \begin{array}{l} i = 1, 2, \cdots, I \\ t = 3, 4, \cdots, T \end{array} \tag{2}$$

where $\hat{\alpha}_{i,t-1}$ and $\hat{\beta}_{i,t-1}$ are the OLS estimators of α_i and β_i, respectively, based on the first $t-1$ observations; and $\bar{R}_{m,t-1} = \frac{1}{t-1} \sum_{j=1}^{t-1} \mathrm{R}_{mj}$. Note that only $T-2$ recursive residuals may be computed. To test for heteroscedasticity in the market model, we compute the recursive residuals $\hat{\varepsilon}_{it}$, $t = 3, \ldots, T$ for each stock i and then apply the tests to these residuals.

Essentially, only two nonparametric tests — "Peak" test of Goldfeld and Quandt [10] and Kendall's rank correlation test (Bey and Pinches [2, p. 306]) —

[1] Analysis of the properties of these residuals in [5] formally justifies the conventional wisdom of using prior period data to estimate residuals in an event studies context (see, e.g. Brown and Warner [7]). Recursive residuals could be very important in the context where independence of successive residuals is crucial, as where inferences are drawn relating to the timing of information releases.

Table I

Percentage of Rejections of Homoscedastic Variance—Traditional Robust Tests

	Kendall's Rank Correction Test		Peak Test	
	Residuals		Residuals	
Alternative	OLS	Recursive	OLS	Recursive
R_{mt}	27	31	7	8
R^2_{mt}	36	39	37	38

have been applied in the literature to test for heteroscedasticity in model (1). The peak test is basically a formalization of heuristic graphical procedures which might be employed when the variance of the disturbance term is thought to increase as a function of some independent variable, say, R_{mt}. It orders R_{mt} from low to high and then orders the (absolute) residuals $|\hat{\varepsilon}_{it}|$, where $\hat{\varepsilon}_{it}$ are estimates of $\tilde{\varepsilon}_{it}$, in the same sequence. A peak is defined when the magnitude of a residual $\hat{\varepsilon}_{it}$ exceeds, for the first time, the magnitude of all the residuals preceding it. The first residual is an exception and is not counted. Kendall's rank correlation test simply computes the product moment correlation coefficient between the ranks of R_{mt} and the corresponding ranks of $|\hat{\varepsilon}_{it}|$. If the residuals are heteroscedastic, and the variance increases with R_{mt}, both the number of peaks and Kendalls' rank correlation coefficient will tend to be large. Both statistics, when standardized by their respective standard errors, are asymptotically normally distributed.

We apply each of these tests to both kinds of residuals —OLS and recursive —twice: first, when the residuals are ordered in terms of R_{mt}; and second, when they are ordered in terms of R^2_{mt}. Table I summarizes our finding when these tests are applied to the stocks in our sample.

An examination of the entries in Table I suggests that the tests may not be very sensitive to the use of OLS residuals. However, the evidence is by no means conclusive, and for this reason we shall use recursive residuals throughout this study.

II. An Alternative Class of Robust Tests

Recently Bickel [3] has proposed a class of robust tests for heteroscedasticity which may have asymptotic power comparable to that of the locally most powerful (LMP) rank tests to be introduced in the next section.[2] To present this class of tests, let $\tau_{it} = E(\tilde{R}_{it})$, the mean of $\tilde{R}_{it}$, and let the error variance be a function of this mean. More specifically, let

$$Var\ (\tilde{\varepsilon}_{it}) = \sigma_i^2 h^2(\tau_{it}) \tag{3}$$

where h is a function of the mean τ_{it}. Assume that $h(\tau_{it})$ can be written as

[2] A quite different robust test of variance increasing with R_{mt}, not studied here, however, has recently been proposed by Koenker and Basset [12].

