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RISK ESTIMATION IN ACTIVE INVESTMENT MANAGEMENT

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Estimating Security Price Risk Using Duration and Price Elasticity

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I. Introduction

IN THIS PAPER WE have developed a model which uses duration and price elasticity to estimate security price risk. The focus is on the volatility of security prices, and how this affects the return to the investor. The investor's return is made up of the dividend yield and the proportional change in the asset price. Historically, dividends have been relatively constant, which would imply that most of the volatility in the investor's return is derived from the volatility in the price of the stock. We are, therefore, interested in the change in price, as measured by its elasticity, and how the elasticity, in turn, affects the investor's return.

Hicks [1939] derived the equivalent "average period" measuring the "elasticity of asset prices with respect to a discount ratio." His interest at the time was focused on fixed income securities such as bonds. Macaulay [1938] had proposed the concept of "duration" to represent the "essence of time element of a loan" with risk proxying properties. Hicks' "average period" and Macaulay's "duration" are equivalent concepts. We have also used duration, or the average period, to measure the elasticity of security prices.

Malkiel [1963] first considered the theoretical relationships of price volatility of a share of common stock. Malkiel concluded that as the general level of share prices change, the prices of growth stocks must fluctuate more than proportionately. Haugen and Wichern [1974] examined the relative response of value in common stocks to changes in the level of interest rates and concluded that the elasticity of share price with respect to parameter changes can be simply and directly related to the "duration" of the stream of payments accruing to the security. Boquist, Racette and Schlarbaum [1975] examined the time-risk relationship as measured by duration.

There is some similarity between the term structure of interest rates and the structure of equity prices. For a given change in interest rates it is known that

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long-term bonds exhibit wider fluctuations than a short issue. In a similar way the response of non-dividend paying growth stocks to a change in the capitalization rate applied to equities in general, are inherently more volatile than the response of the average standard issues of common stocks to changes in the capitalization rate. The volatility of share prices in response to changes in the capitalization rate is an increasing function of the period during which growth continues. The volatility of share prices decreases as dividend payments increase on growth stocks. Volatility of share prices, and therefore the return of the security, is hereby linked to the potential for growth and the extent of time that the expected growth will persist.

In this paper we have introduced a moving average model to generate market expectations of the risk free rate and the market return on a quarterly basis, and derived a formulation for duration in terms of market expectations. An analysis based upon the expectations for growth in the share price which, in turn, is a function of investors' expectations of growth in the return of the security, is developed on the basis of the concept of duration. We then examined the price response to changes in the variables determining price, by deriving their elasticities in relation to duration. The model developed was then tested empirically.

In part II a model of duration is developed from the common stock valuation model, using market expectations of the investor's return. The price response to changes in the variables determining price is examined in part III. The relationship between duration and the elasticities in share prices is discussed in part IV, and the empirical results of the model discussed are presented in part V, with the conclusion given in part VI.

II. The Model

The dividend valuation model with constant growth for stock prices is:

$$P_t = \frac{d_t(1 + g_t)}{r_t - g_t} \tag{1}$$

where P_t = the stock price at the end of period t; d_t = dividends per share paid during period t; g_t = the "expected" growth rate in dividends per share as perceived at end of period t; r_t = the return required by investors to obtain the present value of future dividends. Combining the assumptions of (1): 1) constant growth at rate g_t; and 2) constant discount rate with the usual definition of duration:

$$D = \frac{\sum_{s=1}^{\infty} s \dfrac{d_s}{(1 + r_s)^s}}{\sum_{s=1}^{\infty} \dfrac{d_s}{(1 + r_s)^s}}, \tag{2}$$

by substituting $d_s = d_t\,(1 + g_t)^s$, and $r_s = r_t$ for all s. Equation (2) then reduces to:

$$D = \frac{1 + r_t}{r_t - g_t} \tag{2a}$$

Equation (2a) gives the duration of the stock, as perceived at the end of the t^{th} period if we assume constant dividend growth at rate g_t and a constant rate of discount r_t per period.

