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Difference Systems in Financial Futures Markets

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ABSTRACT

Many financial futures markets allow substitutions for the par grade of security at delivery. Substitutes are deliverable at premiums or discounts—"differences" in commodities parlance—to the futures price. The rule that establishes these differences is called a difference system. This paper characterizes financial futures market equilibrium with yield-based difference systems and investigates particular systems in use. The major finding is that currently used difference systems effectively limit deliverable supply in the futures markets and lead to futures prices which understate the cash market price of the par security.

IN ORDER TO INCREASE deliverable supply, many financial futures markets allow the delivery of a variety of grades (issues) of securities in fulfillment of a financial futures contract.[1] The par grade of security on which the futures contract is based is deliverable at the futures price. Various other grades of securities are deliverable at premiums or discounts to the futures price. In commodities parlance, these premiums and discounts are called "differences", and the rule that establishes the set of premiums and discounts is called a "difference system." [See Hoffman [2] Chapter XV for an excellent discussion of difference systems in commodity futures contracts].

There are two types of difference systems currently in use in financial futures markets. Both are based on security yields. The first is called variable rate or yield maintenance pricing. Under this system, nonpar securities are deliverable at the futures price times a factor which is the price of the delivered security relative to the par security where both are discounted by the yield of the par security selling at the futures price.

The second is called fixed rate or factor pricing. Under this system, nonpar securities are deliverable at the futures price times a "factor" which is the price of the delivered security relative to the par security where both are discounted at the yield of the par security selling at par.

This paper first characterizes financial futures market equilibrium with yield-based difference systems and then investigates theoretically and empirically the particular characteristics of factor pricing and yield maintenance pricing. The major finding is that currently used differences systems produce financial futures prices which systematically understate the spot market price of the par security

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[1] Well-known examples include the Treasury bond and GNMA contracts traded on the Chicago Board of Trade and the Treasury bond contract on the New York Futures Exchange.

and limit deliverable supply in an economic (as opposed to legal) sense to a subset of the set of deliverable issues. Historically, yield maintenance pricing has produced a smaller deliverable supply and a greater distortion of the futures price than has factor pricing. Finally, the issue is raised of whether an ideal difference system is possible, and answered in the negative.

I. Futures Market Equilibrium with a Yield-Based Difference System

To characterize futures market equilibrium with a yield-based difference system, it is convenient to define a present value function $P(c, t, r)$, where c is the coupon rate, t is the time to maturity, and r is the internal rate of return on a security. This function serves to convert price into yield and vice versa. Then

$P(c^f, t^f, r^f)$ = the futures price. c^f and t^f are the coupon rate and time to maturity on the par grade security for the financial futures contract. The futures price is equal to the present value of the payments on the par grade security discounted at r^f, the futures market-determined yield.

$P(c, t, r^s)$ = exogenously given price of the security of grade described by (c, t) in the spot market, where $r^s = r^s(c, t)$ is the spot market discount rate (yield) for a deliverable security of grade (c, t). This specification implies that the cash market price of all deliverable securities can be completely determined by knowledge of c and t. This is broadly true for financial instruments currently traded on U.S. futures exchanges.

$\dfrac{P(c, t, x)}{P(c^f, t^f, x)}$ = difference system conversion factor giving the deliverable price of a security of grade (c, t) relative to the price of the par grade security (c^f, t^f) when the two securities are discounted at the rate (x) specified by the difference system in effect for the futures contract.

With this notation, the spot market purchase of securities of grade (c, t) for immediate delivery against an expiring futures contract results in a profit, π, neglecting transactions costs, of

$$\pi = P(c^f, t^f, r^f)\left[\frac{P(c, t, x)}{P(c^f, t^f, x)}\right] - P(c, t, r^s) \tag{1}$$

Given the futures price, traders will deliver that grade $(\bar{c}, \bar{t})$ of securities which maximizes their profits. Assuming that the deliverable supply includes securities with a continuum of coupons and maturities, this requires $(\bar{c}, \bar{t})$ to satisfy

$$\frac{\partial \pi}{\partial c} = \lambda \tag{2}$$

where $\lambda = 0$ unless a corner solution is reached. If $\bar{c}$ is the highest coupon deliverable security with an optimal maturity, then $\lambda \geq 0$ (i.e., profits might be increased if a yet higher coupon security were deliverable). Similarly, if $\bar{c}$ is the lowest coupon-deliverable security with an optimal maturity, then $\lambda \leq 0$ (i.e. profits might be increased if a yet lower coupon security were deliverable).

$$\frac{\partial \pi}{\partial t} = \gamma \tag{3}$$

where $\gamma = 0$ unless a corner solution is reached. Similarly, in this case $\gamma \geq 0$ (≤ 0) if $\bar{t}$ is the longest (shortest) maturity deliverable security available with an optimal coupon.

