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# Stochastic Dominance: A Note

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IN THE LATE 1960s and 1970s, the Stochastic Dominance (SD) efficiency analysis framework was extensively developed and applied mainly to portfolio selection problems. The three basic decision criteria in this framework are the First, Second, and Third Degree Stochastic Dominance rules (hereafter FSD, SSD, and TSD, respectively). In most previous proofs of FSD, SSD, and TSD, the utility function is assumed to be once, twice, and triple continuous and differential, respectively.

In this note we relax most of the continuity assumption imposed in previous derivations. However, this note contributes not only to the extension of SD rules to a wider set of utility functions, but also to a better understanding of the economic rationale of the SD rules.

## I. The SD Rules

The three main SD rules are as follows:

*Theorem*: $F(z)$ is preferred over $G(z)$ by all investors with $u \in U_i (i = 1, 2, 3)$ if and only if:

a. $[G(z) - F(z)] \geqslant 0$ for all $z$ with a strict inequality for at least one $z$, and $u \in U_1$ where $U_1$ is the set of all $u$ with $u' \geqslant 0$;

b. $\int_o^z [G(t) - F(t)]\, dt \geqslant 0$ for all $z$ with a strict inequality for at east one $z$ and $u \in U_2$ where $U_2$ is the set of all $u$ with $u' \leqslant 0$, and $u'' \leq o$; and

c. $\int_o^z \int_o^v [G(t) - F(t)]\, dt\, dv \geqslant 0$ for all $z$, and $E_F(x) \geqslant E_G(x)$ with at least one strong inequality, and $u \in U_3$ where $U_3$ is the set of all $u$ with $u' \geqslant 0$, $u'' \leqslant 0$ and $u''' \geqslant 0$.

The criteria stated in a, b, and c above are called in the literature FSD, SSD, and TSD rules, respectively. All previous proofs of these rules assume in all the cases that $u$ and its derivatives are bounded and $E(u)$ exists.

FSD was proven by Quirk and Saposnik [7], Fishburn [1], Hadar and Russell [2], and Hanoch and Levy [3]. The proofs were confined to continuous and differentiable utility functions with continuous first derivatives.

SSD was proven by Fishburn [1], Hadar and Russell [2], Hanoch and Levy [3], and Rothchild and Stiglitz [8]. These proofs assume either explicitly or implicitly that the utility functions are twice differentiable *with continuous first and second derivatives.*

TSD was proved by Whitmore [9] for continuous utility function with third-order continuous derivatives. In the next section we shall restate the SD rules

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showing that much less restrictive assumptions on the utility functions are sufficient. The set of less retrictive assumptions sheds new light on the economic rationale of SD rules.

## II. The SD Rules: An Extension

*The Main Claim:* For the three SD rules, it is sufficient to assume:

a. *For FSD,* $U_1$ is the set of all nondecreasing utility function well-defined and bounded over the domain. (There is no need to assume continuous differentiable utility with continuous first derivative);

b. *For SSD,* $U_2$ is a subset of $U_1$ where every $u$ in $U_2$ is differentiable everywhere and has a nonincreasing well-defined and bounded first derivative over the domain. (There is no need to assume that the first derivative is continuous and differentiable); and

c. *For TSD,* $U_3$ is a subset of $U_2$ such that every $u$ in $U_3$ is twice differentiable everywhere and has nondecreasing well-defined and bounded second derivative over the domain. (There is no need to assume that the second derivative is continuous and differentiable).

The proof of Part a is given in the Appendix. Since the proofs of Parts b and c are similar, they are omitted.

*Discussion:* In our proof of FSD we allow utility functions with discontinuity points. This is important when we consider indivisible consumption products for which certain threshold amounts of money are needed. Once one has this critical amount of money, the product can be purchased causing a jump in the utility.

Also note that in our proof we do not assume that the derivative of the utility function is continuous, as was assumed in previous proofs.[1]

In the case of SSD we have to assume differentiability of $u$ because if the utility function has nondifferentiability points, then even if $u''(z) < 0$ at all other points, we may not identify the investor as risk-averse. If $u$ is nondifferentiable, we can find a fair game (and even an unfair game) which will be accepted by the investor, in spite of the fact that $u'' < 0$ at all other points of $u$, thus contradicting the assumption of risk-aversion. A simple illustration of such a situation is given in Figure 1.

The utility function in Figure 1 is monotonically increasing with $u' > 0$ and $u'' < 0$, but has one discontinuity point at $C$ where of course $u$ is also not differentiable. A fair lottery between prizes $oz_1$ and $oz_2$ is given. $Eu$ of the lottery is greater than $u(\bar{z})$, where $\bar{z}$ is the average outcome of the lottery. Therefore, the investor will be willing to pay up to the amount $A$ which is greater than $\bar{z}$ in order to participate in this lottery. Namely, he will accept an unfair game, and this behavior is inconsistent with risk-aversion.

