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Prologue to a Unified Portfolio Theory

By HERBERT F. AYRES and JOHN Y. BARRY*

"A Rose by Any Other Name Will Smell As Sweet"—Shakespeare

I. Introduction

AN OBSERVER OF QUANTITATIVE practices in the financial community of 1980 discovers that what is called "Modern Portfolio Theory" or MPT consists of two classes of models which appear to have little to do with each other although they are applied to the same market price behavior. The two classes of models are long range present value models, which are exemplified by dividend discount models of equity prices and yield to maturity pricing of bonds, and short run models of the behavior of total return which are provided by the various versions of Capital Asset Pricing Theory (CAPT). The former models contain only an ad hoc treatment of risk if it is treated at all, and the latter ones without exception model risk as short run total return volatility. CAPT models use a short time horizon, usually thought of as 30 to 90 days, with the price formation at the horizon implicit and undefined. The present value models of the equities markets have an infinite time horizon. After 90 days, the long term holder has received only 1-2% of the present value. Hence, where the horizon of the one leaves off virtually none of the value measure of the other has been achieved. Clearly, investors have radically varying investment horizons. With few exceptions, there exists no theoretical explanation of how these views can be in equilibrium with each other.

This paper makes a beginning at the construction of a unified framework for discussing both points of view in a logically consistent manner.

It is firmly, even ardently, believed by many in the quantitative financial community that: (1) more than fifteen years of testing have provided ample to copious scientific evidence in the form of statistically significant portfolio betas and market lines which supports the assumptions of CAPT models; and (2) this evidence is so solid that to deny that total return volatility is an appropriate measure of risk is rather like denying that the earth is spherical for all practical purposes because it is about 30 miles out of round.

If the work we present here is correct, the above beliefs have no scientific basis. The issue is not moot: CAPT has been cited in expert testimony before courts of law. Routinely now, the performance of professional investors is "risk adjusted" by volatility measures. Presumably, some are losing business because their performance post adjustment looks particularly poor. Perhaps most disturbing, considering the nature of fiduciary responsibility, is the fact that clients' money in some real life cases is being run using arbitrary volatility constraints which, if

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they have any effect at all, must reduce expected performance. One can only fervently hope that in all such cases the client has himself stipulated that volatility is indeed risk to him given his liabilities and goals. One can further hope with equal fervor that the client really knows what such a stipulation means.

Richard Roll[1] has argued that the usually accepted evidence does not provide support for the assumptions of CAPT. We believe that his work is in no way inconsistent with ours. If we understand Roll's position correctly, he believes that there are structural, but correctable, biases in the presently used risk adjustment procedures. Our position is stronger. If we are correct, there is no evidence supporting the *absolute identification* of Total Rate of Return (TRR) volatility with risk. In fact, there are important cases where it is just plain wrong. Consider a charitable foundation that wants to fund five year grants now. Which investment is more risky, a portfolio of bills rolled over every 90 days or a portfolio of bonds whose cash flow matches the cash disbursements for the grants? Which has a more volatile total return in the next 90 days?

In the process of our work we have discovered the existence of a subtle identity which produces a total return equation which probably explains the statistical evidence attributed to the beta equation of CAPT. This explanation is certainly correct in the U.S. Treasury market, which is one of the world's largest in terms of value outstanding and daily volume of trading. Even if this identity does not completely explain the historical evidence in some capital market, the part attributable to the beta equation will be distinguishable only with considerable difficulty. The historical experiments cannot be accepted as support for CAPT as they now stand. They are at best heavily contaminated by the identity's existence. The experiments will have to be revised and rerun with procedures to correct for its presence.

We emphasize that CAPT is a magnificent technological achievement in our view. One of its outstanding aspects is the demonstration that a general equilibrium solution could be found which is not a horrid tangle of expressions involving the unobservable utility functions of the individual investors. Our quarrel is not with this fine piece of technology. We see it as a very major forward step in the history of the theory of finance and financial markets. Our quarrel is with applying it to markets which its assumptions do not fit, and we most particularly quarrel with pretending that the theory is scientifically validated in major capital markets, which it most certainly is not to date. We are not asserting that CAPT will never be scientifically validated—only that it has not yet been.

