A Multivariate Linear Regression Test for the Arbitrage Pricing Theory
Author(s): J. D. Jobson
Source: *The Journal of Finance*, Vol. 37, No. 4 (Sep., 1982), pp. 1037-1042
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327765
Accessed: 27/11/2009 08:20

THE JOURNAL OF FINANCE • VOL. XXXVII, NO. 4 • SEPTEMBER 1982

A Multivariate Linear Regression Test for the Arbitrage Pricing Theory

J. D. JOBSON*

ABSTRACT

A test for the arbitrage pricing theory which employs a multivariate linear regression model is developed. Given a sample of return premiums for a set of N assets which includes a subset of k linearly independent portfolios, the k factor APT hypothesis is accepted if the intercept term is zero in the multivariate regression of the $(N - k)$ returns on the k portfolios. The test may be carried out simply, by using univariate multiple regression software. The relation of this test to the concept of performance potential and Sharpe's measure of performance is also discussed. If the performance potential of the k portfolios is not significantly less than the performance potential of the complete set of N assets, then the k factor APT hypothesis is accepted.

IN ROLL AND ROSS (RR) [10] an empirical investigation of the Ross [11] arbitrage theory of pricing (APT) was carried out. The central core of the APT as outlined in RR is that the return generating process for a population of N assets is a linear function of only a few, say $k (k << N)$, systematic factors or components and as a consequence many portfolios are close substitutes. The central conclusion of the APT is that the mean premium return vector lies in a k dimensional subspace spanned by the factor loadings. The purpose of the empirical investigation as claimed by RR was to investigate the existence of the factors and their association with the risk premia. As stressed by RR, their methodology does not determine k but is designed to test the hypothesis that there exist several factors whose factor loadings together explain a significant portion of the variation in the mean premium returns. The empirical results in RR, however, indicate the presence of three to four priced factors.

The first empirical test outlined by RR involves two steps. Using time series data, the mean return premium vector is estimated and a factor analysis procedure is used to estimate the k factors and their loadings. A complex weighted least squares procedure is then used to estimate the linear model relating the mean return premium vector to the factor loadings. The weighted least squares procedure requires estimation of the matrix of weights and is carried out for $t = 1, 2, \cdots, T$ cross-sections.

In this paper, the central conclusion of the APT is shown to be equivalent to a statement about the intercept term in the multivariate regression function of a

* Department of Finance and Management Science, University of Alberta, Edmonton, Canada. This research was completed while the author was a visiting scholar in the Department of Biostatistics, University of North Carolina, Chapel Hill, North Carolina 27514. The author is grateful to Richard Roll for helpful comments and to Bob Korkie whose ideas in portfolio performance measurement provided an essential key to this research.

vector of $(N - k)$ returns on a vector of k portfolios, and as such can be tested using a multivariate linear model. This paper will also show that the test can be performed using a univariate multiple regression model. Finally, the relationship of the test to a test of equivalent performance potential for the set of k portfolios and the entire set of N assets developed by Jobson and Korkie (JK) [5], is also discussed.

I. An Alternative Look at the APT Conclusion

As presented by RR, the vector of return premium observations $\mathbf{r}_t(N \times 1)$ on N assets or portfolios is given by the multivariate linear model

$$\mathbf{r}_t = \boldsymbol{\mu} + \mathbf{B}\boldsymbol{\delta}t + \boldsymbol{\epsilon}_t \qquad t = 1, 2, \cdots, T,$$

where

$\boldsymbol{\mu}$ is the $(N \times 1)$ mean premium return vector;

$\mathbf{B}(N \times k)$ a matrix of factor loadings;

$\boldsymbol{\delta}t(k \times 1)$ a vector of scores on the systematic factors; and

$\boldsymbol{\varepsilon}_t(N \times 1)$ a vector of error terms.

The unobserved factor scores $\boldsymbol{\delta}t$ are assumed to have mean zero, to be linearly independent, and to be independent of $\boldsymbol{\epsilon}_t$. The elements of the unobserved error vector $\boldsymbol{\epsilon}_t$ are assumed to have mean zero and to be mutually independent.