In addition to (2) and (3), futures market equilibrium requires that the futures price be such that when the optimal security $(\bar{c}, \bar{t})$ is delivered, no arbitrage profits are possible, i.e.

$$P(c^f, t^f, r^f)\left[\frac{P(\bar{c}, \bar{t}, x)}{P(c^f, t^f, x)}\right] - P(\bar{c}, \bar{t}, r^s) = 0 \tag{4}$$

The simultaneous solution of Equations (2), (3), and (4) determines the equilibrium coupon and maturity security delivered against the futures contract, as well as the equilibrium futures price as functions of spot securities prices and the difference system conversion factor.

From Equations (2), (3), and (4) it follows immediately that if there exists an optimal security to deliver against the futures contract, then the delivery of any suboptimal security must entail losses to the short position. Furthermore, if the par grade security is a suboptimal delivery security, then necessarily the price of the expiring futures contract must be below the spot market price of the par security. For if $(\bar{c}, \bar{t}) \neq (c^f, t^f)$, then

$$P(c^f, t^f, r^f)\left[\frac{P(c^f, t^f, x)}{P(c^f, t^f, x)}\right] - P(c^f, t^f, r^s) < 0$$

or

$$P(c^f, t^f, r^f) < P(c^f, t^f, r^s)$$

This implies that the futures price may display what the British call backwardation, but this backwardation has nothing to do with risk premia and will not disappear at the maturity of the futures contract. It is due to the fact that not all securities may be equally desirable for delivery, and the short futures trader has the option to choose for delivery the security which maximizes his profits.

Thus, the form of the difference system affects not only the extent to which various grades of securities are deliverable in an economic (as opposed to legal) sense, but also the observed price of the future contract. Actual difference systems can thus be compared in terms of their bias toward delivery of certain securities and the extent of the resulting backwardation or difference between the cash price of the par grade security and the futures price at maturity of the contract (called the maturity basis of the futures contract).

II. Difference Systems in Use for Financial Futures—Theoretical Aspects

In U.S. financial futures markets two difference systems exist: yield maintenance pricing and factor pricing.

Yield maintenance pricing, in use for the Chicago Board of Trade (CBT) Certificate Delivery (CD or 'new') GNMA contract, and the New York Futures Exchange (NYFE) Treasury Bond contract, sets x equal to the settlement yield on the expiring futures contract; i.e. $x = r^f$.

With this specification, Equation (4) simplifies to

$$P(\bar{c}, \bar{t}, r^f) = P(\bar{c}, \bar{t}, r^s) \tag{4'}$$

which means that the settlement yield on the futures contract will equal the spot market yield on the optimal delivery security, i.e., $r^f = r^s(\bar{c}, \bar{t})$. The optimal delivery security can be found, as in Section I, from Equations (2) and (3) which in this case are

$$\frac{\partial P(\bar{c}, \bar{t}, r^f)}{\partial c} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial c} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s}\frac{\partial r^s}{\partial c} = \lambda, \quad \text{or using Equation (4')}$$

$$\lambda = -\frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s}\frac{\partial r^s}{\partial c} \tag{2'}$$

$$\frac{\partial P(\bar{c}, \bar{t}, r^f)}{\partial t} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial t} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s}\frac{\partial r^s}{\partial t} = \gamma, \quad \text{or using Equation (4')}$$

$$\gamma = -\frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s}\frac{\partial r^s}{\partial t} \tag{3'}$$

From Equation (2′) it can be seen that if spot market yields are an increasing function of coupon rates $\left(\frac{\partial r^s}{\partial c} > 0\right)$, as is usually the case, then λ will be positive everywhere so that it will be optimal to deliver the highest deliverable coupon security with a given maturity.[2] Similarly, from Equation (3′) the sign of γ can be seen to depend on the slope of the term structure of interest rates: if there is a positively (negatively) sloped term structure; i.e. $\frac{\partial r^s}{\partial t} > (<)\ 0$, then it will be optimal to deliver the longest (shortest) maturity security with a given coupon.

In sum, with yield maintenance pricing it will generally be optimal to deliver the deliverable securities with the highest yield, and the expiring futures yield will equal the spot yield on the optimal delivery securities.