The return, r, at time t reflects investors' expectations about the riskless rate of interest, as well as the risk inherent in the expected growth rate in the dividend stream. The return, therefore, is assumed to be determined by the expected risk free rate and the expected security market rate of return as given in the following relationship:

$$r_t = \hat{R}_{f_{t+1}} + \theta(\hat{R}_{m_{t+1}} - \hat{R}_{f_{t+1}}) \tag{3}$$

where $\hat{R}_{f_{t+1}}$ = expected risk free rate for the quarter, $t + 1$, and thereafter as forecast at the end of quarter, t.

$\hat{R}_{m_{t+1}}$ = expected market rate for the quarter, $t + 1$, as forecast at the end of quarter, t.

θ = the market response factor with respect to the stock.

$\hat{R}_{m_{t+1}} - \hat{R}_{f_{t+1}}$ = the expected excess return above the expected risk free rate.

Investors use market expectations exclusively in determining r_t when $\theta = 1$, and the risk free rate only is used when $\theta = 0$.

We can restate equation (2) by incorporating equation (3) and simplifying to obtain:

$$D = \frac{1 + (1 - \theta)\hat{R}_f + \theta\hat{R}_m}{(1 - \theta)\hat{R}_f + \theta\hat{R}_m - g} \tag{2b}$$

Duration is expressed in terms of both the expected risk free rate, $\hat{R}_f$, the exected market return, $\hat{R}_m$, the excess return response factor, θ, and g.

The next task is to model the market's expectation for R_f and R_m. A moving average model (4) is used to generate the expected values of both variables.

$$\hat{R}_{f_{t+1}} = \bar{R}_f + \gamma_f(R_{f_t} - \hat{R}_{f_t})$$
$$\hat{R}_{m_{t+1}} = \bar{R}_m + \gamma_m(R_{m_t} - \hat{R}_{m_t}) \tag{4}$$

where $\hat{R}_{f_{t+1}}$ and $\hat{R}_{m_{t+1}}$ are the expectations at the end of period t for period $t + 1$, R_{f_t} and R_{m_t} are the realizations during period t, and γ_f, γ_m are the market response factors for the risk free rate and the market portfolio return, respectively.

Equation (4) states that the market expectation at the end of period t equals the market average plus a fraction of the error observed in last periods' expectation.

Note that when $\gamma = 0$, our expectations are not changed by R_{f_t} or R_{m_t}. This implies that the market does not change its expectations when it predicts the market rate. When $\gamma = 1$, our expectations are $\hat{R}_{t+1} = \bar{R}_t + (R_t - \hat{R}_t)$ which says that at each point in time the market puts a large amount of weight on the current observation. The forecast responds immediately to changes in the market return.

Incorporating equation (4) in (2c) we get:

$$D = \frac{1 + (1-\theta)\bar{R}_f + \theta\bar{R}_m + \gamma_f(1-\theta)(R_{f_t} - \hat{R}_{f_t}) + \gamma_m\theta(R_{m_t} - \hat{R}_{m_t})}{(1-\theta)\bar{R}_f + \theta\bar{R}_m + \gamma_f(1-\theta)(R_{f_t} - \hat{R}_{f_t}) + \gamma_m\theta(R_{m_t} - \hat{R}_{m_t}) - g} \tag{5}$$

Equation (5) states that duration changes through time primarily because (1) g changes, and (2) *less* importantly because R_f or R_m changes. Therefore, most of the change in duration comes from changes in g. Changes in P reflect investors' expectations about future changes in g, which indicate that changes in D are related to changes in P.

The relationship between D, duration, and price may be derived using the relationship of g with P. Let us restate equation (1) as an expression for g as follows:

$$g = \frac{Pr - d}{P + d} \tag{6}$$

For given d, P and r, g can be determined from equation (6). We can write equation (2a), an expression for D, in terms of d, P and r by substituting equation (6) in equation (2a):

$$D = \frac{1 + r}{r - \dfrac{Pr - d}{P + d}} \tag{7}$$

and by simplifying r drops out and equation (7) reduces to:

$$D = 1 + P/d \tag{7a}$$

Equation (7a) produces a ready method for computing the duration at each point in time of any stock that is priced on the basis of the constant growth of dividend model given in equation (1). P/d, the price/dividend ratio, is the reciprocal of the dividend yield, a value readily obtainable from almost all financial publications. According to equation (7a), the duration of a stock, assuming a constant growth model, is simply 1 plus the reciprocal of the dividend yield.

