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Source: *The Journal of Finance*, Vol. 37, No. 2, Papers and Proceedings of the Fortieth Annual Meeting of the American Finance Association, Washington, D.C., December 28-30, 1981 (May, 1982), pp. 609-614
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327370
Accessed: 27/11/2009 08:17

THE JOURNAL OF FINANCE • VOL. XXXVII, NO. 2 • MAY 1982

NEW APPROACHES TO PORTFOLIO THEORY AND STOCK MARKET EQUILIBRIUM

Presiding: JOHN LINTNER†

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I. Introduction

IN THIS PAPER we will introduce and develop a nontraditional theory of portfolio selection. The theory assumes that investors choose their portfolios to have a certain minimax property which we will describe in detail below. As we will see in the next two sections, the minimax portfolios turn out to be identical to portfolios that are often recommended on other grounds.

At the outset it seems essential to ask why one needs an alternative to the traditional portfolio-selection theory, which, roughly speaking, says that an investor maximizes his expected utility with respect to a probability distribution that incorporates all available information. A partial answer emerges if one considers the difficulties involved in such an optimization. First of all, the investor must absorb whatever information is available from financial statements, corporate spending plans, etc., concerning the future cashflows of corporations. From this "direct" information the investor would form initial beliefs about what each investment is worth. However, the investor must then incorporate into his probability distributions the "indirect" information that he derives from the market prices of the investments. Specifically, the market prices provide information about how the beliefs of other investors differ from his own.[1] To the extent that the investor thinks his direct information is inferior to that of other market participants, he will modify his own probability distributions.

In practice we know that investors' subjective probability distributions for the returns to the various assets are quite diverse. Perhaps investors do not properly incorporate the "indirect" information into their distributions, or perhaps market prices simply do not convey very much information. At any rate, given this apparent diversity of beliefs one would expect short-selling of assets to be common. Instead, short-selling is rare, and many investors hold the sort of broadly-diversified portfolios that we would expect only if beliefs were homoge-

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[1] See Diamond and Verrecchia [1981] and the references contained in that paper for some theoretical results on this topic.

neous. The theory presented in this paper provides a possible explanation for these observations.

In the next two sections we will assume that an investor forms, by one means or another, a probability distribution P for the cashflows that will derive from the various investments. However, he feels uncomfortable acting on the basis of subjective probability distributions, and he recognizes that because of the difficulties involved in using the available information properly, his probability distribution might not be exactly right. Therefore, he chooses his portfolio so as to provide the highest *guaranteed* level of expected utility, assuming that the true distribution lies in an "ϵ-neighborhood" of P. Formally, his portfolio choice satisfies a minimax property that is mathematically analagous to that developed by Huber [1964, 1981] in an entirely different context.

II. Minimax Portfolio Selection

We will consider the portfolio selection problem for a single investor, using a simple two-period model with n risky assets. Each asset has outstanding one perfectly-diversible share. Let $X_i \geq 0$ $(i = 1, \cdots, n)$ be the second-period value of the i^{th} asset, and let $X = (X_1, \cdots, X_n)$. Let $p_i \geq 0$ be the first-period price of the i^{th} asset, and let $p = (p_1, \cdots, p_n)$.

The investor has initial wealth W, and a smooth Von Neumann-Morgenstern utility function U with $U' > 0$ and $U'' < 0$. In the first period, X is a random variable. Let P be the investor's subjective probability distribution for X.

The investor's portfolio problem is to purchase fractions δ_i $(i = 1, \cdots, n)$ of the n assets. δ_i can be negative, in which case the investor is selling short the i^{th} asset. $\delta = (\delta_1, \cdots, \delta_n)$ must lie in the investor's budget set

$$B = \{\delta \epsilon \mathbb{R}^n : p \cdot \delta = W\}. \tag{2.1}$$

According to expected-utility theory, the investor should choose δ to solve

$$\max_{\delta \epsilon B} E_P U(\delta \cdot X), \tag{2.2}$$

where E_P denotes the expectation with respect to P. However, there is apt to be some question in his mind as to whether P is the correct distribution on which to base his decision. Consequently, he may wish to ensure that his portfolio decision will provide a high expected utility for any distribution Q that is in an ϵ-neighborhood of P. Formally, suppose that the investor chooses $\epsilon > 0$, and considers all distributions Q of the form

