THE JOURNAL OF FINANCE • VOL. XXXVII, NO 5 • DECEMBER 1982

The Journal of FINANCE

VOL. XXXVII | DECEMBER 1982 | NO. 5

The Arbitrage Pricing Theory: Is it Testable?

JAY SHANKEN*

ABSTRACT

This paper challenges the view that the Arbitrage Pricing Theory (APT) is inherently more susceptible to empirical verification than the Capital Asset Pricing Model (CAPM). The usual formulation of the testable implications of the APT is shown to be inadequate, as it precludes the very expected return differentials which the theory attempts to explain. A recent competitive-equilibrium extension of the APT may be testable in principle. In order to implement such a test, however, observation of the return on the true market portfolio appears to be necessary.

THE CAPITAL ASSET PRICING Model (CAPM) has, for many years, been the major framework for analyzing the cross-sectional variation in expected asset returns. The main implication of the theory is that expected return should be linearly related to an asset's covariance with the return on the market portfolio:

$$E_i = \gamma_0 + \gamma_1 \beta_i$$

where

$$\beta_i = \sigma_{im}/\sigma_m^2 \tag{1}$$

is the "beta coefficient" of asset i, E_i its expected return, and γ_0 and γ_1 are constants that do not depend on i.

This simple relation has been the focus of intensive empirical scrutiny for more than a decade. Roll [16], in an influential article, suggests that the CAPM is testable in principle, but he argues that "(a) No correct and unambiguous test has appeared in the literature, and (b) There is practically no possibility that such a test can be accomplished in the future." These conclusions are a consequence of our inability to observe the exact composition of the true market portfolio.

The Arbitrage Pricing Theory (APT) of Ross [18, 19] has been proposed as a testable alternative, and perhaps the natural successor to the CAPM (Ross [21], p. 894). An important intuition in modern portfolio theory is that it is the

* University of California at Berkeley. Thanks to David Babbel, Michael Brennan, Greg Connor, Ken Dunn, Mark Rubinstein, Jeff Skelton, Sheridan Titman, participants in seminars at Berkeley and Stanford and especially Jim Ohlson and Rex Thompson for their helpful comments.

covariability of an asset's return with the return on other assets, rather than its total variability, that is important from the perspective of a risk averse investor who holds a well-diversified portfolio of many assets. Ross's seminal contribution was his insight that this intuition can be transformed into a theory of asset pricing with implications similar to (1).

Whereas derivation of the CAPM requires very specific technical assumptions (quadratic utility or multivariate normality of returns, for example), Ross's theory exploits the concept of a "large" (many assets) security market, consistent with the intuition described above. The market portfolio plays no special role in this theory. Rather, it is the covariability of an asset's return with those random factors which systematically influence the returns on most assets, that is reflected in the expected return relation. This ability of the APT to accommodate several sources of "systematic risk" has been considered by many an advantage in comparison with the CAPM.

Brennan [1] has described the APT as "a minimalist model of security market equilibrium" that is "logically prior to our other utility based models, and should be tested before the predictions of stronger utility specifications are considered." The body of empirical literature concerned with testing the APT is growing at a rapid rate. In addition to the early work of Gehr [8] are studies by Chen [5], Roll and Ross [17], Oldfield and Rogalski [14], P. Brennan [2], Gibbons [9], Reinganum [15], and Brown and Weinstein [3]. The goal of this paper is to provide a critical perspective on this important area of empirical research. Our concern is not with the particular experimental designs and statistical methods used. We address the more fundamental question of what it means to test the APT. The arguments presented below challenge the view that the APT is inherently more susceptible to empirical verification than the CAPM.

The paper is organized as follows. Section I provides an overview of the Ross APT. Section II discusses the inadequacy of the usual formulation of the testable implications of the theory. Section III considers the interpretation of empirical investigations of the APT. Section IV summarizes the main conclusions. Technical arguments have been placed in appendices.