When constant growth is assumed, duration in equation (2a) is a function of only g and r, and if we assume $r = R_f + \theta(\hat{R}_m - \hat{R}_f)$, duration is then a function of θ and g. Invariably $R_f < R_m$, which implies that r increases as θ increase, and is an indication of a negative relationship between θ and duration. The relationship between growth and duration is positive; duration increase as g increase.

III. Price Response to Changes in Variables Determining Price

The dividend valuation model in (1) states that the investor receives a stream of dividends from the share of common stock, plus the appreciation (depreciation) in the price of the share when it is sold. Changes in the price, P, of the share directly results in changes in the rate of return of the share as reflected by investors perception of changes in the expected growth rate.

It is, therefore, necessary to investigate how price would respond to changes in the variables that determine price: g, r, R_{f_t} and R_{m_t}. From equation 1 the change in P resulting from a change in g is:

$$\frac{\partial P}{\partial g} = d(r-g)^{-1} + d(1+g)\,(-1)\,(r-g)^{-2}\,(-1)$$

$$= d\left[\frac{1}{r-g} + \frac{1\,(1+g)}{(r-g)^2}\right] \tag{8}$$

then

$$\frac{\partial P}{P} = d\left[\frac{1}{r-g} + \frac{(1+g)}{(r-g)^2}\right]\left[\frac{r-g}{d\,(1+g)}\right]\partial g$$

$$= \frac{D}{(1+g)}\partial g \quad \text{or} \quad D\frac{\partial g}{1+g} \tag{8a}$$

which says that the percentage change in price from a change in g is the change in g divided by the growth factor, $(1 + g)$, multiplied by the duration.

The change in price resulting from a change in r, the discount rate used by investors to value future dividends is derived in a similar way as (8a).

$$\frac{\partial P}{\partial r} = d(1+g)\,(-1)\,(r-g)^{-2} \tag{9}$$

Then the percentage change in price from changes in the expected return is:

$$\frac{\partial P}{P} = -D\frac{\partial r}{1+r} \tag{9a}$$

The change in P as a result of changes in the expected values of the risk free rate of interest and the rate of return on the market are obtained from equation (3). The changes in the discount rate (∂r) from changes in the expected risk free interest rate ($\partial \hat{R}_{f_{t+1}}$) is:

$$\frac{\partial r_t}{\partial \hat{R}_{f_{t+1}}} = 1-\theta \quad \text{and} \quad \partial r_t = (1-\theta)\partial \hat{R}_{f_{t+1}} \tag{10}$$

Then the percentage change in price from changes in the expected risk free interest rate will be:

$$\frac{\partial P}{P} = \frac{-D\,(1-\theta)\partial \hat{R}_{f_{t+1}}}{1+r} \tag{10a}$$

Similarly, changes in the discount rate (∂r) for changes in the expected market return ($\partial \hat{R}_{m_{t+1}}$) is:

$$\frac{\partial r}{\partial \hat{R}_{m_{t+1}}} = \theta \quad \text{and} \quad \partial r = \theta \partial \hat{R}_{m_{t+1}} \tag{11}$$

then the percentage change in price from changes in the expected market interest rate is:

$$\frac{\partial P}{P} = \frac{-D\theta\partial \hat{R}_{m_{t+1}}}{1+r} \tag{11a}$$

The change in the market price, P, as a result of changes in R_{f_t} and R_{m_t} can be derived in a similar manner and are given below:

$$\frac{\partial P}{P} = \frac{-D(1-\theta)\gamma_f}{1+r}\,\partial R_{f_t} \tag{12}$$

and

$$\frac{\partial P}{P} = \frac{-D\theta\gamma_m}{1+r}\,\partial R_{m_t} \tag{13}$$

The duration, D, is a constant factor in all the expressions for the elasticities of a price of a share with respect to changes in g, r, $\hat{R}_{f_{t+1}}$, $\hat{R}_{m_{t+1}}$, R_{f_t}, and R_{m_t}. D measures the relative volatility of price to changes in g and r. θ measures the volatility of r to changes in $\hat{R}_{f_{t+1}}$, $\hat{R}_{m_{t+1}}$. γ_f, γ_m measure the volatility of $\hat{R}_f$ and $\hat{R}_m$ to changes in R_f and R_m, respectively.

