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The Determination of Fair Profits for the Property-Liability Insurance Firm

ALAN KRAUS and STEPHEN A. ROSS*

ABSTRACT

Single period and dynamic valuation models in continuous time, under certainty and uncertainty, are developed for a property-liability insurance contract to determine the "fair" (competitive) premium and underwriting profit. The intertemporal stochastic model assumes that the claim frequency and the price index of claim settlements are functions of a set of underlying state variables which follow a multivariate Wiener process. The competitive premium is shown to be proportional to the claim frequency and the price index for claim settlements at the time the policy is issued. The factor of proportionality varies directly with the claim settlement rate and the length of coverage, and inversely with the risk-adjusted real interest rate on the dollar-valued claim rate.

ALTHOUGH THE PROPERTY-LIABILITY insurance industry has been subject to price regulation for many years, surprisingly little research has been directed at producing valuation models of a property-liability insurance firm. Although property-liability price regulation has historically involved setting minimum prices to reduce the risk of insolvency, some commissions have begun to move toward setting prices more in the spirit of public utility regulation.[1] If, in an effort to promote efficiency, the working objective of such regulation requires the determination of prices which would prevail under competition, then regulation must be founded on some model of the costs of regulated firms. Under competition with free entry, prices just cover all economic costs, including the opportunity costs of investment by suppliers of capital, and a valuation model is required to explain how the market value of the firm, and therefore the cost of capital, reacts to changes in the prices it charges. Our objective in this paper is to provide such a framework.

A great deal of attention and controversy in discussions of property-liability insurance regulation has been focussed on whether and how investment income is to be considered when setting prices so as to produce a "fair" rate of return. Insurers receive investment income because, on average, premiums are received in advance of the payment of loss claims, and premium funds are invested during this lag. Early insurance regulation ignored investment income in setting rates but this situation has changed in the last two decades. Where investment income has come to be considered in the rate-setting process, it has apparently been, almost always, on an ad hoc basis without much economic rationale. There have recently appeared articles, by Fairley [2] and Hill [5], advancing the analysis of property-liability insurance rates, including investment income, in the framework

* University of British Columbia and Yale University, respectively.

[1] For example, see Hill [5, p. 174, esp. Note 6].

of the Sharpe-Lintner-Mossin mean-variance capital asset pricing model (CAPM). A particular purpose of these studies has been to apply the CAPM notions of systematic and unsystematic (diversifiable) risk to both underwriting and investment income to arrive at profit rates which reflect the systematic risk components of insurance.

The approach we take begins by treating the contract between the insurance firm and the policyholder, as a (nonmarketable) contingent claim whose payoff will be determined by random future events. The insurance firm is thus seen as selling short a bundle of risky securities (policies) and investing the proceeds of the short sales, plus contributed equity capital, in a competitive capital market. The policyholder has in effect simply purchased a security which will pay off contingent on some random event. This perspective makes the valuation problem particularly easy since what is actually done with the proceeds of the short sales in a competitive market is irrelevant to the valuation of the claims sold short. In such a market, the price of a contract is simply the value of the representative claim pattern. But the contract offered by the insurance company is not a simple loan. In particular, claims are paid in current dollars and this subjects the insurance firm to both the expected cost and risk of inflation. In its simplest form, this argument implies that changes in anticipated inflation have no effect on the fair premium if the real rate of interest is unaffected.

Our principal contribution to the study of these issues will be to develop a small dynamic model of the insurance contract and use it to deal rigorously with the effects of both timing of cash flows and uncertainty of basic economic state variables. We first develop the dynamic model in a simplified form and then proceed to incorporate more general assumptions. In order to concentrate on the implications of the dynamic behavior of cash flows and, in particular, of the influence of inflationary effects, we shall study the model in detail under certainty. After analyzing the certainty model, we extend it to a world in which basic economic factors are stochastic. We begin in Section I with the simplest interesting version, a discrete time model and in Section II we extend the model to a truly dynamic setting. Section III examines the one period stochastic model, akin to the Fairley [2] and Hill [5] analyses, and Section IV develops the intertemporal stochastic model. Section V briefly concludes the paper.

I. Discrete Time Certainty Model

Let us begin with the simplest possible model. We assume that each policy is priced competitively and is written for a single period, with the premium for period t, P_t, received at the beginning of the period and the loss, L_t, paid out at the end.[2] If we denote the nominal interest rate by r_t, the fair price for insurance in this certain world is given by

$$P_t = \frac{L_t}{1 + r_t} \tag{1}$$

[2] In this and in subsequent models, all expenses associated with policy sales are netted from the premium and all expenses associated with claims are included in a definition of the loss.