I. An Overview of the Ross APT

The APT assumes returns conform to a K-factor linear model ($K < N$):

$$R_i = E_i + \beta_{i1}\delta_1 + \cdots + \beta_{iK}\delta_K + \epsilon_i \qquad i = 1, N$$

R_i is the random return on asset i, and E_i its expected return. The δ_k are mean zero common factors and the ϵ_i are mean zero asset specific disturbances assumed to be uncorrelated with the δ_k and with each other. In the language of factor analysis, the β_{ik} are the factor loadings. N is the number of assets under consideration. In matrix notation,

$$R = E + B\delta + \epsilon \tag{2}$$

where R, E, and ϵ are $N \times 1$, B is $N \times K$, and δ is $K \times 1$. Let D be the diagonal covariance matrix of ϵ.

A decomposition as in (2) will hold whenever returns are regressed on an arbitrary set of random variables measured as deviations from the mean. In general, however, the ϵ_i will be correlated, thereby violating the factor model definition. An example is the usual "market model" in which δ is the return on a market proxy (see Fama [7], Chapter 3). There is considerable empirical evidence documenting correlation between market model disturbances (see King [12]). Sometimes it is convenient to treat the disturbances as if they were uncorrelated, however. This is often referred to as the Sharpe single index model. Only in this case does the market model constitute a factor model ($K = 1$) in the strict sense used here.

A special case that conveys the basic idea behind the APT, but is too restrictive to be of practical interest, occurs when $\epsilon \equiv 0$, i.e., there are no asset-specific disturbances. In this case, absence of riskless arbitrage implies the existence of a constant γ_0 and a K-vector γ_1 such that

$$E = \gamma_0 1_N + B\gamma_1 \tag{3}$$

Ross's argument is as follows. Consider an arbitrage portfolio with no systematic risk; i.e., an N-vector X such that

$$X'1_N = 0 \quad \text{and} \quad X'B = 0$$

Assuming $\epsilon \equiv 0$ in (2),

$$X'R = X'E$$

Since the portfolio requires no net investment and is riskless we must have, in the absence of arbitrage,

$$X'E = 0$$

In the language of linear algebra, *any* vector orthogonal to 1_N and the columns of B is orthogonal to E. It follows that E must be a linear combination of 1_N and the columns of B, as stated in (3).

When asset specific disturbances are introduced, the situation is complicated considerably. In this case, zero investment and zero systematic risk imply

$$X'R = X'E + X'\epsilon$$

If N is "large" and the arbitrage portfolio is well diversified, then laws of large numbers suggest that the asset specific risk will be *approximately* diversified away so that

$$X'R \approx X'E \quad \text{and hence} \quad X'E \approx 0$$

Even if we overlook the approximation, there is a technical problem. We have considered a well diversified X, while the linear algebra leading to (3) requires that $X'E = 0$ for *any* X orthogonal to 1_N and B.

Ross [18] still manages to prove a result in the spirit of (3) for the general model with asset specific disturbances. The result is considerably weaker than (3), however. Specifically, as the number of assets under consideration approaches infinity, the sum of squared deviations from (3) converges; i.e., there exist γ_0 and γ_1 such that

$$\textstyle\sum_{i=1}^{\infty}[E_i - \gamma_0 - \beta_i\gamma_1]^2 < \infty \tag{4}$$

where β_i is the i^{th} row of B. In order that (4) hold, "most" of the deviations from linearity must be "small," although any particular deviation may be "large."[1]

A test of the APT must, of course, be implemented with a finite set of data. Since any finite sum of squared deviations is clearly finite, (4) is not an empirically testable condition. We should like to know, therefore, whether any empirically testable bound on the deviations is implied by the theory. The arguments in Appendix A suggest that this is not the case. What, then, have empirical investigations of the APT actually tested? This is considered in the next section.

II. The Usual Empirical Formulation

Empirical investigations of the APT have attempted to test the following proposition:

> *If* a set of asset returns conforms to a K-factor model, *then* the expected return vector is equal to a linear combination of a unit vector and the factor loading vectors (5)

i.e., If (2) then (3).