The second difference system in use in U.S. financial futures markets is factor pricing. Factor pricing, in use for the CBT Collateralized Depository Receipt (CDR) delivery GNMA contract and the CBT Treasury bond contract, sets x

[2] A number of considerations affect the size and sign of $\frac{\partial r^s}{\partial c}$: (1) the amortization of security discounts (or premiums) caused by interest rate changes are treated as capital gains (or losses) for tax purposes. Thus, for government securities which are usually issued near par, low-coupon securities usually have a tax advantage over high-coupon securities, and this tax advantage can be expected to show up in relative security yields; (2) possibly acting in the opposite direction is the fact that low-coupon securities are subject to greater interest rate risk than are high coupon securities (see the Appendix for a proof); (3) in the case of GNMA certificates, the mortgagor has the right to refinance or call his mortgage at will. The value of this option obviously varies directly with the coupon rate on the mortgage and so tends to raise relative yields of high coupon certificates.

equal to the coupon rate on the basis grade security, i.e., $x = c^f$.[3] Obviously, if $c^f = r^f = r^s(\bar{c}, \bar{t})$ this difference system is identical to yield maintenance pricing. More generally, with this specification (2), (3) and (4) become

$$\left[\frac{P(c^f, t^f, r^f)}{P(c^f, t^f, c^f)}\right] \frac{\partial P(\bar{c}, \bar{t}, c^f)}{\partial c} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial c} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s} \frac{\partial r^s}{\partial c} = \lambda \tag{2''}$$

$$\left[\frac{P(c^f, t^f, r^f)}{P(c^f, t^f, c^f)}\right] \frac{\partial P(\bar{c}, \bar{t}, c^f)}{\partial t} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial t} - \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s} \frac{\partial r^s}{\partial t} = \gamma \tag{3''}$$

$$\left[\frac{P(c^f, t^f, r^f)}{P(c^f, t^f, c^f)}\right] P(\bar{c}, \bar{t}, c^f) - P(\bar{c}, \bar{t}, r^s) = 0 \tag{4''}$$

Consider first the determination of the optimal delivery coupon for given maturity securities. Define $E_{p\cdot c}(c, t, r) = \dfrac{c}{P(c, t, r)} \dfrac{\partial P(c, t, r)}{\partial c}$ as the elasticity of security price with respect to the coupon rate. Then combining Equations (2″) and (4″) yields

$$E_{p\cdot c}(\bar{c}, \bar{t}, c^f) - E_{p\cdot c}(\bar{c}, \bar{t}, r^s) = \frac{\bar{c}}{P(\bar{c}, \bar{t}, r^s)} \left[\lambda + \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s} \frac{\partial r^s}{\partial c}\right] \tag{5}$$

Now it can be shown (see the Appendix) that for the securities under consideration

$$E_{p\cdot c}(c, t, c^f) \gtrless E_{p\cdot c}(c, t, r^s) \quad \text{as} \quad c^f \gtrless r^s$$

Thus, ignoring $\dfrac{\partial r^s}{\partial c}$, this difference system produces a bias towards the delivery of high (low) coupon securities ($\lambda > (<) 0$) as $c^f > (<) r^s$. Historically, at 8 percent c^f has generally been below spot market rates for deliverable grade securities. Consequently, factor pricing should historically have produced a bias towards the delivery of the lowest coupon security deliverable. But as was pointed out above, $\dfrac{\partial r^s}{\partial c}$ is usually positive. This would tend to counteract or even overpower the bias towards low coupon securities.

A similar analysis applies to the optimal delivery maturity decision under factor pricing. Thus, define $E_{p\cdot t}(c, t, r) = \dfrac{t}{P(c, t, r)} \dfrac{\partial P(c, t, r)}{\partial t}$ as the elasticity of security price with respect to maturity. Then combining equations (3″) and (4″) yields

[3] Actually, for the CDR GNMA contract an adjustment is made to principal rather than to price. For example, if security y has a price of 1.1 relative to the basis grade security under factor pricing, then only \$100,000 ÷ 1.1 principal balance of security y need be delivered, at the quoted futures price, to fulfil the contract. Thus, in practice—and in the empirical section which follows—equations (2″), (3″), and (4″) are multiplied by $\left(\dfrac{P(c^f, t^f, c^f)}{P(\bar{c}, \bar{t}, c^f)}\right)$ for the CDR GNMA contract. This modification affects the scale but not the existence of arbitrage gains or losses for any (c, t) and so (assuming cash-futures convergence) has no effect on the optimal delivery security or on observed futures prices.

$$E_{p\cdot t}(\bar{c}, \bar{t}, c^f) - E_{p\cdot t}(\bar{c}, \bar{t}, r^s) = \frac{\bar{t}}{P(\bar{c}, \bar{t}, r^s)}\left[\gamma + \frac{\partial P(\bar{c}, \bar{t}, r^s)}{\partial r^s}\frac{\partial r^s}{\partial t}\right] \tag{6}$$

Now $E_{p\cdot t}(c, t, r) \gtrless 0$ as $c \gtrless r$. Furthermore, it can be shown that if $c < r$, then $\frac{\partial E_{p\cdot t}}{\partial r} < 0$, so that the elasticity is more negative the higher the discount rate. With market rates above coupon rates this suggests that the left-hand side of Equation (6) is positive so that, with a flat term structure $\left(\frac{\partial r^s}{\partial t} = 0\right)$ factor pricing produces a bias towards the delivery of the longest maturity security available. However, a significant negative term structure could overcome this bias.