$$Q = (1 - \epsilon)P + \epsilon Q', \tag{2.3}$$

where Q' is an *arbitrary* distribution on the nonnegative orthant. The set of all distributions of the form (2.3) is denoted $N_\epsilon(P)$. This definition of an ϵ-neighborhood was popularized by Huber, and has proven useful in robustness theory over many years.[2] For any portfolio δ and any distribution Q, let

$$V(\delta, Q) = E_Q U(\delta \cdot X). \tag{2.4}$$

[2] The topology generated by these neighborhoods is stronger than the topology of weak convergence. In fact, if $Q \in N_\epsilon(P)$, then $|Q\{A\} - P\{A\}| \leq \epsilon$ for every event A, so that Q is near P in "total variation".

We will assume that the investor seeks a pair (δ^*, Q^*), with $Q^* \in N_\epsilon(P)$, such that

$$\sup_{\delta \in B} V(\delta, Q^*) = V(\delta^*, Q^*) = \inf_{Q \in N_\epsilon(P)} V(\delta^*, Q), \tag{2.5}$$

and chooses δ^* as his portfolio.

If one so desires, he can interpret (2.5) in terms of a zero-sum game in which the investor chooses a portfolio in his budget set and nature chooses a distribution for X in $N_\epsilon(P)$; the "payoff" is the investor's expected utility. According to (2.5), (δ^*, Q^*) is a saddlepoint if it exists, and hence δ^* is a minimax portfolio choice for the investor. As we mentioned in the introduction, δ^* provides the highest guaranteed expected utility, given that the true distribution lies in $N_\epsilon(P)$.

We will now show that a saddlepoint exists. Moreover, the investor's minimax portfolio can be alternatively characterized as the solution to the optimization problem (2.2), *if short-sales were prohibited.* Formally, define a portfolio δ_N as the solution to

$$\begin{aligned} &\max_\delta V(\delta, P) \\ &\text{such that} \quad \delta \in \mathrm{B} \quad \text{and} \quad \delta \geq 0. \end{aligned} \tag{2.6}$$

LEMMA: There exists a distribution Q' on the non-negative orthant such that

(i) δ_N is the solution to
$\max_{\delta \in B} V(\delta, (1 - \epsilon)P + \epsilon Q')$.
(ii) If the i^{th} component of δ_N is nonzero, then $X_i = 0$ with probability one under Q'.

Proof: In the appendix.

According to the lemma, there is a distribution in $N_\epsilon(P)$ that would induce an expected-utility-maximizing investor to select δ_N, even *without* the constraint $\delta \geq 0$. Moreover, that distribution can be chosen to have the largest possible probability for $\{X_i = 0\}$ (among distributions in $N_\epsilon(P)$) for those assets that are held in positive amounts under δ_N.

PROPOSITION 1: There exists $Q^* \in N_\epsilon(P)$ such that

$$\sup_{\delta \in B} V(\delta, Q^*) = V(\delta_N, Q^*) = \inf_{Q \in N_\epsilon(P)} V(\delta_N, Q), \tag{2.7}$$

and hence δ_N is a minimax portfolio.

Proof: Let $Q^* = (1 - \epsilon)P + \epsilon Q'$, where Q' is as in the lemma. The left-hand equality in (2.7) holds by property (i) of Q'. On the other hand, for *any* $Q \in N_\epsilon(P)$ we have

$$\begin{aligned} V(\delta_N, Q) &\geq (1 - \epsilon) V(\delta_N, P) + \epsilon \inf_{Q''} V(\delta_N, Q'') \\ &= (1 - \epsilon) V(\delta_N, P) + \epsilon U(0) && [\text{since } \delta_N \geq 0 \quad \text{and} \quad X \geq 0] \\ &= (1 - \epsilon) V(\delta_N, P) + \epsilon V(\delta_N, Q') && [\text{by property } (ii) \text{ of } Q'] \\ &= V(\delta_N, Q^*). \end{aligned}$$

This establishes the right-hand equality in (2.7), and completes the proof. (Note that the infimum on the right in (2.7) is obviously attained by the distribution $(1 - \epsilon)P + \epsilon\, Q^0$, where Q^0 is degenerate on zero. However, this distribution is *not* generally minimax.)