Besides the technical work we present here we have views which we wish to label as opinion. It is our opinion that no general market equilibrium theory will ever be validated in the major capital markets of the world which limits its focus to the asset side of the balance sheet. Were it possible to do so, it would be possible for you to measure the risk associated with portfolios A and B absolutely. Portfolio A consists of billions of dollars of corporate bonds with quite long maturities. Portfolio B consists of the same amount of floating rate and short

[1] See Richard Roll, "A Critique of the Asset Pricing Theory's Tests," *Journal of Financial Economics*, 4, (May, 1977), 129–176, and "Ambiguity When Performance is Measured by the Securities Market Line", *Journal of Finance*, 33, (Sept., 1978), 1051–1069.

term loans to the same corporations. Would you be in any way uncomfortable with your measurements if we told you that A was owned by a bank and B by a life insurance company? We believe the respective CEO's would have very different views about the riskiness of the two portfolios.

The purpose of the above examples is to challenge the oft heard statement that TRR volatility of asset portfolios is intuitively satisfactory as a risk measure. We suggest that examination of the behavior of portfolio managers with very different liabilities and objectives makes the TRR volatility of only asset portfolios a very *unsatisfactory* and *unacceptable intuitive absolute* risk measure.

The analysis in this paper is a version of the case of known cash flows simplified for explanatory purposes. The results generalize to the exact case for certain cash flows and also to the case where only estimates are available.

The reader will see that we believe the *internal* rate of return to be the more fundamental stochastic process in capital markets with the *total* rate of return the secondary or derived one. We expect that the process of the formation of estimates of future cash flows will gain more and more prominence in future research on the equities markets. If we are correct, the primary area of research on equities may shift from academia to those financial institutions where dividend discount forecasts are currently prepared.

In the body of this paper we first exhibit the identities that define total rate of return and internal rate of return. We construct a portfolio model from these definitions; next we derive a "beta" equation which is an identity. Next we present and discuss what we call our testable contention that the TRR of the key driver portfolio in our model is strongly correlated with the TRR's of broadly based market indices. We then discuss the "beta" equations—the one we derive and the one from CAPT. We cite the difficulty in defining the CAPT "market" and show how important it is. We point out how the key driver portfolio in our model presents no such problem. Next we show, if our testable contention is correct, that the identity we derive may well account for all the observed phenomena usually cited in support of CAPT.

2. The Basic Mathematical Identities

In this section we start with the basic definitions of TRR and IRR in the general time varying case. We derive the present value of price formulas; then we develop dynamic equations relating TRR, IRR, duration and the change in IRR. We do this in some detail because of their importance and because of current misconceptions about the general applicability of duration. Next we develop simplified portfolio mathematics for the purpose of exposition. The simplifications introduced cause the structure shown to have the same important properties as the more complicated exact structure. We next derive a beta equation where a portfolio of minimum IRR change variance appears in the same driver role that the "market" portfolio does in CAPT. The existence of this key portfolio is a consequence of the definitions of TRR and IRR. Beta is found to be a duration ratio.

The market price for which a holding can be sold at time n is $p(n)$. $C(n)$ is the cash payment to the owner at time n; $p(n)$ does *not* include $C(n)$. The total rate of return $r(n)$ is defined by Equation (1), the sum of current yield plus fractional capital gains. Algebraic rearrangement gives Equation (2),

$$r(n) = C(n)/p(n-1) + [p(n) - p(n-1)]/p(n-1) \tag{1}$$

a first order difference equation in price.

$$p(n) - [1 + r(n)]p(n-1) = -C(n) \tag{2}$$

The general solution is shown in Equation (3).