Since claims are paid in nominal dollars, L_t reflects the impact of inflation over the period. If the general inflation rate is denoted by i_t and the real interest rate is denoted by ρ_t^*, then by the Fisher equation $(1 + r_t) = (1 + \rho_t^*)(1 + i_t)$, and if the cost of settling claims is expected to inflate at the rate π_t, then $L_t = L_{t-1}(1 + \pi_t)$. If $\pi_t = i_t$, then Equation (1) becomes

$$P_t = \frac{L_{t-1}}{1 + \rho_t^*} \tag{2}$$

If the real rate stays constant, then the equilibrium premium should rise at the actual realized inflation rate. Notice, too, that changes in anticipated inflation affect the equilibrium premium only through changes in the real rate of interest. If ρ_t^* remains constant, then an increase in the nominal interest rate, r_t, with an increase in anticipated inflation, will have no impact on P_t provided the anticipated inflation in average losses, π_t, rises with the general inflation level.

More generally, if we allow for relative price changes between losses and other nominal quantities, then

$$\frac{P_{t+1}}{P_t} = \left[\frac{(1 + \rho_t^*)(1 + \pi_{t+1})}{(1 + \rho_{t+1}^*)(1 + i_{t+1})}\right](1 + i_t) \tag{3}$$

which implies that a wedge can be driven between premium inflation and general inflation through changes either in the real rate or in the real value of the average loss.

Actual cash flows come from combining these policies over time and accounting for changes in the stock of policies and claims. To see the extent to which our general principles carry over to a more realistic world, we shall extend the analysis to a dynamic model which is simple, but sufficiently realistic to accommodate the principal dynamic influences.

II. Dynamic Certainty Model

Our basic dynamic model is given by the following two differential equations:

$$\dot{Q}_t = S_t - \delta Q_t \tag{4}$$

and

$$\dot{C}_t = \alpha_t Q_t - \theta C_t \tag{5}$$

where Q_t denotes the stock of policies in force at time t, S_t is a function giving the current sales of new policies, δ is the rate of attrition of existing policies through nonrenewal, C_t is the stock of unsettled claims, α_t is the accident frequency, and θ is the rate of settlement. These equations say that the policy stock is augmented by new sales and declines by attrition through nonrenewals, while the stock of unsettled claims grows by the rate of accidents among outstanding policies and declines by the rate of claim settlements. By making sales an appropriate function of relative prices, this system could equally well describe a single firm within a competitive industry or the industry itself—whether competitive or not. The reasonableness of this model hinges primarily on the assumed homogeneity of claims which is a central part of our analysis. To the extent to which claims and their attendant losses are too "lumpy," the argument would have to be modified.

The above equations can describe a variety of different dynamic paths for both policies and claims depending upon the assumptions we make on the sales and accident rates. When we consider stochastic generalizations of this model, we shall permit α to vary, but in the present section we treat it as constant. Our point of view, though, permits us to allow completely general sales functions, S_t. Since, under competition, the cohort of policies sold at any time must pay for itself, sales can have no influence on pricing in our model. Consequently, we assume $S = 0$ for the following development. For simplicity, we also assume that the attrition rate, δ, is zero.[3] With these assumptions, our dynamic system consists of (5) alone.

Our problem is to determine the fair or competitive pricing of a given cohort of policies. For convenience, we assume this cohort of policies is issued at time $t = 0$ and offers coverage over a period of length T. For simplicity, we assume that premiums are paid in full at time of issue, but more general payment schedules, like general sales schedules, are easily accommodated by superimposing the analysis of this section; the generalizations are straightforward. For convenience of notation, we set $Q_0 = 1$, so that the cohort consists of a single policy and P_0 is the revenue collected at time 0. Since this policy is issued at time 0, $C_0 = 0$.

Solving (5) for the stock of claims at time t, we have

$$C_t = \begin{cases} \dfrac{\alpha}{\theta}(1 - e^{-\theta t}) & \text{for} \quad 0 \le t \le T \\[2ex] \dfrac{\alpha}{\theta}(1 - e^{-\theta T})e^{-\theta}(t - T) & \text{for} \quad t > T \end{cases} \tag{6}$$

where the break at T occurs because no new claims are generated against these policies after the coverage period, T. (In effect, $\alpha_t = 0$ for $t > T$.)

