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Optimal Sequential Investment When Capital is Not Readily Reversible

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ABSTRACT

When investment opportunities arrive one at a time and are reviewed sequentially, a corporation's optimal policy differs from a standard net present value rule if the corporation exercises control over an industry state variable and control is costly. The first condition presupposes a degree of market power for the firm; the second occurs if corporate investment decisions are imperfectly reversible.

To address the problem of optimal investment in this context, a firm's investment decisions are modeled as a Markov reward process. The causes of economic irreversibility are discussed and general propositions concerning the optimal investment policy are derived. These propositions are then applied to the optimization of an exploration program by an oligopolistic firm (a price leader). Under particular demand and distributional assumptions, solutions for the optimal decision rule and the value of the exploration program are obtained and their properties examined.

CURRENT INVESTMENT MAY AFFECT future opportunities by creating "growth opportunities" (as defined by Myers [16]; see also Spence [25]), or by reducing the value of an existing opportunity set. Despite the known potential for these interactions, capital budgeting techniques such as net present valuation, scenario planning, and Monte Carlo simulation focus exclusively on the "stand alone" value of an investment. The capacity for modeling interactions is present in the multiperiod models of Merton [16], Bogue and Roll [5], and others.[1] The absence of applications however, is evidence that actual economic interactions between current and future opportunities are not well understood.

This paper develops a model of a firm facing a sequence of interdependent growth opportunities. Opportunities have positive net present values because the firm exercises market power in its product market; current investment changes the value of future opportunities through its effect on a product market variable. The model is used to examine the impact of demand, industry structure, and technology on optimal investment rules and on the value of aggregate growth opportunities.

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[1] See, for example, Cox, Ingersoll, and Ross [7], Richards and Sudaresen [20], Constantinides [6].

A small but significant literature exists on the interaction of current decisions and future opportunities. Elton and Gruber [10] demonstrated that the required marginal rate of return depends on the evolution of the investment path over time. Aivasian and Callen [1] considered the effect of market structure on optimal investment: they show that a monopolist or Cournot oligopolist optimally sets the marginal rate of return higher (lower) than the cost of capital if future investment opportuniies are a decreasing (increasing) function of current investment. However, A-C do not specify how future opportunities become a function of present investment.

This paper extends previous research in several ways. First, it introduces a model of the dynamic arrival and sequential review of opportunities. Future investment opportunities are uncertain as to timing and value, and past investments may affect the value of future investments. It is assumed that each firm seeks to maximize the present value of its sequence of opportunities.

A corporation's optimal policy differs from the standard net present value rule (accept if direct NPV > 0)[2] if the corporation exercises control over an industry state variable, but the control is costly. The first condition presupposes the firm operates in an imperfectly competitive industry. Since a competitive firm cannot affect future industry prices or technology, its impact on future state variables is negligible. The competitive firm optimally evaluates sequential opportunities independently of one another and neglects impacts on industry prices and output.

The absence of state transition costs also implies opportunities may be evaluated independently of one another. If transitions are costless then the firm can always achieve the highest value state instantaneously and at zero cost. The impact of investment on the state variable is irrelevant to the firm's decision to accept an opportunity.

On the other hand, if state changes are costly, a particular investment may encourage or discourage access to a higher value state. The former occurs when there are economies of scale in research, production (e.g., the learning curve) or market penetration. This effect was first identified by Lintner [14] and models of this type have been recently studied by Spence [25] and Gilbert and Harris [11].

Investment may also cause transitions to lower valued states: for example, adding new capacity and increasing output may lead to a decrease in price. If an investment results in a transition to a less advantageous state (e.g., one with lower prices), and regaining the initial state is costly, then the investment is termed "irreversible." As discussed in Section II, irreversibility may be caused by technological or environmental factors or by the relative cost advantage of sunk capital over new investment. Irreversibility may be permanent if the initial state is never regained or temporary if the initial state is regained after some time. Whatever its origin or duration, the negative impact of irreversibility on the firm's future opportunities is relevant to investment decisions, and appropriate

[2] Among simultaneously arriving, mutually exclusive investments, the standard rule is to choose the one having the highest NPV. Although some of the methods of Section II can be extended to the more general case, the problem of simultaneous, mutually exclusive investments having differential impacts on future opportunities is not treated in this paper.

adjustments for irreversibility should be incorporated into capital project evaluations.

This paper develops a model of irreversibility within a dynamic sequence of investment opportunities. The model permits opportunities to be explicitly related to demand and to competitive characteristics of the product market. The techniques for introducing demand, technology, and product market assumptions into the basic framework of the model are illustrated in Section II. An important product of the model is a formula for valuing a firm's growth opportunities given the optimal decision rule.

Section I introduces a general model of dynamic investment in a sequence of valuable opportunities. Sources of economic irreversibility are discussed and general propositions concerning the optimal investment policy are derived. Section II applies these propositions to the optimization of an oligopolistic firm's exploration and investment program. Under particular demand and distributional assumptions, closed-form solutions for the optimal decision rule and the value of the exploration program are derived and their properties examined. Section III presents conclusions.

I. A General Model of Imperfectly Reversible Investment

Consider an industry characterized by one state variable Q which could be capacity or output. In principle, Q may be a continuous or a discrete variable; to simplify the exposition I shall treat Q as discrete:

$$Q = 0, 1, 2, 3 \cdots.$$

I assume Q is established in the following manner: a single firm or several firms is engaged in a search for productive opportunities. These might be new oil wells or mineral deposits, new products or design improvements for existing products, new technologies, etc. Opportunities, *if encountered and accepted* result in both a reward $r(\cdot;\, Q)$ to the firm which captures the opportunity, and a change in the state of the industry from Q_i to Q_j with probability p_{ij}.

The reward $r(\cdot;\, Q)$ is defined as the direct net present value of an investment opportunity: it is the amount which could be obtained upon sale of the investment in the capital markets, or equivalently, the amount which is added to the value of the firm's assets in place as a result of taking the opportunity. I assume that capital markets are competitive, frictionless, and fully informed, and that conflicts of interest between debt and equity holders are not present.[3]

Successive rewards are assumed to be drawn from a stationary distribution conditioned on the state variable:

$$F(r(\cdot;\, Q)) = F(\cdot \mid Q)$$

The objective of every firm is assumed to be maximization of the market value of

[3] Thus firms will not forgo profitable investment opportunities because of conflicts of interest between various claimants (see Myers [17]). The firm here can be thought of as having an all-equity capital structure.

its investment program under uncertainty as to the sequence of opportunities each will encounter and the sequence of future industry states.