We shall refer to (5) as the empirical formulation of the APT. Given the discussion of Section I, we know that *(5) is not literally an implication of the APT*. Nonetheless, it might be viewed as a reasonable representation of the *intuitive* content of the theory. Its rejection could not be equated with rejection of the theory. Its acceptance in an empirical test would be consistent with the theory, however, and might (power considerations aside) be interpreted as evidence in favor of the theory. A theory that cannot be rejected is not necessarily preferable to the CAPM, though.

Proponents of the APT have emphasized that, in contrast to the CAPM, the APT may be tested by merely observing subsets of the set of all returns (Roll and Ross [17], p. 1080).[2] Provided that observable returns conform to a factor model, the matrix of factor loadings can be estimated by the statistical technique of factor analysis (see Morrison [13], Chapter 9). The number of factors must be known in advance, though they need not be observable. There is an issue of uniqueness, however. If A is any $K \times K$ nonsingular matrix, then B and δ in (2) may be replaced by BA and $A^{-1}\delta$. The factor model definition is still satisfied, but with factors $A^{-1}\delta$ and loading matrix BA. As Roll and Ross ([17], p. 1084) note, this is of no concern from the perspective of the APT. The empirical formulation of the APT in (5) is a statement about the relation between the expected return vector E and the space spanned by the loading vectors and a unit vector.[3] Since that space is unaltered when B is replaced by BA, there is really no

[1] Formally: for every $\varepsilon > 0$, there exists an integer M such that for all $i > M$, $|E_i - \gamma_0 - \beta_i\gamma_1| < \varepsilon$.

[2] As Roll and Ross note, "the APT yields a statement of relative pricing on subsets of the universe of assets." This contrasts with the CAPM which is a preference based equilibrium model, not an arbitrage model. See Huberman [10] for a clarification of the no-arbitrage condition underlying the APT.

[3] The space spanned by a given set of vectors is the set of all vectors which are linear combinations of those given vectors.

problem. The particular factor analytic estimation technique used simply chooses some basis for that space.

In light of the difficulties in measuring the true market portfolio, a theory which permits estimation of the appropriate risk measures without observation of the corresponding "factors" is certainly appealing. Some might argue that this apparent immunity from measurement problems more than compensates for the ambiguity surrounding the approximate nature of the risk-return relation. Another source of ambiguity should be considered, however.

Let us say that two sets of securities are *equivalent* if the corresponding sets of obtainable portfolio returns are identical. In this case, the two sets of securities are merely different packagings of the same underlying returns. Given perfect markets with no transaction costs, investors would be indifferent between equivalent sets. This simple idea plays an important role in many applications of arbitrage theory (for example, the Modigliani-Miller theory of corporate capital structure, and the theory of option pricing). The return on the equity of a firm may be viewed as the return on a portfolio whose components correspond to the underlying assets (long positions) and liabilities (short positions) of the firm. Alternative packagings of the underlying returns may, of course, be obtained by forming portfolios of stocks. The empirical formulation of the APT in (5) does not discriminate between different packagings. It would seem natural, therefore, to inquire as to the relation between the factors in the respective factor models for two equivalent sets of securities.

If, intuitively, we identify factors with the pervasive forces in the economy, then we might expect the same set of factors to be obtained from equivalent sets of securities. This is not the case, however. The basic idea may be illustrated with a simple example. Consider two securities which conform to the following 1-factor model:

$$R_1 = E_1 + \delta + \epsilon_1$$
$$R_2 = E_2 - \delta + \epsilon_2 \tag{6}$$

where

$$\text{var}(\delta) = 1 \quad \text{and} \quad \text{var}(\epsilon_1) = \text{var}(\epsilon_2) = \sigma^2 > 0$$

Let $R_1^* = R_1$ and $R_2^* = \alpha R_1 + (1 - \alpha) R_2$. Thus R_2^* is the return on a portfolio of the initial securities. R_2^* may be written as

$$R_2^* = [\alpha E_1 + (1 - \alpha) E_2] + (2\alpha - 1)\delta + [\alpha\epsilon_1 + (1 - \alpha)\epsilon_2]$$

Now $\text{cov}[\epsilon_1, \alpha\epsilon_1 + (1 - \alpha)\epsilon_2] = \alpha\sigma^2$. Unless $\alpha = 0$, R_1^* and R_2^* will not conform to a 1-factor model with factor δ. The disturbance term for R_2^*, relative to δ, is a mixture of ϵ_1 and ϵ_2, and is not uncorrelated with ϵ_1.