IV. Duration and the Elasticities in Share Prices

The relationship between duration, the discount rate, and the growth rate is presented in Table 1. For various values of r and g, we present d/P and D

where
$$d/P = \frac{r-g}{1+g} \tag{14}$$

and
$$D = \frac{1+r}{r-g} \tag{15}$$

The importance of D is given in the following relationships of the elasticities of share prices to changes in r, and g.

$$\frac{\partial P/P}{\partial r/(1+r)} = -D \tag{16}$$

Equation (16) shows that the value of a percentage change in price results from a proportional change in the discount rate is a negative function of duration:

$$\frac{\partial P/P}{\partial g/(1+g)} = D \tag{17}$$

while equation (17) shows that the percentage change in price from a proportional change in the growth rate is given by the duration.

Knowledge of the duration, therefore, provides us with ready information on price volatility for proportional changes in either r or g.

The duration given in Table 1 is in quarters. To obtain the duration in years the quarterly figures should be divided by 4. For each discount rate various values of the growth rate are given. As the growth rate increases relative to the discount rate, which means the dividend yield decreases, the duration increases. Since the investor receives only the dividends in this model, as the yield gets smaller, the duration gets longer.

The change in price that results from an instantaneous change in the discount rate, r, or the growth rate, g, can be measured from equations (8a) and (9a).

Table 1

The Relationship of Duration with the Discount Rate and the Growth Rate

$(1+r)^4 - 1$	$(1+g)^4 - 1$	d/P (Qtrly)	D (in Qtrs)
.05	0	.0122719	82.4871
.05	.02	.00727310	138.493
.05	.04	.00239516	418.509
.1	0	.00241134	42.4707
.1	.02	.0190562	53.4765
.1	.04	.0141212	71.8157
.1	.06	.00930343	108.487
.1	.08	.00459775	218.498
.15	0	.03555777	29.1233
.15	.02	.030444	33.8472
.15	.04	.0254538	40.2868
.15	.06	.0205823	49.5855
.15	.08	.015824	64.1952
.15	.1	.0111749	90.4866
.2	0	.0466349	22.4432
.2	.02	.0414664	25.1159
.2	.04	.0364229	28.4553
.2	.06	.0314992	32.7468
.2	.08	.0266901	38.4671
.2	.1	.0219912	46.4727

Although the elasticities are not a function of the absolute values of the dividend and price, the relative changes as given by $\partial P/\partial r$ and $\partial P/\partial g$ are.

Table 2 presents the values of $\partial P/\partial r$, and $\partial P/\partial g$ for various values of r, g, and d. Given the values of r, g, and d, the price of the share is calculated, and $\partial P/\partial r$ and $\partial P/\partial g$ are computed. Quarterly data is used in all the computations. For each value of r, changes in g and d are used to compute P. Then $\partial P/\partial r$ and $\partial P/\partial g$ are computed accordingly. $\partial P/\partial r$ shows how much price will change for each basis point change in r, the discount rate per quarter. For example, in the first line of Table 2, for each basis point change in r, price will change 16.6 cents, when $r = .05$, $g = 0$, $d = .25$, and $P = \$20.3718$. On the other hand, price will change by 47.03 cents for each basis point change in r, when $r = .05$, $g = .02$, $d = .25$, and $P = \$34.37$. Note here how the price change increases as g increases.

Similarly, $\partial P/\partial g$ shows by how much price will change as g changes. The corresponding change in price for a basis point change in g, the growth rate per quarter, is 16.8 cents and 47.4 cents. The change in price is slightly more than a change in g than for a change in r.

The change in the price of a share from changes in the risk free rate, R_f, and the market rate of return, R_m, can be obtained from the following relationships:

$$\frac{\partial P}{\partial R_{f_t}} = \frac{-DP}{1+r}[\gamma_f(1-\theta)] \tag{18}$$

and

$$\frac{\partial P}{\partial R_{m_t}} = \frac{-DP}{1+r}[\gamma_m \theta] \tag{19}$$