Thus, factor pricing, like yield maintenance pricing, creates an optimal subset of securities for delivery against the futures contract. Under both systems, the optimal securities to deliver depend on the term and coupon structures of interest rates—but with factor pricing they also depend on the general level of interest rates relative to the coupon rate on the par security. Therefore, there is greater uncertainty about the identity of the optimal delivery securities under factor pricing than under yield maintenance pricing. The optimal delivery securities have likely had a lower coupon and longer maturity under factor pricing than under yield maintenance pricing.

Finally, factor pricing may be compared with yield maintenance pricing in terms of the equilibrium maturity basis, and hence backwardation, which results from the two systems. The maturity basis under yield maintenance pricing (see Equation (4′) is

$$P(c^f, t^f, r^s(\bar{c}', \bar{t}')) - P(c^f, t^f, r^s, (c^f, t^f)) \leq 0$$

where the prime distinguishes the optimal delivery security here from that under factor pricing. The maturity basis under factor pricing for the same par security (see Equation 4″) is

$$\left[\frac{P(c^f, t^f, c^f)}{P(\bar{c}, \bar{t}, c^f)}\right] P(\bar{c}, \bar{t}, r^s(\bar{c}, \bar{t})) - P(c^f, t^f, r^s(c^f, t^f)) \leq 0$$

Subtracting the former from the latter gives

$$P(\bar{c}, \bar{t}, r^s(\bar{c}, \bar{t}))\left[\frac{P(c^f, t^f, c^f)}{P(\bar{c}, \bar{t}, c^f)} - \frac{P(c^f, t^f, r^s(\bar{c}', \bar{t}'))}{P(\bar{c}, \bar{t}, r^s(\bar{c}, \bar{t}))}\right]$$

which takes the sign of the sum in brackets. Generally it is not possible to sign this sum so there can be no presumption about which system will produce the better estimate of the spot price of the par security. However, in the case of GNMA certificates, which have no well-defined maturities, the situation is clearer. For if the time variable is deleted and it is assumed that $\bar{c} > c^f$, then, since $r^s(\bar{c}', \bar{t}') \geq r^s(\bar{c}, \bar{t})$, the bracket sum must be positive by the theorem stated in the Appendix.[4] In the case of GNMA futures, then, factor pricing should historically have produced a smaller backwardation than yield maintenance pricing.

[4] $r^s(\bar{c}', \bar{t}') \geq r^s(\bar{c}, \bar{t})$ because $(\bar{c}', \bar{t}')$ has already been shown to be the highest yield security deliverable.

III. Difference Systems in Use for Financial Futures—Empirical Aspects

While the above analysis gives qualitative expectations of the differences in equilibrium under factor pricing and yield maintenance pricing, it cannot answer questions of the quantitive significance of these differences. In order to do this, two types of data were assembled. First, data were assembled on spot and futures market prices for expiring futures contracts on settlement days for each contract.[5] Next, the loss or gain resulting from spot-futures arbitrage for each deliverable security was computed. Results were averaged over all expiring contracts in the period 1/1976 to 6/1980 and are presented in Table I.

Column 1 of Table I gives the results for Treasury bonds. This column indicates that a trader would on average have made \$14 per contract by buying the optimal spot security and immediately delivering it against the expiring Treasury bond futures. Choosing the second-best delivery security would have resulted in an average loss of \$163, while the third-best delivery security would have produced an average \$229 loss per contract.[6] The small size of these losses in relation to the \$100,000 principal amount of the contract suggests that factor pricing for Treasury bonds has come quite close to assuring the economic deliverability of at least several grades of securities against the futures contract.[7]

At the same time, Row 5 of Column 1 indicates the average loss from delivering the par grade 8 percent coupon Treasury bond would have been only \$355. Thus the maturity basis of this contract has averaged only $-.355\left(\text{or } -\frac{11}{32}\text{nd}\right)$ point, implying a roughly 4 basis point overstatement of the spot yield on the underlying 8 percent Treasury bond.

[5] For the Treasury bond and CDR GNMA contract, settlement may be any business day of the month in which a contract expires. For the T bond the settlement date was chosen to be the last day of trading of the contract. For the CDR GNMA contract the settlement date was chosen to coincide with that of the CD GNMA contract which is usually the 11th of the month. Prices are as quoted in the *Wall Street Journal.* It should be noted that a deliverable security may not be quoted in the *WSJ* if trading is thin. Thus losses in Table I may be understated since some unquoted security may have been optimal to deliver on occasion. Also, when there are multiple delivery days there is generally an optimal delivery day. Thus, if futures prices reflect optimal delivery days a delivery day chosen at random will likely show losses to the short. Results for the GNMA futures traded on the American Commodity Exchange and the NYFE Treasury bond contract, both subject to yield maintenance pricing, are not reported here because of thin trading in the former contract and a lack of history for the latter contract.