Assume that there exists a subset of k linearly independent[1] portfolios or assets with $(k \times 1)$ return premium vector $\mathbf{r}_{1t}$. The model can be written as two equations

$$\mathbf{r}_{1t} = \boldsymbol{\mu}_1 + \mathbf{B}_1\delta_t + \boldsymbol{\epsilon}_{1t} \tag{1}$$

$$\mathbf{r}_{2t} = \boldsymbol{\mu}_2 + \mathbf{B}_2\boldsymbol{\delta}_t + \boldsymbol{\epsilon}_{2t} \tag{2}$$

where the partitioning of $\boldsymbol{\mu}$, $\mathbf{B}$, and $\boldsymbol{\epsilon}_t$ conform to the partitioning of $\mathbf{r}_t$ into $\mathbf{r}_{1t}$ and $\mathbf{r}_{2t}$.

Since $\mathbf{B}_1$ must be of full rank k, Equation (2) for the $((N - k) \times 1)$ return premium vector $\mathbf{r}_{2t}$ may be written as

$$\mathbf{r}_{2t} = (\boldsymbol{\mu}_2 - \mathbf{B}_2\mathbf{B}_1^{-1}\boldsymbol{\mu}_1) + \mathbf{B}_2\mathbf{B}_1^{-1}\mathbf{r}_{1t} + (\boldsymbol{\epsilon}_{2t} - \mathbf{B}_2\mathbf{B}_1^{-1}\boldsymbol{\epsilon}_{1t}) \tag{3}$$

If the central conclusion of the APT is true, then there exists a $(k \times 1)$ vector $\boldsymbol{\lambda}$ such that $\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\lambda}$. By partitioning this set of equations for $\boldsymbol{\lambda}$ as in the case of $\mathbf{r}_t$, it can be seen that $\boldsymbol{\mu}_2 = \mathbf{B}_2\mathbf{B}_1^{-1}\boldsymbol{\mu}_1$, and therefore the first term in (3) is zero.

The term $[\boldsymbol{\mu}_2 - \mathbf{B}_2\mathbf{B}_1^{-1}\boldsymbol{\mu}_1]$ does not represent the intercept term in the regression function $E[\mathbf{r}_{2t}/\mathbf{r}_{1t}]$ because of the correlation between $\boldsymbol{\epsilon}_{1t}$ and $\mathbf{r}_{1t}$.[2] Estimation of the intercept term in (3) would require an instrumental variables approach (see

[1] The term "k linearly independent portfolios" is equivalent to saying that for all T, $T > k$, the $(T \times k)$ matrix of returns generated by these k portfolios would have full rank k. This term does not refer to statistical independence.

[2] I am indebted to Richard Roll for bringing this problem to my attention.

Wonnacott and Wonnacott [14], Chapter 17) or an errors in variables approach (see Fuller [4]). To avoid this complex estimation problem additional restrictions are required on the vector $\mathbf{r}_{1t}$.

Recall that the vector $\mathbf{r}_{1t}$ could be assumed to represent the return premiums of a set of k linearly independent portfolios. If it can be assumed that these k portfolios represent linear combinations of a large set of assets, it would be reasonable to write Equation (1) without the error term ϵ_{1t}. By employing portfolios which are derived from a large number of assets (say M), the resultant error term ϵ_{1t} in (1) should become negligible if the portfolio weights are of order $\left(\frac{1}{M}\right)$.[3] The error term in (3) therefore becomes ϵ_{2t}, which by assumption is independent of $\mathbf{r}_{1t}$. The intercept term in (3) is now equivalent to the intercept term in the regression function of $\mathbf{r}_{2t}$ on $\mathbf{r}_{1t}$. Therefore, given a linearly independent set of k portfolios with return premium vector $\mathbf{r}_{1t}$ and a set of $(N-k)$ assets with return premium vector $\mathbf{r}_{2t}$, a test of the hypothesis of a k factor APT model can be carried out by testing for a zero intercept term in the multivariate linear regression $\mathbf{r}_{2t}$ on $\mathbf{r}_{1t}$.

To test the hypothesis $H_0: \boldsymbol{\mu}_2 = \mathbf{B}_2\mathbf{B}_1^{-1}\boldsymbol{\mu}_1$, some assumptions must be made about the distribution of $\mathbf{r}_t$. Typically, in regression analysis the assumption of normality is made, or in large samples asymptotic theory is employed to justify the use of conventional tests. The normality assumption is employed for the remainder of the paper.