Table 2

Changes in Share Price Changes in the Discount Rate and the Growth Rate

$(1+r)^4 - 1$	$(1+g)^4 - 1$	d	P	$\partial P/\partial r$	$\partial P/\partial g$
.05	0	.25	20.3718	−1660.04	1680.41
.05	0	.5	40.7436	−3320.07	3360.82
.05	0	.75	61.1153	−4980.11	5041.23
.05	0	1	81.4871	−6640.15	6721.64
.05	.02	.25	34.3732	−4702.74	4736.94
.05	.02	.5	68.7465	−9405.47	9473.88
.05	.02	.75	103.12	−14108.2	14210.8
.05	.02	1	137.493	−18810.9	18947.8
.05	.04	.25	104.377	−43153.3	43256.7
.05	.04	.5	208.755	−86306.6	86513.3
.05	.04	.75	313.132	−129460.	12977.0
.05	.04	1	417.509	−172613.	17302.7
.1	0	.25	10.3677	−429.955	440.322
.1	0	.5	20.7353	−859.909	880.644
.1	0	.75	31.103	−1289.86	1320.97
.1	0	1	41.4707	−1719.82	1761.29
.1	.02	.25	13.1191	−685.045	698.099
.1	.02	.5	26.2382	−1370.09	1396.2
.1	.02	.75	39.3574	−2055.14	2094.3
.1	.02	1	52.4765	−2740.18	2792.4
.1	.04	.25	17.7039	−1241.48	1259.02
.1	.04	.5	35.4079	−2482.97	2518.03
.1	.04	.75	53.1118	−3724.46	3777.05
.1	.04	1	70.8157	−4965.94	5036.06
.1	.06	.25	26.8718	−2846.61	2873.09
.1	.06	.5	53.7436	−5693.21	5746.18
.1	.06	.75	80.6154	−8539.82	8619.27
.1	.06	1	107.487	−11386.4	11492.4
.1	.08	.25	54.3744	−11600.9	11654.3
.1	.08	.5	108.749	−23201.9	23308.5
.1	.08	.75	163.123	−34802.8	34962.8
.1	.08	1	217.498	−46403.7	46617.1
.15	0	.25	7.03082	−197.73	204.76
.15	0	.5	14.0616	−395.459	409.521
.15	0	.75	21.0925	−593.189	614.281
.15	0	1	28.1233	−790.918	819.041
.15	.02	.25	8.21181	−268.403	276.574
.15	.02	.5	16.4236	−536.806	553.148
.15	.02	.75	24.6354	−805.209	829.723
.15	.02	1	32.8472	−1073.61	1106.3
.15	.04	.25	9.82171	−382.099	391.825
.15	.04	.5	19.6434	−764.198	783.65
.15	.04	.75	29.4651	−1146.3	1175.47
.15	.04	1	39.2868	−1528.4	1567.3
.15	.06	.25	12.1464	−581.604	593.575
.15	.06	.5	24.2928	−1163.21	1187.15
.15	.06	.75	36.4391	−1744.81	1780.72
.15	.06	1	48.5855	−2326.42	2374.3
.15	.08	.25	15.7988	−979.381	994.879
.15	.08	.5	31.5976	−1958.76	1989.76
.15	.08	.75	47.3964	−2938.14	2984.64

Table 2—(*Continued*)

$(1+r)^4 - 1$	$(1+g)^4 - 1$	d	P	∂P/∂r	∂P/∂g
.15	.08	1	63.1952	−3917.52	3979.52
.2	0	.25	5.36079	−114.952	120.313
.2	0	.5	10.7216	−229.905	240.626
.2	0	.75	16.0824	−344.857	360.939
.2	0	1	21.4432	−459.809	481.252
.2	.02	.25	6.02897	−144.676	150.675
.2	.02	.5	12.0579	−289.352	301.35
.2	.02	.75	18.0869	−434.028	452.026
.2	.02	1	24.1159	−578.704	602.701
.2	.04	.25	6.86381	−186.609	193.406
.2	.04	.5	13.7276	−373.218	386.812
.2	.04	.75	20.5914	−559.827	580.218
.2	.04	1	27.4553	−746.436	773.623
.2	.06	.25	7.9367	−248.321	256.143
.2	.06	.5	15.8734	−496.642	512.286
.2	.06	.75	23.8101	−744.963	768.429
.2	.06	1	31.7468	−993.285	1024.57
.2	.08	.25	9.36678	−344.258	353.447
.2	.08	.5	18.7336	−688.517	706.893
.2	.08	.75	28.1003	−1030.78	1060.34
.2	.08		37.4671	−1377.08	1413.79

Note that equations (18) and (19) are the product of equation (9a) and $[\gamma_f(1 - \theta)]$, and $[\gamma;_m\theta]$, respectively. To get $\partial P/\partial R_{f_t}$ and $\partial P/\partial R_{m_t}$, we obtain $\partial P/\partial r$ in Table 2 and multiply by $[\gamma_f(1 - \theta)]$ and $[\gamma_m\theta]$, respectively.