$$p(n) = p(n_0) \prod_{i=n_0+1}^{j=n} [1 + r(i)] - \sum_{i=n_0+1}^{i=n} C(i) \prod_{j=i+1}^{j=n} [1 + r(j)], \qquad n > n_0. \tag{3}$$

which expresses the value of $p(n)$ in terms of past cash flows, it is an ex post view of current prices. The solution can be put in ex ante form as shown in Equation (4), where N is the time of the last cash flow.

$$p(n) = \sum_{i=n+1}^{i=N} C(i)/\prod_{j=n+1}^{j=i} [1 + r(j)] \tag{4}$$

Equation (4) says that the price is the sum of the future discounted cash flows. Because it expresses the price now in terms of future cash flows, it is an ex ante point of view. Both equations are mathematically equivalent solutions of Equation (2). They simply express different points of view, ex post and ex ante.

$$p(\gamma, n) = p(n_0)(1 + \gamma)^{n-n_0} - \sum_{i=n_0+1}^{i=n} C(i)(1 + \gamma)^{n-i} \tag{5}$$

$$p(\gamma, n) = \sum_{i=n+1}^{i=N} C(i)/(1 + \gamma)^{i-n} \tag{6}$$

Equation (5) is the definition of the ex post internal rate of return and Equation (6) is the definition of the ex ante internal rate of return. It will be only fortuitous if $\gamma(n)$ from Equation (5) equals $\gamma(n)$ from Equation (6). We treat both cases because although CAPT is an ex ante theory, most tests of the theory use ex post data.

$$\gamma = C(n)/p(\gamma, n-1) + [p(\gamma, n) - p(\gamma, n-1)]/p(\gamma, n-1) \tag{7}$$

It is easy to verify the $p(\gamma, n)$ satisfies the difference Equation (7).

Since $p(n) = p(\gamma(n), n)$ and $p(n-1) = p(\gamma(n-1), n-1)$, we may rewrite equation (1) as shown in equation (8).

$$\begin{aligned} r(n) = C(n)/p(\gamma(n-1), n-1) + [p(\gamma(n), n) \\ - p(\gamma(n-1), n-1)]/p(\gamma(n-1), n-1) \end{aligned} \tag{8}$$

Also we can set $\gamma = \gamma(n-1)$ in Equation (7) as shown in Equation (9).

$$\begin{aligned} \gamma(n-1) = C(n)/p(\gamma(n-1), n-1) + [p(\gamma(n-1), n) \\ - p(\gamma(n-1), n-1)]/p(\gamma(n-1), n-1) \end{aligned} \tag{9}$$

We subtract equation (9) from equation (8) and get equation (10).

$$r(n) = \gamma(n-1) + [p(\gamma(n), n) - p(\gamma(n-1), n]/p(\gamma(n-1), n-1) \tag{10}$$

Observe that the numerator of the second term of the right hand side is a price difference both of whose terms are *contemporaneous at time n*. We define $\Delta\gamma(n-1) = \gamma(n) - \gamma(n-1)$ and we define $H[x, y]$ by Equation (11).

$$H[x, y] \equiv [1/p(y, n-1)][(p(x, n) - p(y, n)]/(x - y), \tag{11}$$

where x and y are dummy variables.
We may now rewrite equation (10) as shown in equation (12).

$$r(n) = \gamma(n-1) + H[\gamma(n), \gamma(n-1)]\Delta\gamma(n-1) \tag{12}$$

To facilitate the interpretation of equation (12) we need the approximation shown in Equation (13).

$$H[\gamma(n), \gamma(n-1)] \doteq [1/p(\gamma(n), n-1)]\partial p(\gamma(n), n)/\partial\gamma(n) \equiv -D(n), \tag{13}$$

where $-D(n)$ is the duration.
Since we are thinking of small time steps—a day, a week, a month—the approximation is very good; in fact, for the very important special case of exponential growth cash flows, it is exact. Using Equation (13), we rewrite Equation (12) in Equation (14). Notice that

$$r(n) = \gamma(n-1) - D(n)\Delta\gamma(n-1) \tag{14}$$

we use an "=" in Equation (14) rather than an "$\doteq$". We do not include any Ito drift terms which will not affect the covariance structure of total-covariance returns.