If we denote the price index for the average claim by q_t and assume that it grows at the exponential rate π, then

$$q_t = q_0 e^{\pi t} \tag{7}$$

It follows that the dollar cash outflows from losses are given by

$$L_t = \theta\, q_0 e^{\pi t} C_t \tag{8}$$

and the fair or competitive price for insurance, P_0, is given by the present value of these dollar losses. Letting r and ρ denote the nominal interest rate and the real rate of return on policies, respectively, where $\rho \equiv \pi - r$, we have

$$\begin{aligned} P_0 &= \int_0^\infty L_t e^{-rt}\, dt \\ &= q_0 \alpha \left(\frac{\theta}{\rho + \theta}\right)\left(\frac{1 - e^{-\rho T}}{\rho}\right) \end{aligned} \tag{9}$$

[3] Equivalently, we can assume that when a policyholder drops coverage, the fair rebate is given. This effect is small and the rebate—even if not fair—makes it negligible.

and

$$P_0 \approx (\alpha T) q_0 \left(\frac{\theta}{\rho + \theta} \right) \left(1 - \frac{1}{2} (\rho T) \right) \tag{10}$$

where the approximation is derived by simple Taylor series expansion. Notice that our previous simple results on the impact of realized inflation and changes in anticipated inflation remain valid. If the real rate of interest is unaltered by changes in the anticipated inflation rate of losses, then premiums should not change. As expected, premiums must rise if the rate of claim settlement, θ, is increased.[4] Similarly, premiums should rise with the price level, q_0.[5] Of course, we

[4] We are implicitly assuming that the settlement rate is bounded from below by regulation or competition since the notion of a competitive premium is meaningless if the insurer is free to make the settlement rate as low as desired.

[5] Another possibility is that the dollar value of the loss is frozen at the time of the accident. Letting C_t^* denote the dollar value of unsettled claims at time t, we have

$$\dot{C}_t^* = \alpha q_t Q_t - \theta C_t^* = \alpha q_0 e^{\pi t} - \theta C_t^*$$

which has the solution

$$C_t^* = \begin{cases} \left(\dfrac{\alpha q_0}{\theta + \pi} \right) (e^{\pi t} - e^{-\theta t}) & \text{for } 0 \leq t \leq T \\ \left(\dfrac{\alpha q_0}{\theta + \pi} \right) (e^{\pi T} - e^{-\theta T}) e^{-\theta (t - T)} & \text{for } t > T \end{cases}$$

Thus, since $L_t = \theta C_t^*$, the present value of losses is

$$PVL = \int_0^\infty L_t e^{-rt}\, dt = \left(\frac{\alpha \theta q_0}{\theta + \pi} \right) \left[\frac{\pi + \theta}{\rho (r + \theta)} \right] (1 - e^{-\rho T})$$

so that the fair premium is

$$P_0 = \left(\frac{\alpha \theta q_0}{\theta + r} \right) \frac{1 - e^{-\rho T}}{\rho} \tag{10a}$$

Now, the original result on premiums rising with inflation remains unaltered, but an increase in anticipated inflation will lower premiums with real rates unaltered. The reason is simply that higher nominal rates accelerate the discounting of claims awaiting settlement.

As a practical matter, though, increases in nominal interest rates from increases in inflation would have a very small impact on premia, since, for $\theta = 1.5$ and $r = .1$, for example,

$$\frac{r}{P} \frac{\partial P}{\partial r} = -\frac{r}{\theta + r} = -.06$$

Raising r to .15, a 50% increase, would lower premiums by only 3%.

In general, we would expect some portion of costs from an accident at time t to continue to inflate while a portion remained fixed. We could model this by reinterpreting r as the nominal rate minus the average inflation rate on costs past the accident. Alternatively, if γ is that proportion subject to inflation at the rate$_\pi$ and $(1 - \gamma)$ is, as above, not affected by inflation, then the fair premium would be a weighted average of (10) and (10a):

$$P_0 = \alpha \theta q_0 \left[\frac{1 - e^{-\rho T}}{\rho} \right] \left[\gamma \left(\frac{1}{\theta + \rho} \right) + (1 - \gamma) \left(\frac{1}{\theta + r} \right) \right]$$

More generally, the latter term would be an average over different inflation rates for different components.

As we have seen, though, none of these modifications alters the dynamic behavior of premia and the magnitude of the influence of changing anticipated inflation rates would be small.

have defined the real rate of return on policies by subtracting the nominal inflation rate on losses rather than the economy-wide inflation rate. If i denotes the economy-wide rate and ρ^* denotes the real rate of interest for the economy, then $\rho = \rho^* - (\pi - i)$, and the differences between ρ and ρ^* can be seen to arise from differences in the two inflation rates. With ρ^* constant, if π and i move together, then changes in the nominal rate of interest will leave premiums unaffected.