A firm exercises control over its investment program in that at the time an opportunity is encountered and the reward of the net present value is revealed, the firm which "owns" the opportunity may reject it: in this case the firm gets no reward, and the state of the industry remains unchanged. Irreversibility becomes a consideration for optimization, if, having accepted the opportunity and initiated the change of state, the firm either cannot return to the initial state immediately, or would incur some cost in doing so. If the destination state's value is less than the initial state's, and if return is costly, then the firm may optimally forgo some opportunities with positive net present value in order to avoid the change in state.

There are two ways investment may result in transitions to lower valued states. First, investment may permanently alter the environment or deplete a resource in fixed supply.[4] Second, if capital cost is "sunk," it may not be cost effective to replace old capital with new.[5] New capital investments will then cause increases in output. If the firm faces a downward sloping demand curve, an increase in output causes price to fall and reduces the value of future opportunities. However, a return to the previous output level is costly, because it requires productive capacity to be shut down, with a resulting loss of profits on the foregone output.

Within an industry characterized by a sequential search for investments, the optimization problem of each firm can be formulated as a Markov decision process. Define $v_Q(s)$ as the value of a particular firm's future investments when the industry is in state Q with time s remaining to the horizon, assuming the firm follows an optimal policy from s onward. Bellman's equation yields:

$$v_Q(s + \Delta s) = E \max_{a \in A} \{r(a; Q) + \beta \textstyle\sum_i p_{Qi}(a) v_i(s)\} \tag{1}$$

where

$A \equiv$ the set of actions open to the firm (a is an element of A);
$r \equiv$ the firm's reward, given action a;[6]
$p_{Qi}(a) \equiv$ the conditional probability of a transition from state Q to state i, given action a;

[4] Models of irreversible change in the environment are considered by Arrow [2], Henry [12], and others. Models of resource depletion are considered in Levhari and Pindyck [13], Pindyck [18], [19], Stiglitz [27], Stewart [26], and others. Models of resource depletion and renewal are found in Baldwin and Meyer [3] and Deshmukh and Pliska [9].

[5] To illustrate sunk cost irreversibility, consider a firm which can produce output Q at cost c_o. Assume a better technology makes the same production possible for $c_N < c_o$. If no investment has been made in the c_o technology, or, if c_o consists only of short run operating costs, c_N is the technology of choice.

However, suppose $c_o = s + o$ where s is the annualized cost of sunk capital and o the operating cost. The new technology will replace the old only if $c_N < o$, that is, only if the total cost of the new opportunity is less than the ongoing (variable) cost of the old. If $c_N > o$ with probability 1, the first investment is effectively irreversible. Old capacity will not be shut down to make way for new technology and new investments will cause at least temporary increases in output.

[6] In general, $r(a; Q)$ may be an expected reward conditioned on action a.

$\beta \equiv$ a discount factor, i.e., $\beta = e^{-\alpha\Delta s}$ where α is an appropriate discount rate;[7] and

$E \equiv$ the expectations operator.

Expectations are taken over the probability of controlled and uncontrolled transitions in the interval Δs (for simplicity these events are assumed to be mutually exclusive) and the reward distribution applicable in state Q.

Although (1) is written in terms of discrete time intervals and a finite horizon, no problems are posed in formulating the problem as a continuous time Markov process or with an infinite horizon. As a matter of formal accounting which becomes important for an infinite horizon, I assume that a reward gained in the interval Δs is distributed to the shareholders, or, more realistically, added to the firm's assets in place. Either way, values of past investments do not enter into the forward-looking value v_Q.[8]

This formulation assumes that current and future investment decisions will not affect the value of a firm's assets-in-place. Although in certain cases this assumption would not be appropriate, the effects of investment on future opportunities are conceptually distinct from effects on assets in place (even though both effects might arise within the same firm). To maintain this distinction and minimize confusion, this paper focuses on investments' impact on future opportunities.[9]

It is assumed that, if an opportunity arrives between time $s + \Delta s$ and s, the firm's actions are constrained to simply accepting or rejecting the opportunity. Let $a = 0$ denote a rejection and $a = 1$ an acceptance. Then from Equation (1), the optimal decision rule is of the form:

$$a^*(Q, s) = \begin{cases} 1 & \text{if} \quad r(Q) > \sum [p_{Qi}(0) - p_{Qi}(1)]\beta v_i(s) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

By previous assumption, $P_{QQ}(0) = 1$, that is, if an opportunity is rejected, no transition occurs. The more general notation in (2) reflects the possibility that

[7] It is assumed that the discount factor β (therefore α) is a constant through time. However, proofs of Propositions 1–3 go through equally well if β is state dependent as long as

$$0 < \beta(Q) < 1$$

In the model developed in Section III, industry actions and risks vary with the state variable and the discount rate in general would be a state-dependent quantity. However, with a state-dependent discount factor, a general equilibrium asset pricing model would be necessary to close the system. A complete model of this type would be much less tractable.

[8] If the value of assets in place grows at the required rate of return α, that value could be incorporated in v_Q. In this case, assuming no dividends, optimization would take place with respect to the firm's terminal value (or in the infinite horizon limit, the growth rate). Both procedures yield the same optimization rule; however, the method adopted here is notationally simpler since values of assets-in-place do not have to be carried from period to period in the dynamic program.

[9] In some cases (e.g. a monopolist), joint effects of investment on future opportunities and assets-in-place can be treated with the framework of (1). Propositions 1–3 apply to those cases. In other cases, treatment of joint effects requires that the state space be expanded to two or more variables (as, for example, when industry behavior depends on total output and on the distribution of capacity among competing forms). Industry dynamics and optimization rules are then more complex.

the act of rejecting an opportunity changes transition probabilities. In most applications this assumption is not necessary, but the more general notation will be used for the moment.