In order to gain more insight into the SSD rule, consider the well-known

[1] The semiquadratic utility function is a continuous utility function with noncontinuous derivative. For further details on the semivariance criterion and the semiquadratic utility function, and their application to portfolio selection, see H. Markowitz [5], Hogan and Warren [4], Porter [6].

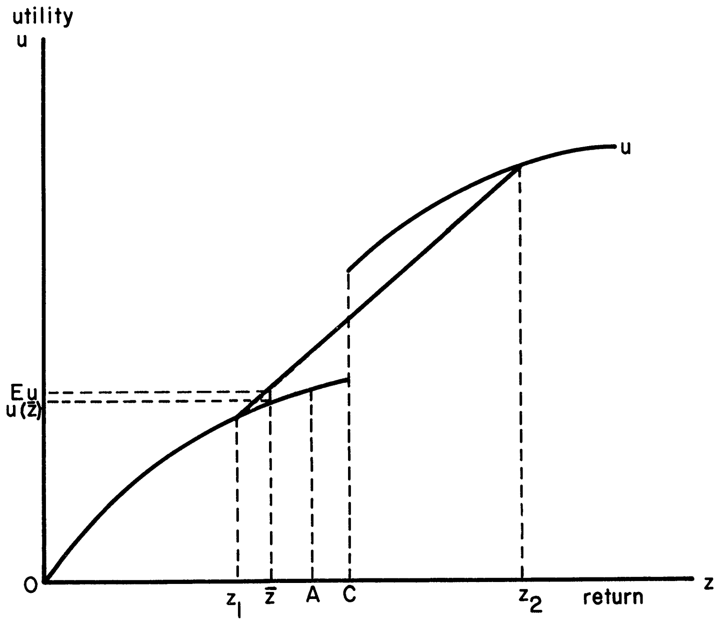


**Figure 1.** A Discontinuous Utility Function.

equation

$$\Delta \equiv E_F(u) - E_G(u) = \int_a^b [G(t) - F(t)]u'(t)\,dt \tag{1}$$

We see that $\Delta$ is equal to the product of $u'$ and the areas under $(G\text{–}F)$. In order to have dominance of $F$ over $G$ by SSD, it is necessary that the areas are positive in the beginnings, although they may be subsequently followed by negative areas. Since $u'$ is nonincreasing, the early positive areas are multiplied by higher marginal utilities than the following negative areas. Recalling that the SD rule requires that the cumulative area is positive up to every point, the products of the areas by the marginal utility also must be positive. Namely, if the SSD criterion holds, multiplication of the $(G\text{–}F)$ cumulative area by a decreasing $u'$ necessarily yields a positive term implying that $\Delta \geqslant 0$. Note that if $u$ is not differentiable, it may be that $u'$ is not everywhere decreasing. Hence, some negative areas will be multiplied by a large value of $u'$, thus causing $\Delta$ to be negative.

In proving TSD, one also has to assume, in addition to differentiability of $u$, that $u'$ is differentiable and $u''$ is bounded and well defined over the domain. The requirement that $u'$ is differentiable for TSD may be explained as follows:

If $u'$ is differentiable, one can use second integration by parts, and write (1) as follows:

$$\begin{aligned}\Delta &\equiv E_F(u) - E_G(u) \qquad (2)\\ &= u'(b)\int_a^b [G(t) - F(t)]\,dt - \int_a^b \int_a^v u''(t)[G(t) - F(t)]\,dt\,dv\end{aligned}$$

According to the TSD rule, the mean of $F$ is greater than or equal to the mean of $G$; thus, $\int_a^b [G(t) - F(t)]\, dt \geqslant 0$. In addition, according to TSD, for every $z$ we have $\int_a^z \int_a^v [G(t) - F(t)]\, dt\, dv \geqslant 0$. This inequality necessarily implies that $\int_a^b \int_a^v - u''(z)[G(t) - F(t)]\, dt \geqslant 0$ only if $-u''(z)$ is positive and nonincreasing. It is easy to show that one should assume $u''' \geqslant 0$, as well as differentiability of $u$, in order to guarantee nonincreasing $-u''(z)$. If $u'(z)$ is not differentiable at all points, one can construct an example in which the integral conditions of TSD for dominance of $F$ over $G$ does hold. Nevertheless, an investor with $u' \geqslant 0$, $u'' < 0$, and $u''' \geqslant 0$ prefers $G$ over $F$, due to the existence of a nondifferentiability point of $u'$.