It is important to recognize that the entire analysis above is nothing but a structure of identities based on the definitions of TRR in Equation (1) and IRR. They apply to individual securities, to arbitrary groupings of securities (such as industry averages), and to any portfolios of securities.

Portfolio Mathematics

Exact portfolio mathematics is arduous and complex. Different programs of portfolio construction and maintenance lead to quite different complex collections of equations, which, as is well known, give rise to surprisingly minor differences in numerical results—the total rates of return of broadly diversified portfolios of the same securities are very highly correlated given almost any conventional portfolio protocols. In this paper we cut through the complexity with two simplifying assumptions *for expositional purposes*, and we assure the reader that the basic results below hold without the assumptions but with considerably more complex mathematics. Thus, the substance of what we say below *does not depend on the accuracy of our expositional assumptions*; but, as a matter of fact, these assumptions are almost always excellent approximations.

The portfolio structure we use is as follows. P is the dollar value of the portfolio. P_j is the unit price of the jth security. We use the convention that subscripted variables apply to individual securities and nonsubscripted variables apply to a portfolio. Let w_j be the number of units of the jth security in the portfolio which is constant over time and x_j be the fraction of the portfolio wealth invested in the jth security. Then $P(n) = \Sigma_j w_j P_j(n)$, $x_j(n) = w_j P_j(n)/P(n)$. Equations (15) and (16) hold in general.

$$r(n) = \Sigma_j x_j(n-1) r_j(n) \tag{15}$$

$$P(\gamma, n) = \Sigma_j w_j p_j(\gamma, n) \tag{16}$$

Equations (17) and (18) are our simplifying assumptions.

$$\gamma(n-1) = \Sigma_j x_j(n-1)\gamma_j(n-1) \tag{17}$$

$$D(n) = \Sigma_j x_j(n-1)D_j(n) \tag{18}$$

We multiply Equation (14) for the jth security by $x_j(n-1)$ and sum to get equation (19).

$$\Sigma_j x_j(n-1)r_j(n) = \Sigma_j x_j(n-1)\gamma_j(n-1) - \Sigma_j x_j(n-1)D_j(n)\Delta\gamma_j(n-1) \tag{19}$$

Using equations (15) and (18) we can rewrite equation (19) as shown in equation (20).

$$r(n) = \gamma(n-1) - \Sigma_j x_j(n-1)D_j(n)\Delta\gamma_j \tag{20}$$

If we subtract equation (14) for the portfolio as a whole from equation (20) and divide through by $D(n)$, Equation (21) results. The definition of a_j is

$$\Delta\gamma(n-1) = \Sigma_j[x_j(n-1)D_j(n)/D(n)]\Delta\gamma_j(n-1) = \Sigma_j a_j \Delta\gamma_j(n-1) \tag{21}$$

shown in equation (22).

$$a_j \equiv x_j(n-1)D_j(n)/D(n); \qquad \Sigma_j a_j = 1. \tag{22}$$

The purpose of maneuvering $\gamma(n-1)$ into this form is so that we may write $\phi = \Delta\gamma(n-1)$ and $\phi_j = \Delta\gamma_j(n-1)$ and write $\phi = \Sigma_j a_j \phi_j$. We explain why in the next section.

Portfolios of Minimum ϕ Variance and "Beta" Equations

It can be shown that for any specific set of securities there exists at least one portfolio of minimum ϕ variance. Appendix A contains theorems on the properties of portfolios of minimum ϕ variance. We set the mathematical structure up this way so that when we develop the exact portfolio mathematics and generalize to uncertain future cash flows ($C(i)$ becomes $C(i, n)$ in Equation (6)). Appendix A plays exactly the same role it does here with somewhat different definitions of ϕ and ϕ_j. The results we are about to derive generalize broadly. In the remainder of this section unsubscripted variables refer to a *portfolio of minimum ϕ variance.*