Consider the covariance between R_1^* and R_2^*:

$$\begin{aligned} \text{cov}(R_1^*, R_2^*) &= \alpha\text{var}(R_1) + (1 - \alpha)\text{cov}(R_1, R_2) \\ &= \alpha(1 + \sigma^2) + (1 - \alpha)(-1) \end{aligned}$$

Let $\alpha = 1/(2 + \sigma^2)$, so that $\text{cov}(R_1^*, R_2^*) = 0$. Since $\alpha \neq 0$, R_1^* and R_2^* violate the

1-factor model. But any set of uncorrelated returns conforms to the simplest possible factor model: a 0-factor model. To see this, write R_1^* and R_2^* as

$$R_1^* = E_1^* + \epsilon_1^*$$
$$R_2^* = E_2^* + \epsilon_2^* \tag{7}$$

where $E_i^* = E(R_i^*)$ and $\epsilon_i^* = R_i^* - E_i^*$, $i = 1, 2$. By the choice of α, $\text{cov}(\epsilon_1^*, \epsilon_2^*) = 0$. Therefore (7) is a legitimate factor model with $K = 0$. It is easily verified that $\{R_1, R_2\}$ and $\{R_1^*, R_2^*\}$ are equivalent sets.[4]

It has been shown that equivalent sets of securities need not conform to the same factor model. In particular, the number of factors in the respective models need not be the same. Therefore, this is not an instance of the phenomenon described earlier, which involved an arbitrary invertible transformation of one set of factors into another basis for the same factor space.[5] Whereas that phenomenon poses no problem, the present consideration does. The empirical formulation of the APT in (5), together with (6), implies the existence of γ_0 and γ_1 such that[6]

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \gamma_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \gamma_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8}$$

On the other hand, (5) and (7) imply the existence of γ_0^* such that

$$\begin{bmatrix} E_1^* \\ E_2^* \end{bmatrix} = \gamma_0^* \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{9}$$

But equivalence means that R_1 and R_2 are equal to portfolios of R_1^* and R_2^*, so that $E_1 = E_2 = \gamma_0^*$ as well. (8) and (9) will not be consistent unless $\gamma_1 = 0$ and $\gamma_0 = \gamma_0^*$.

Let us summarize the observations above. First, equivalent sets of securities may conform to very different factor structures. Second, the usual empirical formulation of the APT, when applied to these structures, may yield different and inconsistent implications concerning expected returns for a given set of securities. The implications will be consistent if and only if all of the securities have the same expected return. While the example above considered only two securities, the conclusions apply, aside from a few mild technical restrictions, to any *finite* set of securities (see Appendix B).[7]

In light of these observations, (5) cannot be considered an adequate formulation of the empirical content of a testable theory of asset pricing. It rules out the very expected return differentials which the theory seeks to explain.[8] We have already noted that the exact risk-return relation of (5) is not literally an implication of the Ross APT. The positing of such a relation might, therefore, be considered the main source of difficulty. But our observations concerning the factor models of

[4] R_2 may be recovered by shorting $\alpha/(1 - \alpha)$ units of R_1^* and buying $1/(1 - \alpha)$ units of R_2^*.

[5] The factor space is the set of random variables which are linear combinations of the given factors.

[6] The case $N = 2$ is clearly without content and is considered for the purpose of illustration only. The same conclusions hold for any finite N. See Appendix B.

[7] Note that the CAPM suggests no particular relation between the expected returns of uncorrelated securities, since their covariances with the market may vary.

[8] Ingersoll [11] has made a similar assertion independently.

equivalent sets of securities are disturbing (and revealing) quite apart from this issue.