$$h(\tau_{it}) = 1 + \theta_i a(\tau_{it}) + 0(\theta_i) \tag{4}$$

where the parameter θ_i does not involve τ_{it}; $a(\tau_{it})$ is a positive function exclusively of τ_{it}; and $0(\theta_i)$ is a function of θ_i such that $0(\theta_i)/\theta_i \rightarrow o$ as $\theta_i \rightarrow o$. Then, the null hypothesis of homoscedasticity can be stated as H_{oi}: $\theta_i = o$, and the alternative H_{ai}: $\theta_i \neq o$; i.e., the error variance corresponding to the return of the ith stock changes with its mean. This formulation of the variance of $\tilde{\varepsilon}_{it}$ allows a variety of types of heteroscedasticity. Setting, for example, $a(\tau_{it}) = |\tau_{it}|^k$ or $ln\,|\tau_{it}|$ allows tests of the hypotheses that the standard deviation of $\tilde{\varepsilon}_{it}$ varies with the *kth* power of the mean return and the natural log of τ_{it}, respectively. If we denote by $b(\varepsilon_{it})$ a measure of the variability of ε_{it}, such as ε_{it}^2 for the normal distribution, then under the heteroscedasticity hypotheses, $b(\varepsilon_{it})$ should be correlated with $a(\tau_{it})$ which is that part of the standard deviation of ε_{it} that depends on the mean τ_{it} alone. The class of tests proposed by Bickel [3] exploits this intuition and is defined by the product moment correlation coefficient between $a(\hat{\tau}_{it})$ and $b(\hat{\varepsilon}_{it})$ where $\hat{\tau}_{it}$ and $\hat{\varepsilon}_{it}$ are the estimated values of τ_{it} and ε_{it} respectively,[3] and $b(\cdot)$ is a transformation reflecting the scale of $\hat{\varepsilon}_{it}$ chosen to maximize the power of the test. More formally, the test statistic B_b is defined by

$$B_b = \frac{\sum_t (a(\hat{\tau}_{it}) - \overline{a(\hat{\tau}_{it})})(b(\hat{\varepsilon}_{it}) - \overline{b(\hat{\varepsilon}_{it})})}{\left\{\sum_t (a(\hat{\tau}_{it}) - \overline{a(\hat{\tau}_{it})})^2 \dfrac{1}{T-2} \sum_t (b(\hat{\varepsilon}_{it}) - \overline{b(\hat{\varepsilon}_{it})})^2\right\}^{1/2}} \tag{5}$$

where $\overline{a(\cdot)}$ and $\overline{b(\cdot)}$ are the sample mean values of $a(\cdot)$ and $b(\cdot)$ respectively. It can be shown that the optimum $b(\cdot)$ is given by

$$b(x) = -xg'(x)/g(x) \tag{6}$$

where $g(x)$ is the probability density function of the distribution of $\tilde{\varepsilon}_{it}$, and $g'(x) = dg(x)/dx$.

It follows that $b(x)$ equals x^2 if the probability distribution of $\tilde{\varepsilon}_{it}$ is normal; $|x|$ if the distribution is double exponential; $x(1 - e^{-x})/(1 + e^{-x})$ for the logistic; and $2x^2/(1 + x^2)$ for the Cauchy distribution.

In large samples, B_b is approximately normally distributed with mean zero and unit variance. To distinguish the statistics generated by the different distributions, we denote by B_N, B_L, B_{DE}, and B_C the statistics corresponding to the function $b(\cdot)$ generated by the normal, logistic, double exponential, and cauchy distribution, respectively. We have selected these distributions because they have increasingly fatter tails, and according to Fama [9] and others, rates of return on common stocks appear to be generated by distributions with heavy tails. The application of these tests (at the 5% level of significance) to the market model shows (see Table II) that for $a(\tau_{it}) = |\tau_{it}|^k$ all the statistics B_N, B_L, B_{DE}, and B_C, reject homoscedasticity approximately 36 and 52 percent of the time for $k = 1$ and 2 respectively. Whereas, for the alternative $a(\tau_{it}) = ln\,|\tau_{it}|$ the rate of rejection is

[3] $\hat{\varepsilon}_{it}$ are taken to be the recursive residuals defined in (2), and $\hat{\tau}_{it} = \hat{\alpha}_i + \hat{\beta}_i R_{mt}$, where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the OLS estimates of α_i and β_i respectively.