An important book quantity in the insurance industry, and in its regulation, is the rate of underwriting profit, or underwriting margin, defined as the ratio of the difference between premium income and total undiscounted dollar losses to premium income. To examine underwriting margin in our model, we first calculate total undiscounted dollar losses:

$$\begin{aligned} UDL &\equiv \int_0^\infty L_t\,dt \\ &= q_0\alpha\left(\frac{\theta}{\theta-\pi}\right)\left(\frac{e^{\pi T}-1}{\pi}\right) \end{aligned} \tag{11}$$

where we assume $\pi < \theta$.

The underwriting margin is given by

$$\begin{aligned} U_0 &\equiv \frac{P_0 - UDL}{P_0} \\ &= 1 - \frac{\left(\frac{\theta}{\theta-\pi}\right)\left(\frac{e^{\pi T}-1}{\pi}\right)}{\left(\frac{\theta}{\theta+\rho}\right)\left(\frac{1-e^{\rho T}}{\rho}\right)} \\ &\approx -\left(\frac{r}{\theta-\pi}\right)\left(1+\frac{1}{2}(\theta+\rho)T\right) \end{aligned} \tag{12}$$

Following Fairley [2, pp. 196–98], if we let k be defined as the investable funds per dollar of premiums, the $U_0 = -kr$ implies

$$k \approx \frac{1 + \frac{1}{2}(\theta+\rho)T}{\theta-\pi} \tag{13}$$

Notice that k is a decreasing function of the settlement rate, θ, and an increasing function of the inflation rate, π. Keeping realistic magnitudes in mind, if we take the average delay in claims, $1/\theta$, to be about one year it will dominate inflation and the real rate in determining k, which will be approximately 1.5. Intuitively, raising the claim delay should increase the investable funds per dollar of premiums, and Equation (13) verifies this result.

The fact that k increases with inflation—even if the nominal rate is held constant and ρ adjusts—is a bit more difficult to fathom. The easiest way to see this result is to recall that as the inflation rate is increased the fair premium stays constant, but the summed losses rise, becoming infinite as the inflation rate approaches the claim settlement rate. As a consequence, the underwriting profit becomes arbitrarily negative. Even if the real rate approaches zero for high inflation rates, the premium is still bounded above by the expected loss, $\alpha T q_t$.[6]

Before we close this section, we should reiterate that the above results apply to quite general sales and claim dynamics. In addition, sales costs, claim expenses, and the like can be easily analyzed, usually by just adjusting the price index and the premium to make them gross of such costs.

We turn now to the situation where losses are stochastic in the aggregate.

III. Single Period Stochastic Model

In introducing uncertainty into our model, we emphasize that it is not actuarial risk that will concern us. By assumption, the firm we consider is large enough for individual actuarial risks to even out in the aggregate. It is only the risk of the firm's aggregate cash flow that matters to suppliers of capital, and only the systematic component of that risk (the portion that is not diversifiable across different firms) that affects the market value of the firm.

It is easiest to study the stochastic case first in a single period world (in which we suppress the time subscript for convenience). Applying the Arbitrage Pricing Theory of Ross [7, 8], we find the competitive premium by considering the covariances of the loss, $\tilde{L}$, with the systematic risk factors which are priced in the economy, $(x_1, \cdots, x_n)$. This risk correction gives

$$P = \frac{E[\tilde{L}] - \sum_i \lambda_i [\mathrm{Cov}(\tilde{L}, \tilde{x}_i)/\mathrm{Var}(\tilde{x}_i)]}{1 + r}$$

where $\tilde{x}_i$ is the ith systematic risk factor and λ_i is the price of the ith component of risk. Letting β_i^L be the regression coefficient of the loss of $\tilde{x}_i$, we have

$$P = \frac{E[\tilde{L}] - \sum_i \lambda_i \beta_i^L}{1 + r}$$

Specializing to the CAPM model of Sharpe [9] and Lintner [6] yields

$$P = \frac{E[\tilde{L}] - \lambda \beta^L}{1 + r}$$

where

$$\beta^L \equiv \frac{\mathrm{Cov}(\tilde{L}, \tilde{R}_m)}{\mathrm{Var}(\tilde{R}_m)}$$

is the regression of the loss of the market return, $\tilde{R}_m$, and

$$\lambda \equiv (E_m - r)$$

is the excess return on the market.

In other words, we alter the certainty theory by correcting for the risk premium. Notice that if, as empirical estimates suggest,[7] the beta coefficient on losses is

[6] As a practical matter, if the evidence on very low real rates (see Fama [3] and Fama and Schwert [4] applies to the insurance sector, then the fair premium will be set at the expected loss, despite the fact that premiums are "loaned" by policyholders ahead of loss receipts.