Equation (2) implies that for every state-date combination (Q, s), there is an optimal "break-even reward":

$$r^*(Q, s) = \sum_i [p_{Qi}(0) - p_{Qi}(1)]\beta v_i(s) \tag{3}$$

Under the optimal policy, opportunities bringing in more than $r^*(Q, s)$ should be accepted, while opportunities bringing less than $r^*(Q, s)$ should be rejected.

$r^*(Q, s)$ can be thought of as a "net present value premium" or discount. If positive, it is an extra amount above zero a project must bring to be acceptable to the firm. It can be seen from the fact that (3) is a difference of discounted expected values that $r^*(Q, s)$ directly measures the expected impact of an individual opportunity on the value of future opportunities the firm will encounter.

The remainder of this section establishes certain useful relationships among r^*_Q, v_Q, and the reward distribution, under economically meaningful restrictions on industry dynamics. Section II applies these results to the optimization and valuation of a sequence of irreversible investments for a firm in an oligopolistic product market.

Proposition 1 relates the NPV premium $r^*(Q, s)$ to the firm's degree of control over the outcome of its actions. Action $a = (0, 1)$ is defined as "*focused*" if the consequence of acceptance is a transition from state Q to state i with probability 1, and the consequence of rejection is no transition. An action a is described as focused and "*incremental*" if acceptance results in a transition $Q \rightarrow Q + 1$ with certainty, and rejection causes no transition.

Proposition 1. (i) If action a taken in state Q is focused,

$$r^*_Q = \beta[v_Q - v_i] \tag{4}$$

where Q is the initial state and i is the destination state;

(ii) If action a taken in state Q is focused and incremental,

$$r^*_Q = \beta[v_Q - v_{Q+1}] \tag{5}$$

Proof. Immediate from (3).

Proposition 2 relates the value of future opportunities v_Q to the optimal net present value premium r^*_Q. Using standard terminology, the state space $\{Q\}$ is defined as "*bounded*" if the number of states is finite or (given a continuous state space) can be represented as a finite segment of the real line. State $N \in \{Q\}$ is called "*absorbing*" if the probability of a transition from N to any other state is 0:

$$P_{Ni} = \begin{Bmatrix} 1 & \text{if} & i = N \\ 0 & \text{if} & \text{otherwise} \end{Bmatrix}$$

The Markov process defined for v_Q on state space $\{Q\}$ is termed "*recurrent*" if there is no absorbing state.

Proposition 2. (i) If for all Q, actions are focused and incremental, and N is an absorbing state, then

$$v_Q(s) = \frac{1}{\beta} \sum_{i=Q}^{N-1} r_i^*(s) + v_N(s) \tag{6}$$

(ii) If actions are focused and incremental, $\{Q\}$ is bounded, and v_Q recurrent, then

$$v_Q(s) = \frac{1}{\beta} \sum_{i=Q}^{N-1} r_i^*(s) + \frac{1}{1-\beta} \sum_{j=0}^{N-1} r_j^*(s) \sum_{k=0}^{j} p_{Nk} \tag{7}$$

Proof. (i) follows from Proposition 1 by observing:

$$v_Q = (v_Q - v_{Q+1}) + (v_{Q+1} - v_{Q+2}) \cdots -v_N + v_N$$

(ii) follows by substituting v_N from Equation (1) into Equation (6) (recognizing that no rewards or upward transitions are possible in this state). Each v_Q then appearing can be expanded into a sum of differences. Applying Proposition 1 and collecting terms obtains (7).

The process on v_Q is defined as "(strictly) *ordered*" if there exists a mapping from Q to the integers (or the real line) such that

$$v_0(s) > v_1(s) \cdots > v_Q(s) \cdots, \qquad \text{all} \quad s$$

In all cases where such an ordering exists, the state descriptor Q can be redefined to reflect the natural order. Propositions 1 (b) (on NPV premiums) and 2 (on the value of future opportunities) hold as long as actions are focused and incremental with respect to this ordering. In these cases the sign of $r^*(Q, s)$ is the same for all Q and is determined by the ordering of the v_Q's.

Proposition 3 establishes sufficient conditions for the sign of $v_Q - v_{Q+1}$ to be known. In order to state the proposition, some definitions are necessary. First, a "*local*" process is defined as one for which in time interval Δs transitions are possible only to neighboring states ($Q \to Q-1$, Q or $Q+1$). Second, the process is called "*stable*" if the probability of the system remaining in the same state over interval Δs is greater than 50%:[10]

$$P(Q(s + \Delta s) = Q(s)) \geq \cdot 5$$

Finally let:

$\lambda_Q \equiv$ the arrival rate of the firm's opportunities, conditioned on state (thus the probability of the firm in state Q encountering an opportunity in time span Δs is $\lambda_Q \Delta s$);

$R(\hat{r}, Q) \equiv \int^{\infty}_{\hat{r}}) \, r\,(\cdot \mid Q)\, \mathrm{dF}(\mathrm{r}(\cdot \mid \mathrm{Q}))$, i.e. $R(\hat{r}, Q)$ is the right expectation of $r(\cdot \mid Q)$ calculated from the point value $\hat{r}$; and

$D(\hat{r}, Q) \equiv {}_{\hat{r}}\int^{\infty} [r(\cdot \mid Q) - \hat{r}]\, dF(r(\cdot \mid Q))$, i.e. $D(\hat{r}, Q)$ is the expected excess reward (over $\hat{r}$) calculated from the point value $\hat{r}$.

[10] For a local, stable Markov process, the transition matrix is tri-diagonal, that is, all entries except those of the diagonal and upper and lower off-diagonals are zero. Row entries sum to one, and the diagonal elements are greater than or equal to .5.