Table II

Percentage of Rejections of Homoscedastic Variance—Bickel Tests

$a(\cdot)$	B_N	B_L	B_{DE}	B_C
$\lvert \hat{\tau}_{it} \rvert$	36.3	36.3	37.1	36.6
$\hat{\tau}^2_{it}$	51.8	51.8	54.7	52.3
$ln \lvert \hat{\tau}_{it} \rvert$	32.6	32.4	32.1	32.6

around 32 percent for all four tests.[4] Thus, the most promising alternative hypothesis seems to be that the variance increases with the square of the mean rate of return.

III. Rank Tests for Heteroscedasticity[5]

As an alternative to the Bickel procedure of the previous section, consider a sequence of independent random variables $\{X_t\}_{t=1}^{T}$ each with distribution function $G\left(\frac{X_t}{\sigma_t}\right)$, where the scale parameter σ_t is unknown. The null hypothesis of homoscedasticity can be written as $H_0: \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_T^2$, and the alternative $H_a: H_0$ not true. Alternatively, let $\sigma_t = e^{\lambda + \nu d_t}$, where $\{d_t\}_{t=1}^{T}$ is a sequence of known constants, then H_0 can be expressed more concisely as $H_0: \nu = 0$, and H_a: $\nu > 0$. The alternative $\nu < 0$ may be obtained by replacing d_t by $-d_t$. This alternative specification is general enough to cover many different types of heteroscedasticity, and in particular allows for a richer class of alternatives than does the Bickel procedure. For example, setting $d_t = t$ is equivalent to assuming that the variance of X_t increases log linearly over time, whereas $d_t = \sin\left(\frac{2\pi}{\theta}t\right)$ specifies a variance with a cyclical pattern as would be the case if the variance takes on different value between bull and bear markets. Last, $d_t = R_{mt}^k$, $k = 1, 2$, represents the alternative that the variance varies with the market.

To introduce the tests we wish to consider, let R_t^* be the rank of X_t among the T observations $(X_1, X_2, \ldots, X_T)$. Then, the most powerful tests of the hypothesis of homoscedasticity $H_0: \nu = 0$ versus H_a: $\nu > 0$ are given by the following rank tests:

$$S_N = \sum_{t=1}^{T} (d_t - \bar{d})\left\{\Phi^{-1}\left(\frac{R_t^*}{T+1}\right)\right\}^2 \tag{7}$$

$$S_L = \sum_{t=1}^{T} (d_t - \bar{d})\left\{\left(R_t^* - \frac{T+1}{2}\right)ln\left(\frac{R_t^*}{T+1-R_t^*}\right)\right\} \tag{8}$$

$$S_{DE} = \sum_{t=1}^{T} (d_t - \bar{d})\left\{-ln\left(1 - \left|\frac{2R_t^*}{T+1} - 1\right|\right)\right\} \tag{9}$$

[4] Since the tests are not independent, the overall level of significance would be different from the chosen level of 5%. Thus, caution is needed for the interpretation of the results.

[5] The discussion that follows is of an expository nature. For a rigorous analysis the reader should consult the excellent book by Hájek and Šidák [11].

and

$$S_C = \sum_{t=1}^{T} (d_t - \bar{d}) \left\{ \frac{\left[\tan\left(\pi\left(\frac{R_t^*}{T+1} - \frac{1}{2}\right)\right)\right]^2}{1 + \left[\tan\left(\pi\left(\frac{R_t^*}{T+1} - \frac{1}{2}\right)\right)\right]^2} \right\} \tag{10}$$

where $\bar{d} = \frac{1}{T}\sum_{t=1}^{T} d_t$, and Φ^{-1} is the inverse function of the cumulative normal distribution. These statistics are derived in an Appendix available from the authors. Under the null hypothesis, each test is asymptotically normally distributed with mean zero, and variance

$$Var\,(S) = \frac{1}{T-1}[\sum_{t=1}^{T} (d_t - \bar{d})^2][\sum_{t=1}^{T} (A_t - \bar{A})^2] \tag{11}$$

where $\bar{A} = \frac{1}{T}\sum_{t=1}^{T} A_t$, and for each test A_t stands for the quantity inside the { } brackets $\left(\text{e.g., for } S_N, A_t = \left\{\Phi^{-1}\left(\frac{R_t^*}{T+1}\right)\right\}^2\right)$.[6] Large positive values of S provide evidence of nonstationary variance.