[7] See Hill [5, p. 185] and Fairley [2, p. 204].

negative, then this will raise the fair premium above its value under certainty.

If the beta coefficient is redefined as the beta coefficient of losses per premium dollar, i.e.,

$$\beta_L \equiv \frac{\operatorname{Cov}\left(\frac{\tilde{L}}{P}, \tilde{R}_m\right)}{\operatorname{Var}(\tilde{R}_m)}$$

then the expected underwriting profit is

$$U \equiv \frac{P - E[\tilde{L}]}{P} = -[r + (E_m - r)\beta_L]$$

which is Fairley's and Hill's estimate of the fair profit rate.[8] (Equivalently, we could also define β_L as the negative of the regression coefficient of underwriting profits on the market portfolio; since premiums are given, this will lead to the same analysis.)

Notice that if β_L is constant, the above formulas produce precisely the same inflation corrections as before. Premiums rise with the dollar value of losses and are not affected by changes in anticipated inflation. To put it somewhat more directly, β^L rises proportionately with the price level for losses, and changes in the anticipated inflation rate are offset by equal changes in the nominal interest rate. We shall study these issues in greater detail in the stochastic intertemporal setting below.

IV. Intertemporal Stochastic Model

Assume, as above in the single period case, that systematic, economy-wide risk is completely described by a state vector $x \equiv (x_1, \cdots, x_n)$. The vector x impounds all exogenous activity affecting the basic economic variables of our system. Knowledge of this vector is sufficient to determine the distribution of future returns. Each of the factors is priced in the market place, resulting in an equilibrium in which the expected excess return (above the riskless rate) on any asset is linearly related to the covariances of the asset's return with these factors. A detailed derivation of these arguments is given in Ross [7] and Cox, Ingersoll, and Ross [1]. To apply this equilibrium relation to our situation, we require some further specification and assumptions.

We shall assume that the state vector x is multivariate lognormal, so that its movement in continuous time is described by

$$d\tilde{x}_i = m_i x_i\, dt + \sigma_i x_i\, d\tilde{z}_i \qquad (i = 1, \cdots, n) \tag{14}$$

where m_i and σ_i are intertemporal constants, and $d\tilde{z} \equiv (d\tilde{z}_1, \cdots, d\tilde{z}_n)$ is a multivariate independent Wiener process.

Define $V = V(x, C, t)$ as the value at time t of the remaining cash outflows from claim settlements on a single policy (i.e., for $Q_t = 1$) when the state vector is x and the unsettled claims against the policy are C at time t. As before, we

[8] For example, see Fairley [2, p. 201].

assume that the policy under consideration is issued at time $t = 0$ and offers coverage up to time T. Hence the fair premium is given by $P_0 = V(x, 0, 0)$, the value when the policy is written.

We complete our specification of the model by assuming that the dynamic system of Section II still governs the claim flow, with both the accident frequency and the price level of claims as functions of the state variables. We parametrize these as

$$\log \alpha = \sum_i \alpha_i \log x_i + \log \alpha_0 \tag{15}$$

and

$$\log q = \sum_i q_i \log x_i + \log q_0 \tag{16}$$

The loglinear functions assumed above are general enough to permit econometric estimation, but simple enough to lend themselves to an analytic solution.

The application to our model of the basic equilibrium relation that the expected excess return on an asset is linearly related to the covariances of its return with the underlying factor gives

$$E\left[\frac{d\tilde{V}}{V}\right] + \left[\frac{\theta q C}{V} - r\right] dt = \sum_i \lambda_i \sigma_i \left[\operatorname{Cov}\left(\frac{d\tilde{V}}{V}, \frac{d\tilde{x}_i}{x_i}\right) \Big/ \operatorname{Var}\left(\frac{d\tilde{x}_i}{x_i}\right)\right] dt \tag{17}$$

where $\lambda_1, \cdots, \lambda_n$ are constants. (Cox, Ingersoll and Ross [1] provide a derivation of this basic equilibrium model.) Claim settlements, $\theta q C$, are analogous to dividend payouts on a stock and thus are part of the total return. The right-hand terms in square brackets are the beta coefficients of the return on the asset with respect to each of the factors. For the special case of the traditional CAPM, only the beta with respect to the single factor of return on the market would be included.