The process governing v_Q will be called *"reward ordered"* if v_Q will be called *"reward ordered"* if

$$\lambda_Q R(\hat{r}, Q) \geq \lambda_{Q+1} R(\hat{r}, Q+1) \qquad \text{all} \quad \hat{r} \quad \text{and} \quad Q$$

or if

$$\lambda_Q D(\hat{r}, Q) \geq \lambda_{Q+1} D(\hat{r}, Q+1) \qquad \text{all} \quad \hat{r} \quad \text{and} \quad Q$$

with a strict inequality holding for some Q, $Q+1$.[11]

Proposition 3. Consider the Markov reward process defined in Equation (1). If the process is bounded with N communicating states, local, stable, and reward-ordered, then $v_Q \geq v_{Q+1}$ all Q. If the time remaining to the horizon $s \geq (N-2)\Delta s$, then v_Q is strictly ordered:

$$v_Q - v_{Q+1} > 0 \quad \text{all } Q \tag{8}$$

Proof. See Appendix A.

Comment. Proposition 3 does not depend on the assumption that actions are focused. However, the assumption of local transitions implies that actions are incremental except for actions which cause transitions to an absorbing state.

Proposition 3 only establishes *sufficient* conditions for v_Q to be strictly ordered: these conditions are by no means necessary. The value of Proposition 3 lies in the fact that it places weak restrictions on the magnitudes of the controlled and uncontrolled transition probabilities. Since magnitudes of controlled transition probabilities are functions of the decision rule and thus are known only after optimization, it is important to have a way of assessing the relative magnitudes of v_Q and the sign of r^*_Q that does not depend on the particular dynamics or rewards of the system. Counterexamples of Proposition 3 can be constructed if the assumptions of boundedness, local movement, or reward-ordering are lifted, thus weaker sufficiency conditions will place significantly more restrictions on both transition rates and rewards.

The assumption of stability is a technical assumption: it is designed to simplify

[11] Ordering in R does not imply ordering in D or vice versa. An example which is R-ordered, and not D-ordered is

$$r(X; Q) = P(Q) - \tilde{X}$$

P is a decreasing function of Q and X a random variable. In this case

$$R(X, Q) = P(Q)F(x) - L(x)$$

$$D(x, Q) = xF(x) - L(x)$$

where $F(x)$ is the left tail of the distribution of x and $L(x)$ is the left truncated expectation over the same distribution. D is independent of Q therefore not ordered in Q.

An example which is D-ordered and not R-ordered is

$$r(X; Q) = X(Q)$$

X uniformly distributed with standard deviation Q. Here, for small values of $\hat{r}$, $R_Q > R_{Q+1}$, but $D_Q < D_{Q+1}$ all Q.

the induction proof by excluding cyclical behavior[12] in a discrete time Markov process. From a modeling standpoint, the stability assumption is satisfied for all continuous time Markov processes, and can frequently be satisfied in discrete time models by subdividing the time interval. The restriction thus does not appear onerous.

Proposition 3 implies that for a large class of sequential economic decision processes, the ordering of values of adjacent states is determined by the state-dependent frequency of opportunities times the right truncated expectation (R) or the right excess reward function (D) of the distribution from which the opportunities are drawn. The excess reward $D(x, Q)$ is an increasing function of *both the mean and the dispersion*[13] of the reward distribution. Propositions 1 and 3 together then imply that investors will generally pay to enter or to increase the probability of entering states where opportunities are on average more frequent, more profitable, *or more variable.*

The last result is well known in the options literature, and is derived with increasing frequency in corporate investment models.[14] Proposition 3 implies that

[12] A cyclical matrix is one of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The assumption of local movement imposes a weak continuity on the process, i.e. the process is required not to move more than a short (one-step) distance in one time interval. The assumption of stability imposes a stricter type of continuity on the process, i.e. the probability of no movement over one time interval is greater than .5.

[13] Proof is found in Rothschild [23] and Venezia and Brenner [28]. Briefly, increases in a parameter Q correspond to increases in dispersion in the sense of Rothschild-Stiglitz [24] if

$$\text{(i)} \quad \int_{-\infty}^{\infty} dF(r, Q) = 1$$

$$\text{(ii)} \quad \int_{-\infty}^{\infty} F_Q(r, Q)\, dr = 0 \quad \text{and}$$

$$\text{(iii)} \quad \int_{-\infty}^{r} F_Q(r, Q)\, dr \geq 0 \qquad -\infty \leq \hat{r} \leq \infty$$

Assume (i)–(iii). Then from the definition of D:

$$D = \int_{\hat{r}}^{\infty} (r - \hat{r})\, dF(R, Q) = \int_{\hat{r}}^{\infty} r\, dF(r, Q) - \hat{r}\,(1 - F(\hat{r}, Q))$$

where $F(\hat{r}, Q)$ is the cumulative distribution function from $-\infty$ to $\hat{r}$. Integrating the first term by parts and differentiating with respect to Q yields:

$$\frac{\partial D}{\partial Q} = -\int_{\hat{r}}^{\infty} F_Q(r, Q)\, dr = \int_{-\infty}^{\hat{r}} F_Q(r, Q)\, dr \geq 0$$

The second equality follows from $R - S$ (ii), the inequality from $R - S$ (iii). An increase in Q corresponding to increased dispersion implies an increase in D.

[14] See, for example, recent papers by Bernanke [4], Venezia and Brenner [28], and Roberts and Weitzman [21]. The processes examined by these authors can usually be reinterpreted as reward-ordered processes in the sense of Proposition 3.

the value of an investment program increases with the opportunity set's variance for all investment processes exhibiting local, stable dynamics. Basically, variance increases value because the corporation's ability to accept or reject opportunities skews the distribution of investments actually undertaken in favor of high payoff outcomes. Since investments which are "losers" (as defined by the optimal policy) will be rejected, the chance to encounter a more highly dispersed sequence of opportunities takes on positive value.

II. Example: The Value of an Exploration Program

The economic interest of Propositions 1–3 arises from their application to the sequential investment decisions of firms facing definable product markets and technological constraints. To illustrate these applications this section develops a model of a firm engaged in mineral exploration.[15]

Consider a company engaged in exploration for mineral deposits in various locations. I assume the company's activity is limited to exploration and development of mine sites: once a mine is brought to the point of being productive, it is sold for the net present value of its output. The capital market is assumed to be competitive, complete, and fully informed.

The industry state variable Q is defined to be total output from all mines. Output is increased by opening new mines: firms control output increases (because they may accept or reject new mine opportunities). Output declines because existing mines run out of ore and eventually are shut down. Companies are assumed not to control output decreases: all productive mines are kept open but total output from existing mines is assumed to be stochastically declining over time.[16]

Further assumptions about the industry fall into five categories: demand; search technology; opportunity valuation; investment duration; and market structure.