## Appendix

### *Proof of Part a of the Main Claim*

The necessity proof of FSD is based on a counter example and for that the previous proofs discussed in Section I hold. Thus we have to prove sufficiency, namely if the rule holds then:

$$\Delta \equiv E_F(u) - E_G(u) \equiv \int_a^b [f(t) - g(t)]u(t)\, dt > 0 \quad \text{for every} \quad u \in U_1 \tag{A.1}$$

where $F$, $G$, $f$, $g$ are the cumulative distributions and density functions of $X$ and $Y$, respectively. As $u$ is well defined and bounded over the domain, there are only finite nondifferentiability points $c_1, c_2, \cdots, c_n$ where $c_i > c_j$ for $i > j$. Also denote $u(c_{i-})$ and $u(c_{i+})$ as the limits from left and right of $u(c_i)$ respectively. The integral in (A.1) can be decomposed into $(n + 1)$ components over differentiable domains to obtain:

$$\Delta = \sum_{i=0}^n \int_{c_{i+}}^{c_{(i+1)-}} ((f(t) - g(t))u(t)\, dt \tag{A.2}$$

where $c_{0+} \equiv a$ and $c_{(n+1)} \equiv b$. Integrating by parts in each differentiable segment we obtain:

$$\Delta = \sum_{i=0}^n \left( [F(t) - G(t)]u(t) \Bigg|_{c_{i+}}^{c_{(i+1)-}} - \int_{c_{i+}}^{c_{(i+1)-}} [F(t) - G(t)]u'(t)\, dt \right) \tag{A.3}$$

$F$ and $G$ are cumulative distributions functions and thus $F$ and $G$ are well defined at each point. Even for discrete variables, this function is continuous from the left. Hence $F(c_{i-}) = F(c_{i+}) \equiv F(c_i)$ also $G(c_{i-}) = G(c_{i+}) \equiv G(c_i)$, and we can rearrange (A.3) to obtain.

$$\begin{aligned}\Delta = &- [F(c_0) - G(c_0)]u(c_{0+}) + [F(c_{n+1}) - G(c_{n+1})]u(c_{(n+1)-}) \\ &+ \sum_{i=1}^n [F(c_i) - G(c_i)](u(c_{i-}) - u(c_{i+})) \\ &+ \sum_{i=0}^n \int_{c_{i+}}^{c_{(i+1)-}} [F(t) - G(t)]u'(t)\, dt \end{aligned} \tag{A.4}$$

where $c_0 \equiv a$ and $c_{n+1} \equiv b$. By definition $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Thus, the first two expressions on the right in (A.4) vanish. The sum of the integrals on the right in (A.4) is the integral over the whole domain $[a, b]$. Thus (A.4) can be written as follows:

$$\Delta = \sum_{i=1}^{n} [F(c_i) - G(c_i)](u(c_{i-}) - u(c_{i+})) + \int_a^b [G(t) - F(t)]u'(t)\, dt \quad \text{(A.5)}$$

where $u$ is nondecreasing, $u'(t) \geqslant 0$, and $(u(c_{i-}) - u(c_{i+})) \leqslant 0$. Thus, $F(t) \leqslant G(t)$ for all $t$ guarantees that each expression under the summation sign is nonnegative and the integrand is nonnegative as well. Thus, $\Delta$ is nonnegtive, implying that the expected utility of $F$ is no smaller than the expected utility of $G$.

## REFERENCES


1. P. C. Fishburn. *Decision and Value Theory*. New York: Wiley, 1964.
2. J. Hadar and W. R. Russell. "Rules of Ordering Uncertain Prospects." *American Economic Review* 59 (1969), 25–34.
3. G. Hanoch and H. Levy. "The Efficiency Analysis of Choices Involving Risk." *Review of Economics Studies* 36 (1969), 35–46.
4. W. W. Hogan, and J. M. Warren. "Computation of the Efficiency Boundary in E-S Portfolio Selection Model." *Journal of Financial and Quantitative Analysis* 7 (1972), 1881–96.
5. H. M. Markowitz. *Portfolio Selection*. New York: Wiley, 1959.
6. R. B. Porter. "Semi-Variance and Stochastic Dominance: A Comparison." *American Economic Review* 64 (1974), 200–204.
7. J. P. Quirk and R. Saposnik. "Admissibility and Measurable Utility Functions." *Review of Economic Studies* 29 (1962), 140–46.
8. M. Rothschild and J. E. Stiglitz. "Increasing Risk: I. Definition." *Journal of Economic Theory* 2 (1970), 225–43.
9. G. A. Whitmore. "Third Degree Stochastic Dominance." *American Economic Review* 60 (1970), 457–9.