II. A Likelihood Ratio Test for APT

The premium return vector $\mathbf{r}_t$, $t = 1, 2, \cdots, T$ is assumed to be multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ where $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$ is partitioned to conform to the earlier partitioning of $\boldsymbol{\mu}$ and $\mathbf{r}_t$. The multivariate linear model required is given by

$$\mathbf{r}_{2t} = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{r}_{1t} + \boldsymbol{\mu}_t \qquad t = 1, 2, \cdots, T$$

where $\boldsymbol{\alpha}$ is $(N-k) \times 1$, $\boldsymbol{\beta}$ is $(N-k) \times k$, and $\boldsymbol{\mu}_t$ is $(N-k) \times 1$. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote unknown parameter vectors while $\boldsymbol{\mu}_t$ is the usual vector of error terms. Under the assumption of multivariate normality $\boldsymbol{\alpha} = \boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1$ and $\boldsymbol{\beta} = \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}$.

The likelihood ratio test of $H_0: \boldsymbol{\alpha} = \mathbf{0}$ results in a special case of Wilks Lambda, Ω (see Finn [3]). A test statistic based on Ω due to Rao [8] is given by $\left[\frac{T-N-1}{N-k}\right]\left[\frac{1-\Omega}{\Omega}\right]$ which has an F distribution with $(N-k)$ and $(T-N-$

[3] While there is no requirement that the k portfolios be mutually uncorrelated, several large correlations could result in a multicollinearity problem, and the resulting large standard errors associated with the estimates of the intercept term in (3). The problem of the construction of the k portfolios from a large set of assets has been addressed recently by Chen [2]. The two stage procedure involves factor analysis of the covariance matrix followed by portfolio formation using mathematical programming. I am indebted to Richard Roll for bringing this reference to my attention.

1) d.f. if H_o is true.[4] In the case of the hypothesis $H_0: \boldsymbol{\alpha} = 0$, the test statistic can be written more simply as

$$\left[\frac{T-N-1}{N-k}\right]\left[\frac{\mathbf{r}'\mathbf{S}^{-1}\mathbf{r} - \mathbf{r}_1'\mathbf{S}_{11}^{-1}\mathbf{r}_1}{1+\mathbf{r}_1'\mathbf{S}_{11}^{-1}\mathbf{r}_1}\right] \tag{4}$$

where $\bar{\mathbf{r}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{r}_t$ and $\mathbf{S} = \frac{1}{T}\sum_{t=1}^{T} (\mathbf{r}_t - \bar{\mathbf{r}})(r_t - \bar{\mathbf{r}})'$ are the conventional maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively, and $\mathbf{r}_1$ and $\mathbf{S}_{11}$ conform to the partitioning of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and are the maximum likelihood estimators of $\boldsymbol{\mu}_1$ and $\boldsymbol{\Sigma}_{11}$ respectively.[5]

III. Using the Univariate Multiple Regression Model

As shown in Jobson [7], the test statistic for the hypothesis $H_0: \boldsymbol{\alpha} = \mathbf{0}$ may be computed using univariate multiple linear regression software. The $(T \times N)$ matrix of return premiums is denoted by $\mathbf{R} = [\mathbf{R}_1\mathbf{R}_2]$ where $\mathbf{R}_1$ is $(T \times k)$ and $\mathbf{R}_2$ is $(T \times (N-k))$, and $\mathbf{i}$ $(T \times 1)$ is a column vector of unities.

By performing two separate regressions with forced zero intercept term, $\mathbf{i}$ on $\mathbf{R}_1$ and $\mathbf{i}$ on $\mathbf{R}$, we obtain the regression sum of squares and error sum of squares SSR and SSE (for $\mathbf{R}$) and SSR_1 and SSE_1 (for $\mathbf{R}_1$). The test statistic can be written[6] as $\left[\frac{T-N-1}{T-k}\right]\left[\frac{SSR - SSR_1}{SSE}\right]$ which is the conventional F statistic for testing that the regression coefficients for the columns for $\mathbf{R}_2$ are not significantly different from zero.