Each of the test statistics is designed to be optimal under a particular probability distribution. Thus, S_N is the optimal test when the distribution generating the observations X_t is normal, whereas S_L, S_{DE}, and S_C are optimal when the generating distribution is logistic, double exponential, and cauchy, respectively.

To give an intuitive explanation of how these tests detect changes in the variance, note that one measure of the variance at time t is the magnitude of the deviation of X_t from its mean. Or, if we replace X_t by its rank R_t^*, the magnitude of the deviation of R_t^* from its mean $(=(T+1)/2)$, can also be taken as a measure of variability at time t. If the variance of X_t is changing according to d_t, we then expect nonzero correlation between d_t and the magnitude of $\{R_t^* - (T+1)/2\}$ or a suitable transformation of R_t^*. Our test statistics, therefore, are based on the correlation between d_t and the transformed R_t^*. Note that this is precisely the intuition of the Bickel tests of the previous section, where residual values were used instead of the ranks. The transformation (such as $\Phi^{-1}(\cdot)$) is chosen so as to maximize the power of each test. It is shown in [11] that under some general conditions on the distribution function G, and the constants $\{d_t\}$, the locally most powerful test of heteroscedasticity is given by the correlation coefficient between d_t and the transformed R_t^*.

To apply the tests presented in this section to the problem at hand, notice that X_t plays the role of $\tilde{\varepsilon}_{it}$ in Equation (1). The unknown values of $\tilde{\varepsilon}_{it}$ are estimated by the recursive residuals $\hat{\varepsilon}_{it}$. We have applied the tests S_N, S_L, S_{DE}, and S_C for several alternative sequences $\{d_t\}_{t=3}^{T}$. We have chosen $d_t = t$; $ln\ t$; $\sin\left(\frac{2\pi}{\theta}t\right)$, for

[6] In the statistical literature A_t is referred to as the "Score" function.

Table III

Percentage of Rejections of Homoscedastic Variance—Rank Tests

$d_t = t$		lnt	$\sin\left(\frac{2\pi}{12}\cdot t\right)$	$\sin\left(\frac{2\pi}{24}\cdot t\right)$	$\sin\left(\frac{2\pi}{48}\cdot t\right)$	R_{mt}	R^2_{mt}
S_N	31.3	41.4	3.6	1.6	6.0	35.2	54.9
S_L	29.9	40.6	4.7	1.8	5.7	34.9	54.4
S_{DE}	30.7	40.4	4.4	1.6	6.0	35.6	53.6
S_C	25.2	33.1	5.7	1.8	4.2	23.2	34.4

$\theta = 12$, 24, and 48 corresponding to cyclical changes in variance of period one, two, or four years; and $d_t = R_{mt}$, and R^2_{mt}. All tests are performed at the 5% level of significance, and in Table III we report the percentage of rejections (out of 384) of the null hypothesis for each of the four tests and each sequence[7] d_t. Judging from the results, one may conclude that the homogeneity of variance assumptions is untenable for the majority of the stocks in our sample.[8, 9] For only 30% of the securities no test rejected the null hypothesis for $d_t = t$, lnt, R_{mt}, and R^2_{mt}. The percentage of rejections ranges from approximately 5% for the cyclical variance alternative, 30% to 40% for a monotonically increasing trend, and 35% and 54% for a variance which varies with the market. Hence, the most likely alternative for modeling the variance of the rates of return, at least for our sample of data, is $d_t = R^2_{mt}$, i.e., the variance moves directly with the market return squared. These results strengthen the conclusion reached by Brown [6], Bey and Pinches [2], and others, that heteroscedasticity is a serious problem in the market model.[10] Moreover, they suggest that previous studies would have found more evidence of nonstationarity in the variance of returns, had they ordered the data by the size of R^2_{mt}. Finally, comparing Table III with Table 4 of Bey and Pinches [2, pp. 318–19] it seems that the rank tests tend to detect heteroscedasticity more often than the tests of Goldfeld and Quandt, Bartlett, or Glejser.