The phenomenon is actually more general than the previous discussion might suggest. The following proposition is proved in Appendix B: given a vector of returns R and (almost) any other vector of random variables δ, there exists an equivalent set of securities with return vector R^*, which conforms to a factor model with factors δ. Given this undesirable degree of flexibility, how are we to identify the "true" factors? Indeed, does such a phrase have a well-defined meaning? The securities we observe in the market constitute a particular packaging of the underlying returns in the economy. Are we to assume that the "relevant" factor model is the one which corresponds to this particular packaging? These issues are addressed in the following section.

III. Interpreting Empirical Studies of the APT

Given a vector of security returns, suppose that, using the best available statistical methods, we are unable to reject the expected return relation (3). Should this be interpreted as evidence in support of the APT? The following discussion suggests that such an interpretation may be inappropriate. Let δ be the return on a mean-variance efficient portfolio of securities, and let R be a vector of returns on a proper subset of the securities which enter δ.[9] The proposition of Appendix B implies the existence of an equivalent vector of returns R^*, which conforms to a one-factor model with δ as the factor.

Suppose the vector of returns used in an empirical test happens to be R^*. Note that the factor loadings on δ are just the usual beta coefficients with respect to δ. Since δ is mean-variance efficient, the expected return relation (3) must hold exactly (see Fama [7], Roll [16], or Ross [20]). An empirical test of the APT based on R^* necessarily will appear to support the theory (given a large enough sample of data). It would generally be wrong to attribute any *economic* significance to such a result, however, since the validity of (3), in this case, is simply a *mathematical* consequence of the mean-variance efficiency of δ.

The scenario described above might seem a bit improbable. It is intended more to illustrate what is possible than what is likely. What does seem plausible is that some of the factors in a given factor model representation of returns might be highly correlated with the return on a mean-variance efficient portfolio. In that case, it would not be surprising to find some of those factors "priced" in an APT empirical investigation. Such an empirical result would be of questionable economic significance, however.

These remarks are very close in spirit to the "Roll Critique" of tests of the CAPM. Roll argues that empirical investigations of the CAPM which use proxies for the true market portfolio are really tests of the mean-variance efficiency of those proxies, not tests of the CAPM. The CAPM implies that a particular portfolio, the market portfolio, is efficient. The theory is not testable unless *that* portfolio is observable and used in the tests.

Similarly, it is argued here that factor-analytic empirical investigations of the

[9] If δ consisted solely of securities in R, then Σ (see Appendix B) would be singular.

APT are not necessarily tests of that theory. In the case of the APT, we are confronted with the task of identifying the relevant factor structure, rather than the true market portfolio.[10] Whereas we have a reasonably clear notion of what is meant by "the true market portfolio," it is not clear in what sense, if any, a uniquely "relevant factor structure" exists. We noted in Section II that there are, in general, many factor structures corresponding to equivalent sets of securities. The APT does not appear to provide a criterion for singling out one structure as the "relevant" one.

The recent work of Connor [6] on "Asset Pricing in Factor Economies" is pertinent in this regard. Building on the earlier work of Ross, Connor obtains an exact pricing relation by introducing assumptions about the aggregate structure of the economy. As he notes, his theory relies on principles of competitive equilibrium rather than on an arbitrage technique. Significant, from our present perspective, is the central role played by the market portfolio. A crucial condition in the Connor "equilibrium APT" is that *idiosyncratic risk, defined relative to a given factor structure, is completely diversified away in the market portfolio.* It is important to appreciate the relative nature of this condition. The same economy might satisfy the condition with respect to one factor representation of returns and fail to satisfy it with respect to an alternative representation.

This diversification condition provides us with a basis for evaluating and interpreting factor-analytic investigations of the APT. We argued earlier against attaching much economic significance to factor-analytic results. Suppose, however, it can be shown that the factors (implicitly) identified in a factor analysis explain all of the variation in the return on the market portfolio.[11] In this case, it might be appropriate to interpret the results of the investigation as reflecting on the validity of the "equilibrium APT."[12] Without observing the return on the true market portfolio, however, it is unlikely that the diversification condition can ever be conclusively verified in practice. Thus the "equilibrium APT" appears to be subject to substantially the same difficulties encountered in testing the CAPM.