IV. The APT and Performance Potential

The term performance potential for a set of N assets was introduced in JK [5] and was defined to be the slope of the tangent line to the efficient frontier in

[4] Using the fact that $-2 \log \lambda$ is asymptotically $a\ \chi^2$ distribution, Ross [10] derived a test for market efficiency in the special case $k = 1$. Using Rao's F test, JK [5] have derived a more general test of performance potential.

[5] The likelihood ratio test in this case yields the ratio $|\hat{\boldsymbol{\Sigma}}_{22.1}|/|\hat{\boldsymbol{\theta}}_{22.1}|$ where $\hat{\boldsymbol{\Sigma}}_{22.1} = [\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}]$, $\mathbf{S} = \begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix}$ and $\hat{\boldsymbol{\theta}}_{22.1} = \left[\frac{1}{T}\mathbf{R}_2'\mathbf{R}_2 - \frac{1}{T}(\mathbf{R}_2'\mathbf{R}_1)(\mathbf{R}_1\mathbf{R}_1)^{-1}(\mathbf{R}_1'\mathbf{R}_2)\right]$. Using the identity Rao [9] (p. 33, no. 2.8) $\hat{\boldsymbol{\theta}}_{22.1}$ $[\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}] + (\bar{\mathbf{r}}_2 - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\bar{\mathbf{r}}_1)(\bar{\mathbf{r}}_2 - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\bar{\mathbf{r}}_1)'/(1 + \bar{\mathbf{r}}_1'\mathbf{S}_{11}^{-1}\bar{\mathbf{r}}_1)$. Then, by employing the same "trick" used for Hotelling's T^2, see Anderson [1], the result follows. The derivation of the likelihood ratio test is given in JK [5] and Jobson [7].

[6] For the regression of $\mathbf{i}$ on $\mathbf{R}$ without intercept, the sums of squares are given by

$$SSR = [\mathbf{i}'\mathbf{R}][\mathbf{R}'\mathbf{R}]^{-1}[\mathbf{R}'\mathbf{i}],$$

$$SSE = T - [\mathbf{i}'\mathbf{R}][\mathbf{R}'\mathbf{R}]^{-1}[\mathbf{R}'\mathbf{i}].$$

Similarly for the regression of i on R_1 without intercept

$$SSE_1 = [\mathbf{i}'\mathbf{R}_1][\mathbf{R}_1'\mathbf{R}_1]^{-1}[\mathbf{R}_1'\mathbf{i}].$$

By employing the identity in Rao [9] (p. 33, no. 2.8), the result can be shown. The use of regression software to make inferences about efficient portfolios is outlined in J.K. [6].

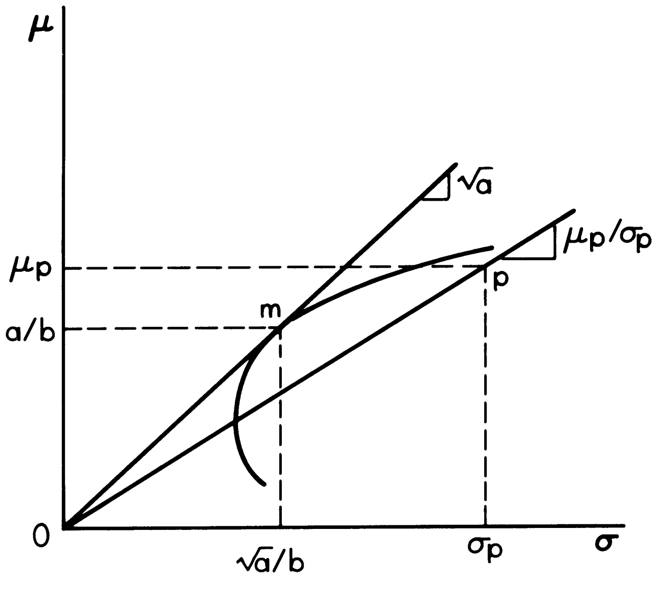
mean return premium and standard deviation space as shown in Figure 1. For the portfolio p with mean μ_p and variance σ_p^2, the slope of the line through p and the origin is given by μ_p/σ_p. If $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ denote the mean premium return vector and covariance matrix for the N assets, the slope of this tangent line is given by $\sqrt{a} = \sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$ and is equivalent to the Sharpe [15] performance measure for the portfolio located at the point of tangency. Using the efficient set notation of Roll [10], the Sharpe measure for portfolio m is given by

$$\frac{\mu_m}{\sigma_m} = \frac{a/b}{\sqrt{a/b^2}} = \sqrt{a}$$

where $a = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $b = \mathbf{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $\mathbf{e}$ ($N \times 1$) is a vector of unities.