We have already observed (see Table I) that Kendall's rank correlation test rejects homoscedasticity for 31 percent and 39 percent of the securities when the absolute value of the (recursive) residuals is hypothesized to be correlated with R_{mt} and R^2_{mt}, respectively. And the Peak test rejects the null hypothesis 8 percent and 38 percent when the residuals are ordered in terms of R_{mt} and R^2_{mt} respectively. Although the power of these tests is less than that of the rank tests for the

[7] Several other alternatives including $d_t = t. \sin\left(\frac{2\pi}{\theta}\cdot t\right)$, and $d_t = R^2_{mt}\cdot\sin\left(\frac{2\pi}{\theta}\cdot t\right)$ were also examined. However, none of these turned out to be highly significant; hence these results are not reported here.

[8] Since the tests are not independent, the overall level of significance would be different from the chosen level of 5%. Thus, caution is needed for the interpretation of the results.

[9] The results of the statistic S_C must be interpreted loosely because for the Cauchy distribution the OLS method for estimating α_i and β_i is not appropriate, since the expected error sum of squares is infinitely large.

[10] Bey and Pinches [2] have recently presented a complete review of the literature on heteroscedasticity in the market model; therefore, no attempt will be made here to compare and contrast past results.

Table IV

Percentage Rejection Rates for the Bickel and the Rank Tests

$a(\hat{\tau}_{it})$ or d_t	Tests							
	B_N	B_L	B_{DE}	B_C	S_N	S_L	S_{DE}	S_C
$\lvert \hat{\tau}_{it} \rvert$	36	36	37	37	37	35	35	26
$\hat{\tau}_{it}^2$	52	52	55	52	54	53	53	36
$ln \lvert \hat{\tau}_{it} \rvert$	33	32	32	33	34	33	34	24

same alternatives (see the last two columns of Table III), they reinforce the conclusion reached earlier that lack of stationary variance is a serious problem of the market model, and that the variance seems to be increasing with R_{mt}^2 rather than with R_{mt}.

In section II we concluded that, according to Bickel tests, heteroscedasticity is a serious problem and it is likely that the error variance increases with the square of the mean rate of return, τ_{it}. As τ_{it} varies linearly with the expected value of R_{mt}, one may conclude that the variance increases with R_{mt}^2. This is consistent with the conclusion reached with the traditional nonparametric tests or the optimum rank tests which are introduced here.

Comparing the entries in Tables II and III, we find that the rejection rates of Bickel tests corresponding to the alternatives specified by $a(\tau_{it}) = \lvert \tau_{it} \rvert$ and τ_{it}^2 (i.e., variance increases with $\lvert \tau_{it} \rvert$ and τ_{it}^2) are comparable to those of the rank tests for alternatives specified by $d_t = R_{mt}$ and R_{mt}^2 (i.e., variance increases with R_{mt} and R_{mt}^2) respectively. Since τ_{it} is linearly related to R_{mt}, this suggests that the Bickel tests and the rank tests may have comparable power. Note, however, that Bickel's formulation allows only those alternatives where the variance is assumed to vary with some function of the mean rate of return, τ_{it}. Thus, the alternatives where $a(\tau_{it}) = R_{mt}$ or R_{mt}^2 are not meaningful. On the other hand, the rank tests are not so limited and we can very well formulate alternatives where $d_t = \lvert \tau_{it} \rvert, \tau_{it}^2$ or $ln \lvert \tau_{it} \rvert$, i.e., d_t can be chosen to be $a(\tau_{it})$. Table IV compares the powers of the Bickel and the rank tests when $a(\tau_{it}) = d_t = \lvert \tau_{it} \rvert, \tau_{it}^2$ and $ln \lvert \tau_{it} \rvert$. As can be seen the powers of these two classes of tests are virtually identical. Thus, given the limitations in formulating the alternatives for the Bickel tests, one may opt for the rank tests.