Applying Ito's Lemma to the value of remaining cash outflows and using the dynamic system assumed, we obtain

$$\begin{aligned} d\tilde{V} &= \sum_i V_i\, d\tilde{x}_i + V_C\, dC + V_t\, dt + \tfrac{1}{2} \sum_i V_{ii} (d\tilde{x}_i)^2 \\ &= \sum_i V_i (m_i x_i\, dt + \sigma_i x_i\, d\tilde{z}_i) + \tfrac{1}{2} \sum_i V_{ii} \sigma_i^2 x_i^2\, dt + V_C(\alpha - \theta C)\, dt + V_t\, dt \end{aligned} \tag{18}$$

where V_i, V_{ii}, V_C, and V_t denote partial derivatives. Using the equilibrium relation (17) the basic valuation equation becomes[9]

$$\begin{aligned} &\tfrac{1}{2} \sum_i \sigma_i^2 x_i^2 V_{ii} + \sum_i (m_i - \lambda_i \sigma_i) x_i V_i + (\alpha - \theta C) V_c \\ &\qquad + V_t - rV + \theta q C = 0, \qquad \text{for } t \leq T \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\tfrac{1}{2} \sum_i \sigma_i^2 x_i^2 V_{ii} + \sum_i (m_i - \lambda_i \sigma_i) x_i V_i - \theta C V_c \\ &\qquad + V_t - rV + \theta q C = 0, \qquad \text{for } t > T \end{aligned} \tag{20}$$

[9] We shall assume, for simplicity, that r is not stochastic. The impact of stochastic interest rates is of interest, but the analysis would be complicated. We are, also, again assuming that losses continue to inflate with the price level after the coverage period. The modifications of the results of this section to the alternative case of Note 5 above can be easily applied to this case.

To solve this system for the fair premium, it is easiest to separate it into two parts. After the coverage period, i.e., $t > T$, the value will be independent of calendar time and depends only upon x and the stock of unsettled claims, C. The dynamic pricing equation (20) is separable in x and C. It is intuitively clear that the value must be linearly homogeneous in C and must have a solution of the form $v(x, C) = g(x)C$.

Substituting this into (20) yields

$$\tfrac{1}{2} \sum_i \sigma_i^2 x_i^2 g_{ii} + \sum_i (m_i - \lambda_i\sigma_i)x_i g_i - (r + \theta)g + \theta q = 0 \tag{21}$$

and solving we obtain

$$g(x) = g_0 \prod_i x_i^{q_i} + \text{complementary solution} \tag{22}$$

where

$$g_0 \equiv -\theta q_0 \left[\tfrac{1}{2} \sum_i \sigma_i^2 q_i(q_i - 1) + \sum_i (m_i - \lambda_i\sigma_i)q_i - (r + \theta)\right]^{-1} \tag{23}$$

We discard the complementary solution (to the homogeneous equation) by considering the value of remaining cash outflows in the case of zero inflation (*i.e.*, $q_i = 0$ for all i).[10]

Our task now is to relate this solution to the solution in the region where policies are still liable for new claims: $t \in [0, T]$. Clearly the stock of existing claims at time t, C, is "sunk" in the economist's way of thinking, and may be separated from the value. Hence, we may write

$$V(x, C, t) = v(x, C) + h(x, t) \tag{24}$$

where $h(x, t)$ denotes the value attached to remaining claims to be filed from t to T (and is independent of C since new claims arise independently of claims already filed). This latter, value, in a risk neutral world, would simply be the discounted expected value of the future claims. In a world with risk premia, it is still the discounted expected value, but with the state variables assumed to move at a drift rate that is net of the risk premia (see Cox, Ingersoll, and Ross [1]). The cost of a marginal increment to the claim stock, $v_c(x, C)$, is simply $g(x)$. Since accidents

[10] The solution to the homogeneous equation from (21) is

$$a_0 \Pi_i x_i^{a_i}$$

where a_0 is an arbitrary parameter and the parameters a_i must satisfy the constraint

$$\tfrac{1}{2} \sum_i \sigma_i^2 a_i^2 + \sum_i (m_i - \lambda_i\sigma_i - \sigma_i^2)a_i - (r + \theta) = 0$$

For $t > T$, $C_{t+s} = C_t e^{-\theta s}$, so that when $q_i = 0$ for all i in (16) (and, thus, $q = q_0$ with zero inflation), it follows that

$$v(x, C) = \int_0^\infty \theta q_0 C e^{-(r+\theta)s}\, ds = \frac{\theta q_0 C}{r + \theta}$$

However, $q_i = 0$ for all i also implies $g_0 \Pi_i x_i^{q_i} = \dfrac{\theta q_0}{r + \theta}$ so that $v(x, C) = g(x)C$ can only be satisfied if $a_0 = 0$ in the complementary solution. For the same solution to (21) to govern the case of zero inflation as well as the general case, therefore, requires $a_0 = 0$, so that the complementary solution vanishes.