Beta Equations

Equation (23) is equation (14) rearranged for a portfolio of minimum ϕ

$$-\Delta\gamma(n-1) = (r(n) - \gamma(n-1))/D(n) \tag{23}$$

variance. Equation (24) is equation (14) applied to the jth security. In

$$r_j(n) = \gamma_j(n-1) - D_j(n)\Delta\gamma_j(n-1) \tag{24}$$

Equation (25) $\Delta\gamma(n-1)$ is added and subtracted, and the spread s_j is defined

$$r_j(n) = \gamma_j(n-1) - D_j(n)[\Delta\gamma(n-1) + \Delta\gamma_j(n-1) - \Delta\gamma(n-1)] \tag{25}$$

in equation (26). We operate on equation (25) as follows. We use equation

$$s_j \equiv \gamma_j - \gamma \tag{26}$$

(23) to replace the first $\Delta\gamma(n-1)$, we subtract $\gamma(n-1)$ from both sides and use the definition of s_j in equation (26) where possible. The result is equation (27). By Appendix A equation (28) holds, so the second and

$$r_j(n) - \gamma(n-1) = s_j(n-1) + (D_j(n)/D(n))[r(n) - \gamma(n-1)] - D_j(n)\Delta s_j(n-1) \tag{27}$$

$$\text{cov}[\Delta\gamma(n-1), \Delta s_j(n-1)] = 0 \tag{28}$$

third terms of the right hand side of equation (27) are uncorrelated. Equation (29) is the famous beta equation of CAPT in the form usually used for statistical estimation where r_m is the TRR of the "market" and r_f is the "risk free" rate. Equation (30) is also a result of CAPT which says that the

$$r_j(n) - r_f(n) = \alpha_j + \beta_j[r_m(n) - r_f(n)] + \epsilon_j \tag{29}$$

$$\text{cov}[r_m(n), \epsilon_j] = 0 \tag{30}$$

second and third terms of equation (29) are uncorrelated. Thus there is a remarkable similarity between pair of equations (27), (28) and pair of equations (29), (30) which we discuss at some length in section 4.

3. Our Testable Contention

In this section we present some of the reasoning behind our testable contention. This contention is the only issue with economic rather than definitional content in this paper. It is key to our argument.

It is clear that the Cov$[r_i, r_j]$ is essentially determined by Cov$[\Delta\gamma_i, \Delta\gamma_j]$ = Cov$[\phi_i, \phi_j]$ from equation (24). Experience shows that the total returns of any two broadly diversified equity portfolios are very highly correlated. One major financial institution even built a provision for printing out all funds with a correlation coefficient with the S & P 500 Index of less than 0.9 into its performance measurement system for data error checking purposes! Thus it is highly likely that the TRR on a portfolio of minimum ϕ variance in the U.S. equities market is strongly, even very strongly, correlated with the TRR of the S & P 500 Index.

The above argument and our work in the U.S. Treasury market,[2] which we discuss below, leads us to our

Testable Contention: In any major capital market the TRR of a portfolio of minimum ϕ variance will be found to be highly correlated with the TRR of broadly based market indices.

We call our contention testable because equation (28) can be used to identify

[2] See Herbert F. Ayres and John Y. Barry, "The Equilibrium Yield Curve for Government Securities," *Financial Analysts Journal*, May/June, 1979. See also John Y. Barry and Herbert F. Ayres, "A Theory Of The U.S. Treasury Market Equilibrium," *Management Science*, June, 1980.

a portfolio of minimum ϕ variance. The indicated correlation tests with widely used market indices are perfectly straight forward.

The portfolio of minimum ϕ variance in the equities market has not yet been isolated and studied insofar as we know. We should like to pose the following question to you, the reader. How much would you like to bet, even money, that when such a portfolio is found and studied its TRR will *not* be highly correlated with that of the S & P 500? We want to point out that if your answer is "not much", *then the whole point of this paper is that you must logically be unenthusiastic about the experimental evidence supporting the unqualified identification of beta with risk.*

4. Discussion of the Beta Equations (27) and (29)

In this section we discuss the important and serious identification problem of the "market" portfolio of CAPT. No such problem exists for the portfolio of minimum ϕ variance; a definitive test exists for its identification. We then present an example of a major capital market in which the arguments of this paper are valid.