V. Empirical Results

The price of a share of common stock is postulated to be a function of the following variables:

$$P_t = f(\theta, \gamma_f, \gamma_m, d_t, R_{f_t}, R_{m_t}, g_t) \tag{20}$$

with d_t, θ, γ_f, γ_m, and g_t being the characteristics peculiar to a stock that determine its price. Both P_t and d_t are directly observable, and θ, γ_f and γ_m can be estimated from historical data.

Changes in price result directly from changes in the rate of discount and changes in the investors' perception of growth. The expected growth rate is not directly observable, and the model does not contain a functional relationship to estimate the expected growth rate. An estimate of the changes in the growth rate is obtained by using a least squares regression to fit the model. The least squares estimating technique provides an estimate of the parameters of the model while allowing us to minimize the changes in g. For each stock the following parameters: θ, γ_1, and γ_2 are chosen to minimize the sum of the squared errors (SSE):

$$\textstyle\sum_{t=1}^{n} (g_t - \bar{g})^2 \tag{21}$$

where $\bar{g} = \sum_{t=1}^{36} g_t/36$ is the average quarterly growth for the 36 quarterly data on each of the 154 common stocks used in the study.

Table 3

Least Squares Estimation of the Parameters

CUSIP No.	γ_f	γ_m	θ	SSE × 10,000	$\bar{g} \times 100$	CUSIP No.	γ_f	γ_m	θ	SSE × 10,000	$\bar{g} \times 100$
453258	.26	.05	.084	48.6	−.18	2824	−.19	.01	.452	5.2	1.18
374280	−.01	.00	−25.470	35.6	−34.68	26609	−.51	−.20	.023	24.0	.45
546268	−.02	−.02	−.161	27.1	−.13	110097	−.24	.04	.251	15.3	.71
853734	.05	.01	.230	10.0	.73	532457	−.25	−.06	.062	23.5	.65
674786	−.12	−.07	.036	3.6	.99	589331	−.14	.01	−.049	6.4	.50
802037	−.16	−.08	.016	6.3	.85	717081	−.09	.00	−.623	12.0	−.43
815246	−.16	−.05	.061	3.8	1.05	806605	−.40	−.11	.034	26.7	.69
74077	−.51	−.12	.025	15.3	.30	832377	.14	.02	.543	17.4	.94
369856	−.29	.04	.103	11.2	−.09	852245	−.25	.02	.276	15.7	.80
370334	−.38	−.10	.057	15.0	.51	859264	−.12	.00	−.485	27.5	−.40
629527	−.25	.99	.020	14.0	.05	934488	−.58	−.27	.009	27.7	.41
853139	−.28	−.05	.022	13.3	.14	189054	−.52	−.10	.222	60.7	.39
99599	−.24	−.00	−1.094	8.1	−.58	279029	−.46	−.09	.118	24.5	.85
191216	−.48	−.17	.038	49.8	.43	628850	−.33	−.05	.123	24.5	.89
718167	−.29	.01	−.381	14.7	.13	742718	−.34	−.09	.097	10.3	.68
121691	−.51	−.17	.096	37.3	−.16	54303	−.19	.00	−2.951	45.4	−3.64
194828	−.25	−.01	−2.070	55.1	−3.00	375766	−.18	−.00	−1.421	28.7	−1.93
955465	−.33	.09	−.155	46.0	−.63	761525	−.21	.05	.006	6.9	.59
373298	−.32	.00	−.176	9.8	27	48825	−.10	−.11	.037	13.1	.34
546347	−.32	−.05	.063	15.3	.88	302290	−.12	.00	−.763	24.1	−1.40
962166	−.26	−.10	.014	7.4	.58	402460	.04	.03	.153	22.0	−.45
608030	−.31	−.04	−.384	116.3	−.88	607059	−.05	.20	.030	51.3	−.36
228669	−.14	.02	.300	17.2	.27	626717	.00	.00	−11.630	14.6	−15.64
391090	−.07	.05	−.024	8.9	.04	718507	.01	.00	−5.615	10.0	−7.59
460146	−.31	−.16	.006	13.2	.12	822635	.16	.02	.352	23.2	.21
793453	−.24	.00	−.324	14.3	−.48	853683	.05	.03	.254	40.0	−.05
905530	−.40	−.15	.039	17.7	.38	318315	−.02	.00	−21.440	39.4	−29.59
139861	−.04	.03	−.014	.9	.99	382550	−.61	−.68	.003	53.2	−.24
364730	−.37	−.07	.063	20.1	.75	781088	−.34	.08	−.051	9.7	.64
499040	−.26	−.11	.032	13.5	.73	690768	−.35	.24	.012	17.5	.08
887360	−.25	.10	.010	11.8	.50	542290	−.09	.04	.205	38.6	−.08
441560	−.20	−.00	−.346	27.3	−.35	478124	−.55	−.11	−.091	20.0	−.26
26375	−.35	.14	−.051	25.9	.50	87509	−.17	.07	.082	46.9	−.25
25321	−.15	.03	.305	15.7	.15	637844	−.23	.00	−.927	15.9	−1.82
260543	−.44	−.11	.024	25.8	.49	211452	−.34	.11	.006	10.0	−.42
263534	−.57	.04	−.312	123.7	−.35	200273	−.01	.00	−17.230	30.1	−23.62
302491	−.22	.96	.014	7.9	.13	244199	−.07	.00	−.723	14.0	−.87
427056	−.48	−.12	.041	47.7	.32	118745	−.16	.00	−.001	16.4	.23
905581	−.42	.40	.020	22.7	−.11	149123	−.18	.00	−.203	4.8	.14
9158	−.26	−.17	.029	4.5	.99	181396	−.22	.00	−.832	28.8	−1.08