Denoting the mean premium return vector and covariance matrix for a subset of k of the N assets by $\boldsymbol{\mu}_1$ and $\boldsymbol{\Sigma}_{11}$ respectively, the performance potential for the subset is denoted by $\sqrt{a_1} = \sqrt{\boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1}$. In JK [5], the null hypothesis $H_1: \sqrt{a} = \sqrt{a_1}$ is shown to be equivalent to the test $H_0: \boldsymbol{\mu}_2 = \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1$ where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are partitioned as in Section II. The multivariate regression test of the k factor APT is therefore equivalent to the test of equality of performance potential between the set of N assets and the subset of k portfolios.

If $k = 1$, the performance potential of the single portfolio is given by its Sharpe measure μ_1/σ_1. In this case, the test of H_1 is a test for the efficiency of the portfolio, as in Ross [12]. Clearly, the tangency portfolio for the subset of k



Potential Performance for a Set of Assets

$$\text{Potential Performance} = \text{Max}\ \frac{\mu_p}{\sigma_p} = \frac{\mu_m}{\sigma_m} = \sqrt{a}$$

Figure 1

portfolios if known a priori could be used in place of the k portfolios to test the hypothesis that $k = 1$. As pointed out by RR in their footnote 5, knowing the parameters in advance allows us to determine the single factor which explains premium returns. "In this trivial sense the number of factors does not matter. Without further assumptions, though, this begs the question." Similarly, here, knowledge of μ and Σ allows the tangency portfolio to be determined.

REFERENCES

1. T. W. Anderson. *An Introduction to Multivariate Analysis.* New York: John Wiley and Sons, 1958.
2. Nai Fu Chen. "The Arbitrage Pricing Theory: Estimation and Applications." Working Paper, UCLA, 1979.
3. Jeremy Finn. *A General Model for Multivariate Analysis.* New York: Holt, Rinehart and Winston, 1974.
4. Wayne A. Fuller. "Properties of Some Estimators for the Errors in Variables Model." *The Annals of Statistics* 8 (June 1980), 407–22.
5. J. D. Jobson and Bob Korkie. "On the Measurement and Testing of Performance Potential." Unpublished Manuscript, 1981.
6. ———. "Statistical Inference in Two Parameter Portfolio Theory with Multiple Regression Software." Unpublished Manuscript, 1981.
7. J. D. Jobson. "Testing For Zero Intercept in Multivariate Normal Regression Using the Univariate Regression Model." Unpublished Manuscript, 1981.
8. C. R. Rao. "An Asymptotic Expansion of the Distribution of Wilks' Criterion." *Bulletin of International Statistical Institute* 33 (1951), 177–80.
9. ———. *Linear Statistical Inference and Its Applications.* New York: John Wiley and Sons, 1973.
10. Richard Roll. "A Critique of the Asset Pricing Theory's Tests, Part I: On Past and Potential Testability of the Theory." *Journal of Financial Economics* 4 (1977), 129–76.
11. ——— and Stephen Ross. "An Empirical Investigation of the Arbitrage Pricing Theory." *The Journal of Finance* 35 (December 1980), 1073–1103.
12. Stephen A. Ross. "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory* 13 (December 1976), 341–60.
13. ———. "A Test of the Efficiency of a Given Portfolio." Presented at the World Econometrics meetings, Aix-en Provence, 1980.
14. Ronald J. Wonnacott and Thomas H. Wonnacott. *Econometrics.* New York: John Wiley and Sons, 1970.
15. William F. Sharpe. *Portfolio Theory and Capital Markets.* New York: McGraw-Hill Inc., 1970.