Believers in CAPT must believe that the second term of equation (29) represents the very important "undiversifiable risk". This belief focuses attention on the issue of how to measure r_m, the dominant variable in that term, and on the issue of how to measure r of equation (27) by analogy.

Little or No Problem In Equation (27)

The important point is that equation (28) provides the experimental basis to identify a portfolio of minimum ϕ variance. At least one such portfolio must exist in any set of securities.

Although equation (28) can serve as basis of a definitive statistical test to identify a portfolio of minimum ϕ variance, analogous equation (30) cannot provide the definitive test to identify the "market" portfolio of CAPT because the Ej of equation (29) are merely regression residuals which are uncorrelated with r_m *by construction. CAPT gives us no independent way to calculate them.*

CAPT provides no operational test which can be used to identify the "market portfolio." As a consequence, the market portfolio is a matter of debate whereas the determination of the portfolio of minimum ϕ variance is a matter of measurement.

Many students of quantitative market theory believe that CAPT can be applied to particular markets where by law or by convention many of the participants are isolated de facto if not de jure. They correctly point out that if the TRR of one broadly based index is regressed on the TRR of some other broadly based index, the correlation is found to be very high and the beta insignificantly different from one. As a consequence, they argue, it must be a matter of minor concern which broadly diversified portfolio is used as a market proxy.

But if any broadly based index will do, so will a portfolio of minimum ϕ variance if our testable contention is correct. If it is correct, there is no logical position one can assume which makes the historical evidence based on estimating equation (29) logically acceptable as supporting evidence for CAPT. The evidence is utterly contaminated by the identity (27). It is important to realize that a particular portfolio of minimum ϕ variance need not be a "nice looking" portfolio like the S & P 500 or a typical investment account. It can be "ugly as sin" with many short sales and peculiar weights on individual securities. So long as its TRR is highly correlated with the TRR of conventional market indices, our argument holds. The conventional beta equation is then largely a statistical proxy for an identity and quite possibly nothing more.

The U.S. Treasury Market

The U.S. Treasury Market is a major capital market in which the arguments we present in this paper are clearly correct.

In our theory of the equilibrium of the U.S. Treasury market the identity in equation (27) is easily applied. The implicit consol is the portfolio of minimum variance using tests based on equation (28). The durations can then be computed exactly. A formula for current coupon bonds is shown in Equation (31). The consol duration is shown in Equation (32).

$$D_j = [1 - \exp(-m_j\gamma_j)]/\gamma_j \tag{31}$$

$$D = D(\infty) = 1/\gamma(\infty) \tag{32}$$

The "betas" in Equation (27) are of course the D_j/D, the duration of the jth bond divided by the duration of the consol.

It is clearly silly to even talk about running regressions to estimate these "betas", for they are calculable. If one persists in this silliness and runs the regressions anyway, he finds that he gets estimates of the D_j/D, with enormous t-statistics on zero, which are insignificantly different from the calculated values.

It is clear then that the behavior of U.S. Government Bond total returns are explained by the identity (27) where the variables and parameters are calculated or observed quantities. Since Equation (27) is an identity it is senseless to argue that the D_j/D are measures of anything other than volatility in any absolute sense. To a bond dealer who has financed his position with call money, they obviously are risk measures. To a holder with long term liabilities, such as life insurance companies, they may not mean a thing with regard to risk.

Thus, there is no experimental support for CAPT in the U.S. Treasury market, one of the largest, most liquid, and simplest markets on earth. We suggest that this lack of experimental support should quell the enthusiasm of proponents of CAPT.