[6] In an actual squeeze, the cash price of an optimal delivery security would be driven up, and along with it the futures price, until the second-best delivery security could be purchased and delivered without economic loss. I thank Bradford Cornell for pointing out that the fact that losses would be incurred by the delivery of a nonoptimal security is evidence that squeezes of the optimal security have not been a feature of these markets. The possible effect of the futures market on the cash market suggests that in some cases cash prices may not be exogenous in this system. In such cases, however, the above analysis still holds as long as cash prices are exogenous from the point of view of the individual trader. The futures price would always reflect the cash price of the marginally optimal delivery security.

[7] On the other hand, it would have been possible to lose a remarkable average \$27,778 per contract by delivering the worst possible deliverable security.

Table I

Dollar Mean and Standard Deviation (in Parentheses) of Arbitrage Loss in Spot Purchase of Securities with Immediate Delivery Against Futures Position

Delivery Security	Treasury Bonds (CBT)	CDR GNMA (CBT)	CD GNMA (CBT)	CDR GNMA[a] (CBT)
Optimal	−14 (294)	114 (322)	8 (1081)	374 (266)
2nd best	163 (174)	672 (595)	1083 (1186)	1192 (600)
3rd best	229 (170)	1278 (813)	1921 (1483)	1958 (770)
Worst	27,778 (4051)	2682 (1165)	3719 (1780)	3592 (1051)
Par grade	355 (119)	1689 (1497)	3347 (1941)	3156 (1348)
Number of observations	11	18	7	7

[a] For the 7 dates of CD GNMA expirations.

Column 2 presents the same analysis for the CDR GNMA contract. Here it appears that factor pricing has been less successful than for Treasury bonds in limiting the basis which could result before alternative security deliveries would occur. For example, the short interest conceivably might be squeezed as much as an average $672 per contract before having recourse to the second-best delivery security. Correspondingly, the maturity basis of this contract (which is stated in terms of an 8 percent coupon GNMA) has been an average of almost minus 1.70 points, implying a roughly 25 basis point average overstatement of actual spot yields on 8 percent coupon GNMAs. Column 3 indicates that the potential losses are even higher for the CD GNMA contract subject to yield maintenance pricing. However, this appears to result more from sample bias than from the effects of yield maintenance pricing. Column 4 gives the results for the CDR GNMA contract over the same 7 sample periods during which the CD GNMA contract existed. Comparison of Columns 4 and 5 suggests that factor pricing and yield maintenance pricing have actually produced fairly similar results over a comparable period. Note that significant average arbitrage losses ($374) do appear for the CDR contract which are not present for the CD contract.[8] But in spite of the apparent underpricing of the CDR futures contract the maturity basis for the CDR contract is less than for the CD contract as predicted.[9]

Finally, Table II gives the percentage of optimal deliveries which were the highest-coupon deliverable security.[10] As predicted with the CD GNMA contract subject to yield maintenance pricing it was apparently always optimal to deliver

[8] This may be partly explained by the existence of unquoted (but more profitable) delivery securities for the CDR GNMA contract. Another factor which plays a very significant role in CDR GNMA futures pricing and deliveries is an allowance for an up to 2½% "overage" on the required principal balance of GNMA certificates which is billed to the long at its face value rather than at its present value. When certificates with significant overage are available, long will demand, and short accept, a lower futures price.

[9] A prominent difference exists in the standard deviations in Columns 4 and 5. This may reflect the relatively thin trading in the CD GNMA contract.

[10] GNMA certificates which have no well-defined actual maturity are conventionally classified only by coupon.

Table II

Percentage of Optimal Deliveries which were Highest-Coupon Deliverable Security

Treasury Bonds	CDR GNMA	CD GNMA
64	78	100

the highest coupon security deliverable, whereas with the factor pricing contracts this was not always the case.

The above data have the advantage of being actual market data. But, for a variety of reasons (many mentioned in previous footnotes) there is sufficient noise in the data to largely obscure the effects of difference systems on financial futures.[11]

One way to isolate the effects of difference systems is to simulate futures prices using the no-arbitrage conditions – Equations (2), (3), and (4) – and actual cash market prices of deliverable securities. This is done below for two hypothetical futures contracts each written on a par 8% 20-year U.S. Treasury bond and each allowing delivery of any U.S. Treasury bond with greater than 15 years to first call, but one using factor pricing and the other using yield maintenance pricing.[12] Delivery on each contract is assumed to be permissible on any business day in March 1981 and the no cash/futures arbitrage condition is imposed each day in this month.[13]

Table III gives results for the factor pricing contract. Column 1 gives the actual cash price of the bond which is closest to the par grade (the 8's of 2001). Column 2 gives the equilibrium futures price which satisfies Equations (2), (3), and (4). The optimal delivery bond is identified in Column 3. Column 4 gives the price to which the futures contract would have to rise before the second-best bond would be deliverable without arbitrage loss. The difference between Columns 2 and 4, given in Column 5, is the extent to which the short could be squeezed by a distortion in the optimal bond's price. The average maximum squeeze of $88 per $100,000 contract is quite low and is closely related to the fact that a variety of bond issues became optimal to deliver over the course of this month, frequently as a result of minor price changes. In addition, the extent to which the simulated futures price understates the cash price of the par security (Column 6) is quite low for this contract, averaging only .80 points.