Demand. The spot price of the mineral is assumed to be a declining function of industry output. For the investment process considered here (i.e. a bounded local, stable Markov process), this is sufficient to imply that $P(Q)$, the market value of the revenue stream of one mine, is also a declining function of output.[17] For

[15] A model of optimization and valuation of an exploration program is found in Roberts and Weitzman [21]. They also formulate the problem as a sequential decision process, but use different structural assumptions.

[16] Using the definitions of Section II, mines are assumed to exhibit sunk cost irreversibility. That is, operating costs of old mines are assumed low relative to the capital plus operating costs of new mines. The opportunity loss associated with lost contribution from shutting down a productive mine makes a mine investment irreversible in the short run. However, technological losses in productivity cause the industry to return eventually to lower output levels.

[17] To see this, write the value of a standard revenue stream as the sum of the current spot value of output ($P_s(Q)$) and the discounted expected value of the stream one period from now:

$$P(Q) = P_s(Q) + \beta \textstyle\sum_{i=0}^{N} p_{Qi} P(i)$$

Except for a probability that the mine shuts down next period, the transition probabilities p_{Qi} above are identical to those in the v_Q process: therefore $P(Q)$ satisfies the dynamic assumptions of Proposition 3 if v_Q does. If $P_s(Q) > P_s(Q+1)$, some Q, the $P(Q)$ process is reward ordered. The result then follows immediately from Proposition 3.

simplicity, I assume the asset value function $P(Q)$ is linear in Q:

$$P(Q) = (a - bQ)\Delta \tag{9}$$

where Δ is the output of one mine.[18]

Search Technology. The probability that a new opportunity arrives in the interval Δs is $\lambda\Delta s$ where λ is assumed independent of Q. The development cost of a new mine is a random variable $\tilde{X}$; successive $\tilde{X}$'s are drawn independently from a known, stationary distribution $F(\tilde{X})$, which does not depend on Q. For modeling purposes, $F(X)$ is assumed to be uniform on the interval $(0, a)$:

$$F'(x) = f(x) = \frac{1}{a}$$

Valuation. The net present value of a mine opportunity to the exploration company is $P(Q) - \tilde{X}$. This is the reward the company earns as a result of opening a mine:

$$r(\tilde{X}; Q) = NPV = P(Q) - \tilde{X} \tag{10}$$

Since the random component of the reward is the cost $\tilde{X}$, it is convenient to consider the expected reward, R, as a function of a maximum cost criterion; i.e. the firm's rule is to accept any opportunity whose cost is *below* a hurdle value $x\Delta$. Consistent with (9), the hurdle value has components cost per unit output x and average mine output Δ.

From (10), the right expectation of $r(X; Q)$ for given x and Q is:

$$R(x, Q) = P(Q)F(x) - L(x) \tag{11}$$

where $F(x) = \int_0^x dF$ and $L(x) = \int_0^x X\, dF$ are respectively, the left truncated distribution and expected cost functions. From (11), if $P(Q)$ declines in Q, $R(x, Q)$ declines in Q, all x; thus, *for any declining spot demand function, the exploration and investment process is reward ordered.* For linear asset demand and a uniform distribution of cost:

$$R(x, Q) = \left\{[a - bQ]\frac{x}{a} - \frac{1}{2}\frac{x^2}{a}\right\}\Delta$$

Investment Duration. All mines are assumed to have an expected life of T years (this is reflected in the asset valuation function $P(Q)$). However, in aggregate, actual mine closings and decreases in output are stochastic. At the level of the industry, the rate of decrease in output given no new openings could be estimated as a state dependent rate $\mu(Q)$. It is sensible to expect the decay rate of output to be an increasing function of output itself (this would be the case, for example, if there were a discrete number of mine sites with independent factors affecting output decreases at each site). For simplicity, I assume the (probabilistic) rate of

[18] $P(Q)$ is an equilibrium price paid by rational, fully-informed investors in a competitive asset market. It in general depends on a large number of economic variables, including the oligopolistic firm's decision rule. However, it is assumed that prices are close enough to equilibrium so that the firm is able to estimate a and b from market data. Consistency of the resulting optimal decision rule and the asset pricing relationship can then be verified from knowledge of the spot demand function and the transition probabilities. Equilibrium properties of spot and asset demand functions under various durability and market structure assumptions are the subject of ongoing research.

decay is linear in Q:

$$\mu(Q) = \mu Q$$

where μ is a constant.

Market Structure. The firm's competitors can cause output to increase by finding and opening new mines. The probability that a competitor reviews an opportunity in the interval Δs is $\delta\Delta s$; like λ, δ is assumed to be independent of Q. For purposes of this illustration, I assume that all firms in the industry draw from the same opportunity pool and that all other firms behave as price takers and will open any mine with $NPV > 0$. These assumptions imply that $\delta\bar{F}(Q)$, the probabilistic rate of uncontrolled increases in Q, is

$$\delta\bar{F}(Q) = \delta\left(1 - \frac{b}{a}Q\right)$$

Infinite Horizon Decision Rule. In modeling corporate investment it is usually not realistic to assume that the process of search and investment will be exogenously terminated at some particular future date. Of primary economic interest is the firm's optimal decision rule for an unbounded planning horizon. To solve for the optimal infinite horizon policy, I make use of the following theorem, stated here without proof:[19]

Theorem 1. Let $v(Q)$ be the present value under the optimal policy of a Markov decision process with positive rewards, positive discounting, and an unbounded planning horizon. Let $f(Q)$ be a stationary policy which when the process is in state Q selects action a so that

$$v_{f(Q)}(Q) = \max_a\{r(Q, a) + \beta \textstyle\sum_{j=0}^{\infty} P_{Qj}(a)v(j)\}$$

Then $v_{f(Q)}(Q) = v(Q)$, and $f(Q)$ is an optimal policy. (The optimal policy is not necessarily unique).