occur at the rate α, we have, using (15) and (16),

$$h(x, t) = \left[E \int_t^T \alpha(x_\tau) v_c(x_\tau, 0) e^{-r(\tau - t)}\, d\tau\right]$$

subject to $d\tilde{x} = (m - \lambda\sigma)x\, dt + \sigma x\, d\tilde{z}$

$$= \alpha_0 g_0 E\left[\int_t^T \textstyle\prod_i (x_i)^{\alpha_i} \prod_i (x_i)^{q_i} e^{-r(\tau-t)}\, d\tau\right]$$

$$h(x, t) = \alpha g[\textstyle\sum_i (m_i - \lambda_i\sigma_i)(q_i + \alpha_i) + \tfrac{1}{2}\sigma_i^2(q_i + \alpha_i)(q_i + \alpha_i - 1) - r]^{-1}$$

$$\cdot [e^{(\sum_i (m_i - \lambda_i\sigma_i)(q_i+\alpha_i) + 1/2\sigma_i^2(q_i+\alpha_i)(q_i+\alpha_i-1) - r)(T-t)} - 1] \tag{25}$$

Hence, at $t = 0$, we have for the fair premium, using (16), (22), and (23),

$$P_0 = V(x, 0, 0) = v(x, 0) + h(x, 0) = g(x)\cdot 0 + h(x, 0) = h(x, 0)$$

$$= \theta\alpha q[\textstyle\sum_i \{(m_i - \lambda_i\sigma_i)(q_i + \alpha_i) + \tfrac{1}{2}\sigma_i^2(q_i + \alpha_i)(q_i + \alpha_i - 1)\} - r]^{-1}$$

$$\cdot[(r + \theta) - \tfrac{1}{2} \textstyle\sum_i \sigma_i^2 q_i(q_i - 1) - \sum_i (m_i - \lambda_i\sigma_i)q_i]^{-1}$$

$$\cdot [e^{(\sum_i [(m_i - \lambda_i\sigma_i)(q_i+\alpha_i) + 1/2\sigma_i^2(q_i+\alpha_i)(q_i+\alpha_i-1)] - r)T} - 1] \tag{26}$$

As a check on our result, notice that if there is no uncertainty ($\sigma_i = 0$ for all i), the accident rate is constant ($\sum_i m_i\alpha_i = 0$), and the inflation rate is $\pi = \sum_i m_i q_i$ then (26) yields

$$P_0 = \frac{\theta\alpha_0 q_0}{(\rho + \theta)}\left[\frac{1 - e^{-\rho T}}{\rho}\right]$$

as we had before in Equation (9).

With uncertainty, the price index for the average loss follows the diffusion

$$\frac{d\tilde{q}}{q} = \left[\textstyle\sum_i \left(\frac{1}{2}\sigma_i^2 q_i(q_i - 1) + q_i m_i\right)\right] dt + \textstyle\sum_i q_i\sigma_i\, d\tilde{z}_i$$

$$\equiv \pi\, dt + \sigma_q\, d\tilde{z}_q \tag{27}$$

where π is the expected inflation rate for losses.

If we define the dollar frequency rate for accidents as $\ell = \alpha q$, then

$$\frac{d\tilde{\ell}}{\ell} = \left[\textstyle\sum_i \left(\frac{1}{2}\sigma_i^2(\alpha_i + q_i)(\alpha_i + q_i - 1) + (\alpha_i + q_i)m_i\right)\right] dt + \textstyle\sum_i (\alpha_i + q_i)\sigma_i\, d\tilde{z}$$

$$\equiv \pi_\alpha\, dt + \sigma_\alpha\, d\tilde{z}_\alpha \tag{28}$$

Substituting (27) and (28) into (26) we obtain a remarkably simple expression analogous to the result for the riskless case (9).

$$P_0 = \left(\frac{\theta\alpha q}{\rho + \theta}\right)\left[\frac{1 - e^{-\rho_\alpha T}}{\rho_\alpha}\right] \tag{29}$$

where

$$\rho \equiv r - \pi + \sum_i \lambda_i \sigma_i q_i,$$

and

$$\rho_\alpha \equiv r - \pi_\alpha + \sum_i \lambda_i \sigma_i (q_i + \alpha_i),$$

are risk adjusted real rates.

We can now verify that our previous analysis of the influence of past and anticipated inflation carries over to the uncertain world. If the inflation rate, π, increases and real rates remain unaltered, then the competitive premium is unaltered. Similarly, if the real accident frequency is unaffected by the price level, then the premium will rise with the price level. In the special case when $\alpha_i = 0$ (the accident frequency is held constant), we have $\rho_\alpha = \rho$.