In retrospect it is obvious that the minimax portfolio cannot feature short sales (though the precise characterization of the minimax portfolio as δ_N is not so obvious). The reason is that the nonnegative orthant (in which X must lie) is bounded below but not above. If some asset, say the i^{th}, were sold short, then nature could choose a $Q \in N_\epsilon(P)$ whose marginal distribution for X_i had a very long right-hand tail. In this way nature could make the investor's expected utility arbitrarily small. To put it differently, the investor avoids short-selling because he is concerned that the true distribution differs by a small amount from his subjectively-assessed distribution. If he sells an asset short, even low-probability events can greatly lower his expected utility.

III. Minimax Under Alternative Assumptions

Under the portfolio-selection process described in the last section, the investor must assess a subjective distribution P for X. We have said nothing about the process by which this would be done. Possibly the investor would assess P in two stages, first assessing a distribution for the market as a whole, and then assessing each individual investment relative to the market.

Formally, let

$$G = X_1 + \cdots + X_n \tag{3.1}$$

be total market value in the second period, and let

$$f_i = X_i/G \qquad i = 1, \cdots, n \tag{3.2}$$

be the fraction of the total second-period market value represented by the i^{th} asset. Let $f = (f_1, \cdots, f_n)$; then $f_i \geq 0$ for all i, and $\sum_{i=1}^{n} f_i = 1$. Note that $X = Gf$. We will suppose that the investor assesses a distribution P_1 for G and distribution P_2 for f, *and that he treats these as independent.* Under independence, the subjective distribution for (G, f) is just the product measure $P_1 \times P_2$. We will modify our notation for the expected-utility function V so that

$$V(\delta, Q_1 \times Q_2) = E_{Q_1 \times Q_2}\, U(G\delta \cdot f). \tag{3.3}$$

Let $\bar{f}$ be the expected value of the vector f under P_2.

Before we move on to study minimax behavior in this context, it is worth asking what the market equilibrium would be like if each investor had the *same* subjective distribution P_2 for f. It turns out that the equilibrium price vector p would be proportional to $\bar{f}$, and that each investor would hold a market portfolio. (This means that all assets would have the same expected return, and shows that the subjective independence of G and f is not an innocuous assumption.) To establish this result, note that the first-order conditions for the investor's expected-utility maximization require that there be a Lagrange multiplier λ such that

$$E_{P_1 \times P_2} U'(G\delta \cdot f)Gf = \lambda p. \tag{3.4}$$

The market portfolio δ_M has, by definition, all of its components equal to $k = W/$

Σp_i. Note that $\delta_M \cdot f = k$ for all f, and hence we have

$$E_{P_1 \times P_2} U'(G\delta_M \cdot f) Gf = [E_{P_1} U'(Gk)G]\bar{f}. \tag{3.5}$$

It follows that if p is proportional to $\bar{f}$, say $p = \alpha \bar{f}$ for some $\alpha > 0$, then (3.4) will hold with $\delta = \delta_M$ and $\lambda = [E_{P_1} U'(Gk)G]/\alpha$. For the correct choice of α, there will be a general equilibrium in which $p = \alpha \bar{f}$ and each investor holds a market portfolio, as claimed. We emphasize that this result will not hold in the general case in which expectations are heterogeneous. However, the result does suggest that p might be *nearly* proportional to $\bar{f}$. As the next result shows, minimax behavior with respect to P_2 would still lead many investors to hold a market portfolio.

PROPOSITION 2: Suppose there exists a non-negative vector $a = (a_1, \cdots, a_n)$ satisfying $\sum_{i=1}^{n} a_i = 1$ such that $(1 - \epsilon)\bar{f} + \epsilon a$ is proportional to p. Then there exists a distribution $Q_2^* \in N_\epsilon(P_2)$ such that

$$\begin{aligned} \sup_{\delta \in B} V(\delta, P_1 \times Q_2^*) &= V(\delta_M, P_1 \times Q_2^*) \\ &= \inf Q_2 \in N_\epsilon(P_2)\ V(\delta_M, P_1 \times Q_2) \end{aligned} \tag{3.6}$$