Theorem 1 states that an optimal stationary policy exists, and that a stationary policy satisfying (12) is necessarily optimal. For the particular irreversible investment process defined above, the theorem implies:

$$\begin{aligned} v_Q = {} & \lambda\Delta s(P(Q)F(x_Q^*) - L(x_Q^*)) \\ & -(\lambda\Delta sF(x_Q^*) + \delta\Delta s\bar{F}(Q))\beta(v_Q - v_{Q+1}) \\ & + \beta v_Q \\ & + \mu\Delta sQ\beta(v_{Q-1} - v_Q) \end{aligned} \tag{13}$$

where x_Q^* is the cost hurdle which maximizes the right-hand side of (13).

The first term in (13) is the firm's expected reward in the interval Δs. Its components are the probability that a new opportunity is encountered ($\lambda\Delta s$) times the expected value of the investment conditioned on the opportunity clearing the hurdle x_Q^*.

The remaining terms are the expected values of future opportunities adjusted by the discount factor β. For example, in the interval Δs, the industry may go from state Q to $Q + 1$ as a result of the firm's investment or as a result of

[19] For a proof, see S. M. Ross [22], Theorem (6.3) and Corollary (6.6).

investment by a competitor: these events have probability $\lambda\Delta s F(x_Q^*)$ and $\delta\Delta s\bar{F}(Q)$ respectively. The discounted expected change in v from this transition is the second term of (13). The third term is the discounted value of future opportunities given no change of Q. The fourth term is the discounted expected change in v resulting from a decrease in Q by one unit.

By Propositions 1 and 3 of Section I above, the maximum is attained if

$$r_Q^* = P(Q) - x_Q^*\Delta = \beta(v_Q - v_{Q+1}) > 0 \tag{14}$$

If investments are imperfectly reversible, investment results in a transition to a lower valued state ($Q \rightarrow Q + 1$) and the optimal policy is not to accept all opportunities with positive net present value. Because current investment negatively affects the value of future opportunities, the optimal policy is to establish minimum *NPV* standards; these standards vary with the state of the industry, Q. $P(Q) - x\Delta$ is monotonic in the decision variable x, therefore for any declining demand function $P(Q)$, optimal x_Q^* is unique.

Under the specific demand, distribution and market structure assumptions given above a closed-form expression for the optimal stationary policy can be obtained. For Q bounded, (13) and (14) form a system of $2N + 1$ equations in $2N + 1$ unknowns $(v_0 \cdots v_N, r_0^*, \cdots r_{N-1}^*)$. Substituting (14) into (13) and taking first differences, the system is reduced to N equations in N unknowns (the r_Q^*'s) subject to the constraint that the a solution r_Q^* is admissible as a difference between successive v_Q's. This constraint requires

$$r_Q^* > 0 \quad \text{all} \quad Q \tag{15}$$

A solution r_Q^* to the reduced system which satisfies constraint (15) is necessarily a solution to the larger system described by Equations (13) and (14). Theorem 1 implies that the stationary policy which applies hurdle r_Q^* (equivalently x_Q^*) to opportunities encountered in state Q is an optimal policy. It is already known that the optimal policy is unique.

Under the assumption of linear demand, uniform distribution of opportunities and price-taking competitors, the optimal policy is linear in Q:[20]

$$x_Q^* = (c - dQ)\Delta \tag{16}$$

where

$$d = \frac{1}{2\lambda}$$

$$\{-\alpha a - 2\mu a - 2\delta b + \sqrt{(\alpha a + 2\mu a + 2\delta b)^2 + 4\lambda b(\alpha a + 2\mu a + 2\delta b)}\} \tag{16a}$$

and

$$c = \frac{a\left\{\alpha a + \mu a + \delta b + \delta(b - d) + \frac{1\lambda}{2a}d^2\Delta - \delta\frac{b}{a}(b - d)\Delta\right\}}{\alpha a + \mu a + \delta b + \lambda d} \tag{16b}$$

[20] Details of the derivation can be obtained from the author. A linear solution is obtained for a large number of initial market structures in addition to the price-leader structure used here. However, boundary conditions (i.e., capital size restrictions) must be verified for each specific case.

(Recall α is the discount rate associated with discount factor β). The linear solution given by (16) holds for all positive parameter values if the size of a capital unit:

$$\Delta < \frac{a}{b}\left[\frac{1}{\frac{\alpha T}{2}+1}\right]$$

where $T \equiv 1/\mu$ is the average duration of a capital unit and a/b is the value of Q at which industry demand is saturated (see Figure 1). In other words, the linear solution may not hold for relatively large capital increments *if* the discount rate is high and the average lifespan of a capital investment is long. However, a continuous discount rate of .3 (equivalent to an annualized rate of 35%) and an expected capital life of 50 years only restricts units of capital to be less than 12% of saturated demand. Thus, for most cases of economic interest the linear solution is likely to hold.[21]

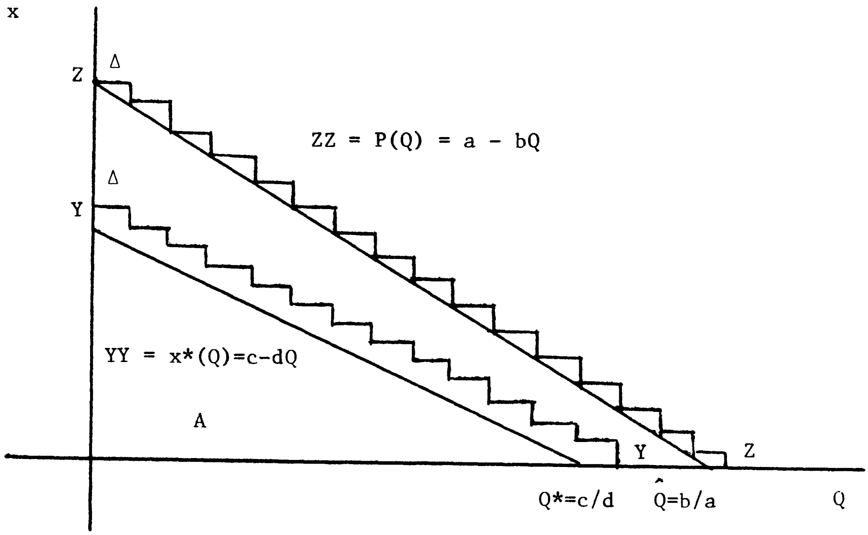
Figure 1 shows the relationship between the asset demand curve $P(Q) = a - bQ$ and the optimal cost hurdle $x^*(Q) = c - dQ$ for Q continuous and discrete. The slope of $x^*(Q)$ is invariant to the differencing interval Δ; however, the intercept c rises as Δ increases. It can be shown by completing the square in (16a) that $d \leq b$, that is, the slope of the optimal hurdle $x^*(Q)$ is less than that of the assumed asset demand curve. Although the firm accepts fewer projects as Q increases, the required *NPV* decreases as industry output increases. However, an oligopolistic firm will stop investing before optimally accepting a project with a zero *NPV*.