Table IV gives results for the yield maintenance pricing contracts in the same format as Table III. In contrast to the factor pricing contract, the yield maintenance pricing contract displays a strong tracking of a single issue of bonds. Accordingly, the consequences of a corner of this security in the cash market in

[11] Additional reasons include the fact that reported cash prices are not synchronous with reported futures prices and that the averages reported span periods of widely varying market conditions.

[12] In the case of factor pricing, these contract specifications conform exactly to the bond contract on the Chicago Board of Trade. In the case of yield maintenance pricing, they differ only moderately from the bond contract on the New York Futures Exchange.

[13] In general, with delivery on any of a number of days at seller's option, this arbitrage need not hold on all delivery days, but this assumption is useful for the purpose at hand.

Table III

Selected Computations Relating to Factor Pricing
Par Bond = 8% Coupon, 20-Year Maturity

	(1) Cash Price of 8's of 2001	(2) Futures Price	(3) Optimal Delivery Issue: coupon	(3) Optimal Delivery Issue: maturity	(4) Futures Price Reflecting Second-best Delivery	(5) Col(2) − Col(4)	(6) Col(1) − Col(2)
2 March 1981	64.13	63.56	7⅞'s	2007	63.67	−.11	.57
3	65.25	64.78	8¼'s	2005	64.82	−.04	.47
4	65.16	64.56	10's	2010	64.56	.00	.60
5	65.31	64.84	8¼'s	2005	64.91	−.07	.47
6	66.94	66.26	8⅜'s	2008	66.39	−.13	.68
9	67.56	66.86	9⅛'s	2009	66.92	−.06	.70
10	67.22	66.56	9⅛'s	2009	66.63	−.07	.66
11	67.13	66.32	10's	2010	66.33	−.01	.81
12	68.00	68.89	9⅛'s	2009	66.90	−.01	1.11
13	68.94	67.68	10's	2010	67.86	−.18	1.26
16	69.19	68.14	8⅛'s	2008	68.17	−.03	1.05
17	69.63	68.46	9⅛'s	2009	68.53	−.07	1.17
18	70.56	69.53	10⅜'s	2009	69.54	−.01	1.03
19	69.84	68.88	9⅛'s	2009	68.93	−.05	.96
20	69.56	68.40	9⅛'s	2009	68.43	−.03	1.16
23	66.75	66.15	7⅞'s	2007	66.35	−.20	.60
24	66.13	65.65	10's	2010	65.68	−.03	.48
25	67.13	65.96	10's	2010	66.08	−.12	1.17
26	65.91	65.27	9⅛'s	2009	65.41	−.14	.64
27	65.63	65.12	8⅜'s	2008	65.19	−.07	.51
30	66.19	65.49	9⅛'s	2009	65.66	−.17	.70
31	67.13	66.37	7⅞'s	2007	66.68	−.31	.76
						Average = −.088(−$88)	Average = .80

Data Source = DRI, Inc.

Table IV

Selected Computations Relating to Yield Maintenance Pricing
Par Bond = 8% Coupon, 20-Year Maturity