Additional insight into the relation between Connor's work and the inadequacy of the usual empirical formulation of the APT may be obtained by reconsidering the case of the 0-factor model, i.e., a set of mutually uncorrelated returns. If there are no "systematic factors," then all risk is, by definition, idiosyncratic. The condition that idiosyncratic risk be diversified away in the market portfolio requires, in this case, that the variance of the market return be zero. Given the substantial variation in all commonly observed market proxies, we can reject this condition with some confidence. Thus, from the perspective of the "equilibrium

[10] An alternative would be to abandon the notion of a "relevant" factor structure, and view the APT as having implications for approximate asset pricing relative to any set of "factors." The implied degree of approximation would presumably differ for different sets of factors. Appendix A notes some problems with the view, but is, by no means, conclusive.

[11] See Shanken and Tajirian [22] for some empirical evidence related to this condition.

[12] Connor employs a more general concept of "factor structure" than that used here and in the empirical APT literature. Since he does not require that the factor model disturbances be uncorrelated, the empirical implications of his work are not limited to factors obtained (implicitly) by factor analysis.

APT," the existence of a 0-factor representation of returns (appropriately) fails to take on any economic significance.

IV. Summary and Conclusions

It is generally accepted that the Capital Asset Pricing Model (CAPM) is not truly testable in a strict sense. Much of this acceptance can be attributed to the persuasive analysis of Roll, who argues that the CAPM is not testable unless the market portfolio of all assets is used in the empirical test. The Arbitrage Pricing Theory (APT) of Ross has been proposed as a testable alternative to the CAPM. Its proponents suggest that it suffices to merely consider subsets of the universe of existing assets to test the APT. The rapidly growing volume of empirical analysis purporting to test the theory indicates that this view has achieved a significant level of acceptance in the finance research community. Our previous observations suggest that this acceptance may not be warranted.

Ross's theory does not (even in the limit as the number of assets $\rightarrow \infty$) imply an exact linear risk-return relation. The testability of the theory could reasonably be questioned on this ground alone. Perhaps of greater concern is the inadequacy of the usual empirical formulation of the intuitive content of the theory. This formulation states that if a (large) set of asset returns conforms to a factor model, then the expected return vector should be equal to a linear combination of the loading vectors and a unit vector. This proposition is appealing in that it appears to capture the spirit of the theory, and is susceptible to statistical testing via factor analytic methods. But taken literally, it is actually equivalent to the proposition that all securities have the same expected return.

This surprising conclusion is a consequence of a previously unnoticed property of the factor model representation of returns. The factor model can be manipulated rather arbitrarily by repackaging a given set of securities. A new set of returns and a corresponding factor model can be produced, with virtually any prespecified random variables as the factors. By itself, therefore, factor analysis is not an adequate tool for identifying the random components of returns that should be relevant to asset pricing.

This conclusion is compatible with the recent work of Connor, who extends the earlier work of Ross. Connor's competitive equilibrium analysis highlights the role of certain aggregate features of the economy in asset pricing. Factor analysis is merely concerned with statistical correlations and is blind to aggregate economic considerations. The failure of the usual empirical formulation of the APT to discriminate between alternative factor representations on the basis of such considerations is its fundamental weakness. Unfortunately, since the market portfolio plays a prominent role in Connor's "equilibrium APT," it appears to be subject to substantially the same difficulties encountered in testing the CAPM.

Appendix A

In this appendix it is argued that Ross's APT does not imply an empirically testable bound on the sum of squared deviations in (4). It is necessary to first review some essential features of the proof in Ross [18]. Consider the problem of minimizing the variance of an arbitrage portfolio with no systematic risk and expected return $c > 0$; i.e.,

$$\text{minimize } X'DX$$

$$\text{subject to}$$

$$X'1_N = 0$$

$$X'\beta = 0$$

$$\text{and} \quad X'E = c \tag{A.1}$$

More specifically, consider an infinite sequence of such problems implicitly indexed by N, the number of assets in the subset under consideration. Let a be the minimum variance obtained in (A.1). Given some mild assumptions on preferences and boundedness of the elements of D, Ross shows that utility maximization implies the sequence of a values must be bounded away from zero. Intuitively, a sequence of a's approaching zero would constitute a sort of arbitrage opportunity in the limit.