The point $Q^* = c/d$ represents the highest level of industry output at which the oligopolistic firm will accept a new opportunity. Given asymmetric behavior, other firms may push the industry to output levels above Q^* but less than $\hat{Q} = a/b$ where demand is saturated. Nevertheless the price-leading firm optimally rejects all opportunities which arrive when output is higher than Q^*.

Comparative Statics. Table I summarizes the impact on the optimal decision rule of changes in the various parameters. A higher opportunity arrival rate λ allows the firm to be more choosy and reduces the optimal cutoff cost x_Q^* for all Q. An increase in the discount rate reduces the value of future cash inflows and makes an opportunity "in the hand" more acceptable. An increase in competition measured by δ increases the probability of transitions to less favorable states: this increases the firm's incentives to take opportunities as they arise, and increases cutoff costs x_Q^*. An increase in the average durability of investments, measured by $T = 1/\mu$, has an indeterminate effect on optimal policy: increasing capital durability may cause cutoff costs to rise in some states and fall in others.

Certain limits of the irreversible opportunities model yield decision rules familiar from classical production theory. For example, pure competition corresponds to a case where $\lambda/(\lambda + \delta) \rightarrow 0$. As λ decreases, decision parameters c and

[21] However, if the change in Q is permanent ($T = \infty$), the linear solution only holds for a continuous state variable ($\Delta = 0$). In this case, without some external source of uncertainty, the system is nonstochastic. With exogenous uncertainty, the system can be made to converge to a diffusion process.



$P(Q) - x$ is the NPV (per unit capacity) when the cost per unit is x.

Under the standard NPV rule, projects are accepted if the point (Q, x) falls belows the asset demand schedule $ZZ = P(Q) = a - bQ$.

An oligopolist applying the optimal dynamic rule accepts projects only if (Q, x) falls in region A, below the schedule $YY = x^*(Q) = c - dQ$. The distance $P(Q) - x^*(Q)$ is the optimal NPV premium in state Q. $x^*(Q) < 0$ if $Q > Q^*$, therefore the oligopolist accepts no opportunities in states $Q > Q^*$.

Figure 1. The Asset Demand Curve and Optimal Decision Rule.

Table I

Comparative Statics

Parameter		Sign of 1st Derivative of		
		d	c	Q*
Frequency of New Opportunities	λ	–	–	–
Competition	δ	+	+	+
Discount Rate	α	+	+	+
Capital Duration	T(=1/μ)	–	indet.	indet.

d increase, and they are bounded from above by a and b. Therefore, as the firm's share of opportunities approaches zero, the limit yields the competitive rule "invest (produce) if $NPV \geq 0$, i.e. if cost is less than price."

Classical monopoly corresponds to a continuous flow of opportunities ($\lambda \to \infty$) of zero duration ($\mu \to \infty$) subject to $\mu/\lambda = 0$ (i.e. there are infinite opportunities per unit output). Opportunities (products) are of zero size ($\Delta = 0$) and there is no competition ($\delta = 0$). Rewriting (16a) and (16b) and taking limits yields

$$c \to 0; \; d \to 0$$

$$c/d = Q^* \to \frac{a}{2b}$$

With an infinite number of opportunities per unit output, a monopolist only accepts those with minimum cost ($x^* = 0$). A monopolist limits production to the profit maximizing quantity $a/2b$, by accepting no opportunities above the point $Q^* = a/2b$.

I emphasize the partial equilibrium character of the preceding comparative statics analysis. The conclusions are based on parameter changes holding $P(Q)$ constant. In a general equilibrium context, changes in parameters will affect $P(Q)$ both directly, because μ and α are implicit in the value of a mine asset, and indirectly, because changes in $x(Q)$ alter the laws of motion of the system, which in turn affect $P(Q)$.[22] A systematic study of the dynamic equilibrium properties of an industry with durable capital is a topic of future research.

Value of Future Opportunities. The total value of the firm's future investment opportunities is calculated by applying Proposition 2:

$$v_Q = \frac{1}{\beta} \sum_{i=Q}^{\hat{Q}-\Delta} r_i^* + \frac{\mu \hat{Q}}{\alpha} r_{\hat{Q}-\Delta}^* \qquad i = Q, Q + \Delta \cdots \hat{Q} - \Delta \tag{17}$$

Applying (13), (14), and (16) and assuming that capital units, Δ, are small, v_Q is approximately

$$v_Q = \int_Q^{\hat{Q}} [a - c - (b - d)x]dx + \frac{\mu \hat{Q}}{\alpha}[a - c - (b - d)\hat{Q}] \tag{18}$$

If $Q = 0$, (18) yields the value of a firm in an industry just starting up:

$$v_0 = \frac{a}{b}\left\{(a - c)\left(1 + \frac{\mu}{\alpha}\right) - (b - d)\frac{a}{b}\left(\frac{1}{2} + \frac{\mu}{\alpha}\right)\right\} \tag{19}$$

Referring to Figure 1, the aggregate value of a firm's growth opportunities in state Q is the area between ZZ and YY from Q to $\hat{Q}$[23] plus a boundary adjustment proportional to $\mu\hat{Q}$, the rate of transition from $\hat{Q}$ to lower states.