	(1)	(2)	(3)		(4)	(5)	(6)
	Case Price of 8's of 2001	Futures Price	Optimal Delivery Issue		Futures Price Reflecting Second-best Delivery	Col(2) − Col(4)	Col(1) − Col(2)
			coupon	maturity			
2 March 1981	64.13	62.97	$11\frac{3}{4}$'s	2001	63.77	−.80	2.16
3	65.25	63.72	$11\frac{3}{4}$'s	2001	64.71	−.99	1.53
4	65.16	63.54	$11\frac{3}{4}$'s	2001	64.54	−1.00	1.62
5	65.31	63.65	$11\frac{3}{4}$'s	2001	64.71	−1.06	1.66
6	66.94	64.86	$11\frac{3}{4}$'s	2001	66.04	−1.18	2.08
9	67.56	65.30	$11\frac{3}{4}$'s	2001	66.54	−1.24	2.26
10	67.22	65.05	$11\frac{3}{4}$'s	2001	66.29	−1.49	2.17
11	67.13	65.00	$11\frac{3}{4}$'s	2001	66.04	−1.04	2.13
12	68.00	65.42	$11\frac{3}{4}$'s	2001	66.83	−1.41	2.58
13	68.94	66.33	$11\frac{3}{4}$'s	2001	67.70	−1.37	2.61
16	69.19	66.70	$11\frac{3}{4}$'s	2001	67.79	−1.09	2.49
17	69.63	67.06	$11\frac{3}{4}$'s	2001	68.64	−1.58	2.57
18	70.56	67.75	$11\frac{3}{4}$'s	2001	69.12	−1.37	2.81
19	69.84	67.27	$11\frac{3}{4}$'s	2001	68.49	−1.22	2.57
20	69.56	66.75	$11\frac{3}{4}$'s	2001	68.16	−1.51	2.81
23	66.75	64.95	$11\frac{3}{4}$'s	2001	66.21	−1.26	1.80
24	66.13	64.37	$11\frac{3}{4}$'s	2001	65.54	−1.17	1.76
25	67.13	64.53	$11\frac{3}{4}$'s	2001	65.92	−1.39	2.60
26	65.91	63.97	$11\frac{3}{4}$'s	2001	65.16	−1.19	1.94
27	65.63	63.97	$11\frac{3}{4}$'s	2001	65.08	−1.11	1.66
30	66.19	64.68	$11\frac{3}{4}$'s	2001	65.75	−1.07	1.51
31	67.13	65.29	$11\frac{3}{4}$'s	2001	66.50	−1.21	1.84
						Avg. = −1.21(−$1210)	Avg. = 2.14

Data Source = DRI, Inc.

this month would have been an average loss to the futures short in excess of \$1,200 (Column 5). And the extend to which this simulated futures price understates the cash price of the par grade security is also a relatively large 2.14 points (Column 6).

In sum, under cash market conditions that existed in early 1981 and an 8% par bond, factor pricing would have produced a much larger economically deliverable supply than yield maintenance pricing. Factor pricing would have produced futures prices which were less subject to cash price distortions and more accurate reflectors of the cash price of the par security than would have yield maintenance pricing.

It is interesting to recall from the theoretical section that factor pricing and yield maintenance pricing are identical if the par grade security has a coupon equal to the highest cash market yield of the deliverable securities. The superior performance of factor pricing here is wholly a consequence of the par grade security having a below current market rate coupon. Thus, the efficient functioning of some of the largest financial futures markets may be to some extent the consequence of the historical accident of rising market interest rates.

IV. On the Existence of an Ideal Difference System

The above findings make clear that under both factor pricing and yield maintenance pricing the actual delivery of a variety of securities against a financial futures contract would generally involve economic loss. Thus, though strictly speaking they permit the delivery of a variety of securities, these difference systems are perhaps best described as bounding the short interest loss which might result from spot market aberrations in the optimal delivery securities. Moreover, the fact that the optimal delivery securities may change with changing market conditions imparts risk to these contracts, and the backwardation in both systems obscures the interest rate information conveyed by futures prices. The question arises as to whether some difference system exists which avoids these shortcomings.

Referring to Section I it can be seen that this would require $r^f = r^s(c^f, t^f)$ and Equations (2) and (3) to be identically equal to zero for any deliverable (c, t). But the satisfaction of these conditions for an arbitrary function $r^s(c, t)$ requires that

$$x = x(c, t) = r^s(c, t) \tag{7}$$

That is, that deliverable relative prices equal spot market relative prices.

Equation (7) is a necessary but not sufficient condition for what might be termed an ideal difference system. Sufficiency would require as well that spot market relative prices be in some sense "normal". In particular, it must not be possible for some short interest to "corner" the spot market in one grade of security, run up its price and deliver the security at a high marginal spot valuation against the futures contract.[14] But if spot market prices were always "normal",

[14] This point is apparently not well-understood in some circles. Federal law requires that just such a system, called commercial differencing, be used for cotton futures contracts. It is interesting to note that there reportedly have been large swings in the relative spot prices of various grades of deliverable cotton (see Hoffman [2]).

there would be no advantages to a difference system in the first place. Thus, it would appear that the only conditions under which a difference system could be ideal are those for which a difference is not needed.

V. Conclusion

In this paper the two difference systems in use for financial futures contracts have been discussed on a theoretical and empirical basis. Both systems produce optimal, though not constant, delivery securities and both systems produce backwardation. It was shown that optimal delivery securities may be lower-coupon longer-maturity securities under factor pricing than under yield maintenance pricing and that the backwardation is normally greater for the latter than the former system—at least for GNMA futures. It was also shown that no difference system can completely eliminate these biases and still serve as a safeguard against market failures. This does not mean that the present difference systems cannot be improved, but only that unavoidable tradeoffs in desirable features must be made in their design.

APPENDIX

All securities considered here pay a coupon annuity, $A = A(c, T)$, and a final balloon payment, $B = B(c, t, T)$ where c is the stated interest rate on the security, T is the stated maturity of the security, and t is the actual time to maturity of the security.