Now let

$$e \equiv E - \gamma_0 1_N - \gamma_1 \beta$$

with γ_0 and γ_1 chosen so as to minimize the expression $e'D^{-1}e$. γ_0 and γ_1 so defined are identical to the coefficients from a generalized least squares regression of E on 1_N and β with nonsingular covariance matrix D. e is the corresponding vector of residuals. A key result in Ross's analysis (Ross [18], p. 349, 357) is that

$$e'D^{-1}e = c^2/a \tag{A.2}$$

If the elements of D are bounded above, say by u, then

$$e'e \leqq uc^2/a \tag{A.3}$$

It might appear that (A.3) provides an upper bound on the finite sum of squared deviations $e'e$, which could conceivably be tested. But (A.2) and (A.3) are purely algebraic facts, devoid of any economic content. The economics enters when we recall that utility maximization implies the sequence of a's is bounded away from zero. It follows from (A.3) that the sum of squared deviations remains bounded above as the number of assets approaches infinity. Therefore, the theory yields an economic restriction on the expected returns in an infinite sequence of economies. There does not appear to be a restriction on any particular economy in the sequence.

Appendix B

This appendix generalizes the results of the two asset example of Section II. Let R be an N-vector of security returns with positive-definite covariance matrix V. Let δ be a K-vector of mean zero random variables jointly distributed with R. As noted in Section I, we can regress each of the components of R on δ to obtain the following representation:

$$R = E + B\delta + \epsilon \tag{B.1}$$

where $E(\delta) = E(\epsilon) = 0$, so that $E = E(R)$. Let $\sum$ be the covariance matrix of ϵ. In general, $\sum$ will not be diagonal, so that (B.1) is not a factor model representation. We shall assume $\sum$ is positive definite. While this rules out some potential random vectors δ, it is a fairly general assumption which should encompass many cases. In particular, in the $K = 0$ case $\sum = V$ and hence is positive definite.

Let Q be an $N \times N$ nonsingular matrix such that

$$Q'Q = \textstyle\sum^{-1}$$

Such a matrix always exists, given our assumptions (this fact is used to show that a generalized least squares regression is equivalent to an ordinary least squares regression on transformed variables (see Theil [23], p. 23). Assume that the row sums of Q are all nonzero (if we imagine that the parameters of $\sum$ have been generated by some random continuous process, then the row sums will be nonzero with probability one). Let D be the diagonal matrix with i^{th} entry equal to the reciprocal of row sum i of matrix Q, and let $P \equiv DQ$. Then $P1_N = 1_N$, i.e., the row sums of P are all equal to one.

Premultiplication of a return vector by P generates a vector of portfolio returns. Transforming the representation (B.1) we obtain

$$R^* = E^* + B^*\delta + \epsilon^*$$

where

$$R^* = PR, \qquad E^* = PE, \qquad B^* = PB \quad \text{and} \quad \epsilon^* = P\epsilon \tag{B.2}$$

P has been constructed so that

$$\text{Var}(\epsilon^*) = P \textstyle\sum P' = DQ \textstyle\sum Q'D' = D^2$$

is diagonal. Therefore, the representation (B.2) is a legitimate K-factor model. The usual empirical formulation of the APT (see (5) of Section II) implies that there exist γ_0 and γ_1 such that

$$E^* = \gamma_0 1_N + B^*\gamma_1 \tag{B.3}$$

Premultiplication of both sides of (B.3) by P^{-1} gives

$$P^{-1}E^* = \gamma_0 P^{-1}1_N + P^{-1}B^*\gamma_1$$

Using (B.2) and noting that $P1_N = 1_N$ implies $P^{-1}1_N = 1_N$, we obtain

$$E = \gamma_0 1_N + B\gamma_1 \tag{B.4}$$

(B.4) is the usual APT expected return relation, with risk measured relative to the random vector δ. Since δ is essentially arbitrary, many such relations may be deduced for the same expected return vector E. The existence of many distinct representations for E is not, in itself, a problem (recall that in the mean-variance context, a different representation exists for each mean-variance efficient portfolio of the securities in R). To the contrary, it might appear that the APT may be tested using any random vector δ. Unfortunately, this view is untenable, since it leads to a logical contradiction.