IV. Conclusion

In this study we have introduced two classes of optimal distribution-free tests for heteroscedasticity, and have applied them to the market model. The major advantages of our nonparametric tests over the more conventional tests of Goldfeld-Quandt and Bartlett are that they are robust to non-normality, and they allow investigations of a wide variety of alternative hypotheses. These hypotheses may be tested simply by specifying a different $\{d_t\}$ sequence in the case of the rank tests and $a(\tau_{it})$ in the case of Bickel tests. This is not true of the traditional tests mentioned above in that the conventional tests require a different ordering of the observations, and therefore a new regression, for each different specification of the alternative hypothesis. This may possibly account for the lack of tests in the literature of the hypothesis that the variance of returns varies with the square of the market return, or with the dummy variable time.

Furthermore, the flexibility of the rank tests makes them prime candidates for applications in other areas of finance, such as in the analysis of the impact of special events (i.e., changes in capital structure, dividend policy, etc.,) on the behavior of the variance of residuals. In these cases, d_t would be set equal to zero for all t prior to the event, and 1 everywhere else.

The results of our analysis imply that the hypothesis of constant variance is untenable for the majority of the stocks in our sample and possibly the variance is increasing with R^2_{mt}. This finding has implications for the efficient estimation of the parameters of the market model. If the behavior of the variance over the sample period follows a predictable pattern, e.g., $\sigma_t = e^{\lambda+\nu d_t}$, then one should incorporate this knowledge in formulating an appropriate variance-covariance matrix for $\tilde{\varepsilon}_{it}$, and obtaining efficient estimates of α_i, β_i, and unbiased estimates of $\sigma^2_{i,T+1}$. As an example of the possible gain in efficiency, suppose the variance of returns varies with the square of the Market Return, i.e., $\sigma^2_{it} = \sigma^2_i\ R^2_{mt}$. Then, the OLS estimator of β_i and its variance are given by (all summations over t = 1, 2, . . . T),

$$\hat{\beta}_i = \sum(R_{it} - \bar{R}_i)(R_{mt} - \bar{R}_m)/\sum(R_{mt} - \bar{R}_m)^2 \tag{12}$$

where $\bar{R}_i = \sum R_{it}/T$, $\bar{R}_m = \sum R_{mt}/T$, and

$$V(\hat{\beta}_i) = \frac{\sigma^2[(\sum R^2_{mt})(\sum R_{mt})^2 - 2T(\sum R^3_{mt})(\sum R_{mt}) + T^2(\sum R^4_{mt})]}{T\sum R^2_{mt} - [(\sum R_{mt})^2]^2} \tag{13}$$

whereas the GLS estimator and its variance are

$$\hat{b}_i = \sum(R_{it}/R_{mt})/T - \hat{\alpha}_i \sum(1/R_{mt})/T$$

where

$$\hat{\alpha}_i = \frac{\sum(R_{it}/R_{mt} - \sum(R_{it}/R_{mt})/T(1/R_{mt} - \sum(1/R_{mt})/T)}{\sum(1/R_{mt} - \sum(1/R_{mt})/T)^2} \tag{14}$$

and

$$Var(\hat{b}_i) = \frac{\sigma^2 \sum(1/R^2_{mt})}{T\sum(1/R^2_{mt}) - [\sum(1/R_{mt})]^2} \tag{15}$$

For the sample period used in this study, we find

$$\frac{Var(\hat{b}_i)}{Var(\hat{\beta}_i)} = \frac{.006448}{.024333} = .26499 \tag{16}$$

Thus, the simple least-squares estimator is only 26 percent as efficient as the GLS estimator.

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