What, then is the influence of risk on the competitive premium? It is easiest to see the comparison with the single period beta effect by simplifying our assumptions.

Suppose that there is only one source of uncertainty, x_1, which influences inflation and the accident rate and which is priced in the market place—i.e., for which $\lambda_1 > 0$. For comparison purposes, $\frac{d\tilde{x}_1}{x_1}$ could be the return on the market portfolio. Now, the risk premia for inflation and losses are of the form $\lambda_1\sigma_1 q_1$, and $\lambda_1\sigma_1(q_1 + \alpha_1)$, respectively. From (27) and (28) we have that

$$\beta_q \equiv \frac{\operatorname{Cov}\left(\frac{d\tilde{q}}{q}, \frac{d\tilde{x}_1}{x_1}\right)}{\operatorname{Var}\left(\frac{d\tilde{x}_1}{x_1}\right)} = q_1 \tag{30}$$

and

$$\beta_\ell \equiv \frac{\operatorname{Cov}\left(\frac{d\tilde{\ell}}{\ell}, \frac{d\tilde{x}_1}{x_1}\right)}{\operatorname{Var}\left(\frac{d\tilde{x}_1}{x_1}\right)} = \alpha_1 + q_1 \tag{31}$$

Hence, letting $\lambda_1 = (E_m - r)/\sigma_1$, we have

$$\rho = r - \tau + (E_m - r)\beta_q$$

and

$$\rho_\alpha = r - \pi_\alpha + (E_m - r)\beta_\ell$$

Notice from (29) that each beta will influence the fair premium separately, and that if the betas are negative, then the risk adjustment raises the competitive premium.

It is also easy to compute the expected total undiscounted loss to get the competitive premium in terms of that loss and the resultant underwriting profit.

Without discounting (i.e., $r = 0$ so $\rho = -\pi$ and $\rho_\alpha = -\pi_\alpha$) and with $\lambda_\iota = 0$, (29) gives this loss,

$$UDL \equiv E\left\{\int_0^\infty \theta q C\, dt\right\}$$

$$= \left(\frac{\theta \alpha q}{\theta - \pi}\right)\left[\frac{e^{\pi_\alpha T} - 1}{\pi_\alpha}\right] \tag{32}$$

and the underwriting profit is

$$U = \frac{P_0 - UDL}{P_0}$$

$$= 1 - \left(\frac{\rho + \theta}{\theta - \pi}\right)\left[\frac{e^{\pi_\alpha T} - 1}{\pi_\alpha}\right]\left[\frac{1 - e^{-\rho_\alpha T}}{\rho_\alpha}\right]^{-1}$$

$$\approx -\left(\frac{r}{\theta - \pi}\right)\left[1 + \frac{1}{2}(\theta + \rho)T\right] - \left(\frac{E_m - r}{\theta - \pi}\right)\left[\beta_q + \frac{1}{2}(\theta + \rho)T\beta_\ell\right] \tag{33}$$

where, again, approximations are by simple Taylor series expansion. This final result makes the risk adjustment to the underwriting profit transparent. If ρ is small and if the inflation beta, β_q, together with the dollar accident beta, β_ℓ, weighted by the average number of claims per policy is negative, then the underwriting profit will be adjusted upward along with the fair premium, above the riskless level.

V. Conclusions

To summarize our findings, we have basically shown that since the property-liability insurance contract gives payouts in nominal dollars, the competitive premium is insulated against changes in the nominal rate through changes in the anticipated inflation rate. In other words, the premium is affected by inflation only in so far as real rates of interest are impacted. Of course, over time this same analysis will imply that since competitive premia are denominated in current dollars, they will rise with inflation.

The introduction of risk means that the policyholder receives a contract whose risk-free value is modified by the betas of both the accident rate and the inflation rate. The competitive premium will vary inversely with these betas since the rate of return to the policyholder will increase as they are increased. This simply says that if insurance losses are a hedge against systematic economic risk (i.e., betas are negative) then fair premiums will be higher than discounted expected losses, thereby reflecting the increased value of the insurance contract. Further theoretical research should examine these effects with an explicitly stochastic interest rate—this would be an important extension of the present work on the impact of inflation.

Empirically testing and implementing this theory requires estimation of the claim delay, of the stochastic structure of the average accident frequency, and of the average inflation rates for losses by insurance lines. These can then be used to determine the relationship between actual premia and market determined premia, and among other uses, they would provide a test of the competitive structure of the industry.

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