This may be seen by letting $K = 0$ in the argument leading to (B.4); i.e., let δ be the empty set (vector). In this case, we have the existence of a single number γ_0, such that

$$E = \gamma_0 1_N \tag{B.5}$$

Thus the empirical formulation of the APT, from which (B.5) was deduced, rules out the very expected return differentials which the theory attempts to explain.

REFERENCES

1. M. Brennan. "Discussion." *Journal of Finance* 36 (May 1981), 352–3.
2. P. Brennan. "A Test of the Arbitrage Pricing Model." Unpublished manuscript, University of British Columbia, Vancouver, Canada, 1981.
3. S. Brown and M. Weinstein. "A New Approach to Testing Asset Pricing Models: The Bilinear Paradigm." Unpublished manuscript, Bell Laboratories, Holmdell, NJ: 1982.
4. G. Chamberlain and M. Rothschild. "Arbitrage and Mean-Variance Analysis on Large Asset Markets." Unpublished manuscript, University of Wisconsin, 1981.
5. N. Chen. "Some Empirical Tests of the Theory of Arbitrage Pricing." Working Paper No. 69, University of Chicago, 1982.
6. G. Connor. "Asset Pricing in Factor Economies." Doctoral dissertation, Yale University, 1982.
7. E. Fama. *Foundations of Finance.* New York: Basic Books, Inc., 1976.
8. A. Gehr. "Some Tests of the Arbitrage Pricing Theory." *Journal of the Midwest Finance Association*, 1975.
9. M. Gibbons. "Empirical Examination of the Return Generating Process of the Arbitrage Pricing Theory." Unpublished manuscript, Stanford University, 1981.
10. G. Huberman. "A Simple Approach to Arbitrage Pricing Theory." Working Paper No. 44, University of Chicago, 1980.
11. J. Ingersoll. "Some Results in the Theory of Arbitrage Pricing." Working Paper No. 67, University of Chicago, 1982.
12. B. King. "Market and Industry Factors in Stock Price Behavior." *Journal of Business* 39 (1966), 139–90.
13. D. Morrison. *Multivariate Statistical Methods.* 2nd edition. New York: McGraw-Hill, 1976.
14. G. Oldfield and R. Rogalski. "Treasury Bill Factors and Common Stock Returns." *Journal of Finance* 36 (May 1981), 337–50.
15. M. Reinganum. "The Arbitrage Pricing Theory: Some Empirical Results." *Journal of Finance* 36 (May 1981), 313–21.
16. R. Roll. "A Critique of the Asset Pricing Theory's Tests." *Journal of Financial Economics* (May 1977), 129–76.
17. ——— and S. Ross. "An Empirical Investigation of the Arbitrage Pricing Theory." *Journal of Finance* 35 (December 1980), 1073–1103.
18. S. Ross. "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory*, December 1976, 341–60.
19. ———. "Return, Risk, and Arbitrage." In I. Friend and J. Bicksler (eds.), *Risk and Return in Finance.* Cambridge: Ballinger, 1977.
20. ———. "The Capital Asset Pricing Model (CAPM), Short-Sale Restrictions and Related Issues." *Journal of Finance* 32 (March 1977), 177–83.
21. ———. "The Current Status of the Capital Asset Pricing Model (CAPM)." *Journal of Finance* 33 (June 1978), 885–901.
22. J. Shanken and A. Tajirian. "Equity Factors and the Market Portfolio." Unpublished manuscript, University of California, Berkeley, 1982.
23. H. Theil. *Principles of Econometrics.* New York: John Wiley & Sons, Inc., 1971.