The valuation of a future stream of opportunities is among the most troublesome issues in modern finance. Although the same principles of valuation apply to *R & D* and exploration programs as to other investments, standard capital budgeting techniques are difficult to apply in practice. In general, the programs' payoffs consist of opportunities arriving at uncertain times; the value of these opportunities will depend on then-current industry conditions, as well as on the corporation's decision on what to do with each one. In these situations, investment models based on scenario planning or Monte Carlo techniques quickly become intractable.

[22] Cf. footnote 17.

[23] To calculate v_Q, the schedule YY is extended beyond Q^* using the linear formula $YY = x^*(Q) = c - dQ$.

Equation (17) is a valuation formula applicable in exactly this context. Given estimates of the asset demand function $P(Q)$, technological and market structure parameters, λ, μ, δ, $\bar{F}$, and the discount rate α, x_Q^* can be calculated. The parameter estimates and x_Q^* can be applied in (17) to estimate v_Q. The resulting value can then be compared to proposed *R & D* or exploration expenditures to determine if they are worthwhile investments. This valuation procedure will reflect interactions between demand, market structure, opportunities, and investments which are difficult to handle within standard valuation methods.

III. Conclusion

This paper develops a model of optimal sequential investment for firms which have market power and make investments not readily reversible in the short run. It was demonstrated that when the relation between prices and output is negative, such firms will optimally require investments to have direct *NPV*'s in excess of some positive number. The "*NPV* premium" is necessary to compensate the firm for the loss in the value of future opportunities implicit in accepting a particular investment. In the example developed in Section II, the optimal *NPV* premium depended on industry characteristics including the (derived) demand for assets; the arrival frequency and cost distribution of opportunities; competition; the average durability of capital; and the market discount rate. Optimal *NPV* premiums are closely related to the value of future opportunities: the aggregate value of an investment program can be calculated as the sum of *NPV* premiums plus a boundary adjustment.

The paper's primary contribution is a rigorous, choice-theoretic analysis of investment when commitments to specific projects are economically irreversible in the short-run. In the general form of Section I, the analysis can be applied to actual investment problems of firms to determine which opportunities should be accepted within a dynamic investment program, and to evaluate allocations of corporate resources to research and exploration investments.

APPENDIX A

Proof of Proposition 3

Define:

$\theta_Q(r_Q) \equiv$ the probability of a controlled or uncontrolled transition from Q to $Q + 1$, in interval Δs, conditional on hurdle r_Q being applied; and

$\gamma_Q(r_Q) \equiv$ the probability of a transition from Q to $Q - 1$ in interval Δs, conditional on hurdle r_Q being applied.

When actions are unfocused ϕ and γ will be sums of probabilities of individual events and acts; for the system to be stable it is sufficient to have

$$(\lambda_Q + u_Q + d_Q)\Delta s < \cdot 5 \quad \text{all} \quad Q,$$

where λ_Q is the arrival rate of new opportunities, and u_Q, d_Q are respectively the rates of uncontrolled transitions to $Q + 1$ and $Q - 1$.

For a local process, Equation (1) can be written out and terms rearranged:

$$\begin{aligned} v_Q(s + \Delta s) = {} & \lambda_Q \Delta s R(r_Q^*, Q) + \gamma_Q(r_Q^*)\beta(v_{Q-1} - v_Q) \\ & + \beta v_Q - \phi_Q(r_Q^*)\beta(v_Q - v_{Q+1}) \end{aligned} \tag{A.1}$$

where stars (*) indicate optimization with respect to state Q. (To simplify notation, time arguments on the right-hand side have been suppressed.)

Define

$$\begin{aligned} \hat{v}_Q(s + \Delta s) = {} & \lambda_Q \Delta s R(r_{Q+1}^*, Q) + \gamma_Q(R_{Q+1}^*)\beta(v_{Q-1} - v_Q) \\ & + \beta v_Q - \phi_Q(r_{Q+1}^*)\beta(v_Q - v_{Q+1}) \end{aligned} \tag{A.2}$$

i.e. $\hat{v}_Q(s + \Delta s)$ is the value of the dynamic process if the "wrong" *NPV* criterion is applied in state Q over the next interval Δs. By definition, v_Q is a maximum

$$v_Q(s + \Delta s) \geq \hat{v}_Q(s + \Delta s)$$

Subtracting v_{Q+1} from $\hat{v}_Q$:

$$\begin{aligned} & \hat{v}_Q(s + \Delta s) - v_{Q+1}(s + \Delta s) \\ & = [\lambda_Q \Delta s R(r_{Q+1}^*, Q) - \lambda_{Q+1}\Delta s R(r_{Q+1}^*, Q + 1)] \\ & + \gamma_Q(r_{Q+1}^*)\beta(v_{Q-1} - v_Q) \\ & + [1 - \phi_Q(r_{Q+1}^*) - \gamma_{Q+1}(r_{Q+1}^*)]\beta(v_Q - v_{Q+1}) \\ & + \phi_{Q+1}(r_{Q+1}^*)\beta(v_{Q+1} - v_{Q+2}) \end{aligned} \tag{A.3}$$

Let the process be reward-ordered in R i.e., $\lambda_Q R_Q \geq \lambda_{Q+1} R_{Q+1}$ all Q, holding strictly for some Q. The first term of (A.3) is nonnegative from the definition. From the assumption of stability ϕ_Q and γ_{Q+1} are each less than .5. The expression is then nonnegative if the $v_Q - v_{Q+1}$ terms are nonnegative.

To complete the proof, consider the process with one instant left to go. Recall $v_Q(0) = v_{Q+1}(0) = 0$:

$$v_Q(\Delta s) = \max_r \lambda_Q \Delta s R(r, Q)$$

For all Q the maximum is attained at $r = 0$. Again by the property of reward ordering

$$v_Q(\Delta s) - v_{Q+1}(\Delta s) \geq 0$$

and, for at least one Q, $Q + 1$

$$v_Q(\Delta s) - v_{Q+1}(\Delta s) > 0 \quad \text{(strict inequality)}$$

Given that the system is bounded and has N communicating states, the positive element will be present in (A.3) within at most $N - 2$ steps. At that point

$$\hat{v}_Q - v_{Q+1} > 0$$

implying $v_Q - v_{Q+1} > 0$ all Q.

For a process reward ordered in D, the induction proof is similar. Details of the proof are available from the author.

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