Let $F(r, t) = \int_o^t e^{-rs}ds = \frac{1}{r}(1 - e^{rt})$ be the present value annuity factor and let $D(r, t) = e^{-rt}$ be the present value discount factor for a single future payment, where r is the discount rate. Then the present value of a security, P, is

$$P(c, r, T, t) = AF + BD \tag{A.1}$$

As defined in the text, let $E_{p.c}$ be the elasticity of present value with respect to the coupon rate. Thus,

$$E_{p.c} = \frac{P^*}{c^*} = x\frac{A^*}{c^*} + (1 - x)\frac{B^*}{c^*} \tag{A.2}$$

where $x = \frac{AF}{P}$ and where an asterisk indicates the logarithmic derivative of the variable it superscripts.[15]

This appendix proves that for this type security,

$$E_{p.c}\,|\,r_1 \gtreqless E_{p.c}\,|\,r_2 \quad \text{as } r_1 \gtreqless r_2 \tag{A.3}$$

i.e. that the price of high coupon securities will rise relative to the price of low coupon securities as higher discount rates are applied.

[15] See [1] for an algebra of logarithmic derivatives.

As a corollary it is proved that

$$|E_{p.r}|\,|c_1 \gtrless |E_{p.r}|\,|c_2 \quad \text{as } c_1 \lesseqgtr c_2, \tag{A.4}$$

i.e. that low coupon securities are more volatile with respect to interest rate changes than are high coupon securities. To prove (A.3) is sufficient to show that $\frac{(E_{p.c})^*}{r^*} > 0$.

Define $v = \frac{x}{E_{p.c}} \frac{A^*}{c^*}$. Then, from Equation (A.2) it follows that

$$\begin{aligned}
\frac{(E_{p.c})^*}{r^*} &= v\left[(1-x)\left(\frac{F^*}{r^*} - \frac{D^*}{r^*}\right)\right] - (1-v)\left[x\left(\frac{F^*}{r^*} - \frac{D^*}{r^*}\right)\right] \\
&= (v - x)\left(\frac{F^*}{r^*} - \frac{D^*}{r^*}\right) \\
&= \frac{x(1-x)}{E_{p.c}}\left(\frac{F^*}{r^*} - \frac{D^*}{r^*}\right)\left(\frac{A^*}{c^*} - \frac{B^*}{c^*}\right)
\end{aligned} \tag{A.5}$$

Now, from the definitions of F and D it can be seen that $\left(\frac{F^*}{r^*} - \frac{D^*}{r^*}\right) = \left[\left(\frac{rte^{-rt}}{1-e^{-rt}} - 1\right) - (-rt)\right] > 0$ for $rt > 0$. Therefore, the sign of (A.5) depends on the sign of $\left(\frac{A^*}{c^*} - \frac{B^*}{c^*}\right)$.

In the case of Treasury bonds $A = c$ and B is a fixed principal repayment so that $\frac{A^*}{c^*} - \frac{B^*}{c^*} = 1$ and Equation (A.5) is clearly positive.

In the case of GNMA certificates, if the underlying mortgages are not prepaid then $B = 0 \Rightarrow x = 1$ so that Equation (A.5) equals zero. However, mortgages are usually prepaid:

GNMA certificates are in fact quoted on the basis of a thirty-year mortgage prepaid in the twelfth year. Under these circumstances, the annuity, A, is given by $A = \frac{c}{1-e^{-cT}}$, and the balloon prepayment is expected to be $B = \frac{1-e^{-ct}}{1-e^{-cT}}$, where t is the expected term to prepayment and T is the stated maturity of the underlying mortgages. Therefore, $1 < \frac{A^*}{c^*} = 1 + \frac{cTe^{-cT}}{1-e^{-cT}} < 2$ for $cT > 0$, and $1 > \frac{B^*}{c^*} = \frac{cte^{-ct}}{1-e^{-ct}} - \frac{cTe^{-cT}}{1-e^{-cT}} > 0$ for $t < T$. Thus in this case as well $\left(\frac{A^*}{c^*} - \frac{B^*}{c^*}\right) > 0$ so that Equation (A.5) is positive. Q.E.D.

As a corollary, it follows immediately that $\frac{(E_{p.r})^*}{c^*} = \frac{x(1-x)}{E_{p.r}}\left(\frac{F^*}{r^*} - \frac{D^*}{r^*}\right)\left(\frac{A^*}{c^*} - \frac{B^*}{c^*}\right) < 0$, since $E_{p.r} < 0$. This establishes Equation (A.4).

REFERENCES

1. R.G.D. Allen. *Mathematical Analysis for Economists.* New York: St. Martin's Press, 1938, pp. 246–8.
2. G. Wright Hoffman. *Futures Trading Upon Organized Commodity Markets in the United States.* Philadelphia: University of Pennsylvania Press, 1932, pp. 277–306.