Proof: Let $Q_2^* = (1 - \epsilon)P_2 + \epsilon Q_2^a$, where Q_2^a is degenerate on a. δ_M will be optimal against $P_1 \times Q_2^* = P_1 \times [(1 - \epsilon)P_2 + \epsilon Q_2^a]$ (thereby yielding the left-hand equality in (3.6)) if and only if there exists a Lagrange multiplier λ such that

$$(1 - \epsilon)E_{P_1 \times P_2} U'(G\delta_M \cdot f)Gf + \epsilon E_{P_1 \times Q_2^a} U'(G\delta_M \cdot f)Gf = \lambda p. \tag{3.7}$$

As we noted in the discussion preceeding the statement of this proposition, $\delta_M \cdot f$ is constant for all f; say $\delta_M \cdot f \equiv k$. We can therefore write (3.7) as

$$(1 - \epsilon)[E_{P_1} U'(Gk)G]\bar{f} + \epsilon[E_{P_1} U'(Gk)G]a = \lambda p.$$

The positive factor $E_{P_1} U'(Gk)G$ can be absorbed into λ; hence if there exists a non-negative a with $\Sigma a_i = 1$ such that $(1 - \epsilon)\bar{f} + \epsilon a$ is proportional to p, δ_M will be optimal against $P_1 \times Q_2^*$. This establishes the left-hand equality in (3.6). The right-hand equality holds trivially since $V(\delta_M, P_1 \times Q_2)$ is independent of Q_2.

IV. Summary and Conclusions

In this paper we have examined the implications of certain forms of minimax behavior in portfolio selection. Though minimax behavior is not truly optimal, there are reasons to suppose that it might approximate the behavior of some investors. One reason is the obvious discomfort that many decisionmakers feel when forced to make decisions based on subjective probability distributions, particularly when those distributions were formed on the basis of vague information. One would not be surprised if a decisionmaker sought to protect himself against the possibility that his distribution is incorrect.

Portfolio selection seems a natural context for minimax behavior because of the enormous amount of information, both direct and indirect, that an investor must incorporate into his probability distributions. The results of this paper, which show that minimax behavior leads to reasonable portfolios of the sort we observe in practice, lend some support to the minimax hypothesis.

V. Appendix

Proof of the Lemma:

Assume without loss of generality that the first j coordinates of δ_N are zero, and the others positive. Let e_i be the i^{th} standard basis vector. Then by definition of δ_N, δ_N solves

$$\begin{aligned} &\max_\delta\ (1-\epsilon)E_P U(\delta\cdot X) \\ &\text{subject to } p\cdot\delta \leq W \\ &\qquad -e_i\cdot\delta \leq 0\ (i = 1, \cdots, n) \end{aligned}$$

and there must exist Lagrange multipliers λ, μ_1, $\cdots$, μ_n, all non-negative, such that

$$(1-\epsilon)E_P U'(\delta_N\cdot X)X = \lambda p - \mu_1 e_1 - \cdots - \mu_n e_n \tag{A.1}$$

In fact, $\mu_i = 0$ for $i > j$, since the corresponding constraints are non-binding.

For any non-negative n-vector a, let Q^a be the distribution that is degenerate on a. We will construct an a such that δ_N is optimal against $(1-\epsilon)P + \epsilon Q^a$ among $\delta \in B$. For this it is necessary and sufficient that a satisfy

$$(1-\epsilon)E_P U'(\delta_N\cdot X)X + \epsilon U'(\delta_N\cdot a)a = \lambda p. \tag{A.2}$$

Define

$$a_i = \frac{\mu_i}{\epsilon U'(0)} \qquad (i = 1, \cdots, n)$$

and set $a = (a_1, \cdots, a_n)$; *then by* $(A.1)$,

$$(1-\epsilon)E_P U'(\delta_N\cdot X)X + \epsilon U'(0)a = \lambda p.$$

But the first j coordinates of δ_N are zero, and only the first j coordinates of a are non-zero; hence $\delta_N\cdot a = 0$ and (A.2) must hold. If we set $Q' = Q^a$, then Q' certainly satisfies condition (i) of the lemma; also, since $a_i = 0$ for $i > j$, condition (ii) is satisfied.

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