5. CONCLUDING REMARKS

Pretend for a moment that the theoretical work of all the scientists from Markowitz and Sharpe to Roll and Ross had never existed. Then there would be no CAPT, no efficient frontiers, no formal identification of volatility with risk, etc. Notice that nothing in our analysis above would change one whit. Definitions are definitions, deductions are deductions and historical phenomena are historical phenomena. Then suppose that you were talking to a graduate student who told you that he was running a series of regressions of the type shown in equation (29)—for whatever motivation—to find out if there is a linear relationship between rates of return and beta. He's using the S & P 500 to compute r_m. You would have to tell him that there must be such a relationship because of the identity in equation (27) and because the S & P Index is a good proxy for a portfolio of minimum ϕ variance. Our analysis is nothing but a textbook argument from multi-variate statistics; it has nothing necessarily to do with economic behavior.

There are a variety of mysterious phenomena which were reported in the literature which become less mysterious given equation (27) and our testable contention. (1) Statistical estimates of the "risk free" rate have tended to be too high. We see in equation (27) that γ appears where the "risk free" rate does in the β equations; γ represents long term expectations of returns in the equities market which is on average greater than the Treasury Bill rate. (2) The alphas of individual stocks have been found to be sometimes significantly positive and sometimes significantly negative and sometimes insignificant. Equation (27) suggests that the reason may be explained by the fact that s_j appears where α_j appears in the beta equation. The spreads would be expected to be sometimes positive, sometimes negative, sometimes large, and sometimes small. (3) A number of investigators have reported that betas are random walks with a tendency to regress to one. This also is not surprising since $D_j(n)/D(n)$ appears in the analgous position to β_j. Of course, this ratio is going to exhibit random walk-like behavior due to the changes in γ and γ_j. Also, historically we know that exceptional stocks tend over time to look more and more like the herd. (4) Friend and Westerfield have reported that "diversifiable risk" is, if anything, a slightly better explanatory variable than "undiversifiable risk".[3] Equation (27) suggests that this is perfectly plausible because the duration $D_j(n)$ is in both the terms analgous to the undiversifiable and diversifiable risk terms in Equation (29).

In short, if it flys like a duck, and it waddles like a duck, and it quacks like a duck, and it has webbed feet like a duck, (maybe) it is a duck.

APPENDIX A

Properties of Portfolios of Minimum ϕ Variance

We define a *portfolio combination* $\phi(a) = \sum_i a_i\phi_i$ on a set of N securities ($i = 1, 2, \ldots, N$) where the ϕ_i are random variables, with $\sigma_{ij} \equiv \mathrm{COV}[\phi_i, \phi_j]$, but the a_i are not. $\sum_i a_i = 1$. In this sense, we use the vectors a and b to define various portfolio combinations in the analysis below; we use the vector α to define the portfolio combination of minimum variance.

The proofs in our Management Science article can be easily modified to show:

THEOREM: Proposition A and Proposition B are logical equivalents. Proposition A: $\phi(\alpha)$ is a portfolio combination of minimum variance. Proposition B: $\mathrm{COV}[(\phi(\alpha), \phi_k - \phi(\alpha)] = 0$ for all k.

COROLLARY: If σ^{-1} exists, the portfolio combination of minimum variance $\phi(\alpha)$ exists and is unique.

[3] Irving Friend and R. Westerfield, "Risk and Capital Asset Pricing," Rodney L. White Center for Financial Research, Working Paper, #5-79, May, 1979.

Proof: The proof is by exhibition. If σ^{-1} exists, both σ and σ^{-1} are positive definite. Consider the vector equation $\sigma Z = w$, where w is a vector such that $w_i = 1 (i = 1, 2, \ldots, N)$. The unique solution is $Z = \sigma^{-1}w$. $\sum_i Z_i = w^t Z = w^t \sigma^{-1} w > 0$ since σ^{-1} is positive definite. Let $b \equiv Z/\sum_i Z_i$, then $\sigma b = w/\sum_i Z_i$ and $b^t \sigma b = g^t w/(\sum_i Z_i)^2 = 1/\sum Z_i$. We have proved Equation (6) which is Equation (4a), a necessary and sufficient condition for the portfolio

$$\sigma b = (b^t \sigma b) w \tag{6}$$

combination of minimum variance.