57264	−.02	.00	−.862	2.9	−.45	731095	−.46	−.17	.028	18.8	.87
444492	−.13	.16	−.003	8.2	.86	1688	−.22	1.01	.034	12.0	.42
832110	−.10	.01	.176	3.1	.98	247361	−.16	.13	−.056	5.0	.63
91797	−.26	.08	−.081	12.3	.53	667281	−.19	.00	−.319	4.7	.18
481196	−.31	−.29	.019	18.4	.29	30177	−.51	−.17	.033	14.3	−.45
294098	−.15	.00	−4.337	97.0	−4.98	371028	−.15	.00	−2.600	24.0	−4.07
456866	−.19	.00	−.873	18.9	−.99	913025	−.18	.00	−1.253	21.7	2.17
122781	−.49	−.11	.058	12.9	1.00	24735	−.25	.00	−.392	20.6	−.12
212363	−.08	.05	−.016	2.4	.99	124845	−.27	−.01	−.569	7.0	−.54
459200	−.62	.16	−.072	110.6	.28	882387	−.06	.04	.100	27.8	.08
984121	−.46	−.11	.066	25.0	.77	45573	−.47	−.25	−.029	53.2	−.16
237688	.00	.00	.007	.0	1.05	146227	−.25	.00	−.823	32.0	−.92
253849	.00	.00	.007	.0	1.05	314099	−.23	.00	−1.439	15.6	−1.58
428236	−.05	.00	−.287	.7	.57	572342	−.18	.00	−1.580	31.9	−2.37
438506	−.01	.00	−16.760	25.6	−22.81	708160	−.64	.20	−.070	52.7	.34
848355	−.23	.01	.208	6.6	.87	812387	−.59	.04	−.353	57.1	−.12
902878	−.32	.00	−1.145	67.0	−2.02	482584	−.55	−.11	.067	16.1	.96
291011	−.37	−.10	.086	9.4	.66	477196	−.17	.04	.034	23.7	−.17
369604	−.19	.00	−1.245	15.9	−1.39	549577	−.37	−.00	−.331	12.4	−.41
852206	−.20	−.00	−1.360	29.9	−1.83	786514	−.49	.03	−.189	21.3	−.23
578592	−.24	−.00	−1.801	19.9	−2.53	515174	−.24	.12	.018	43.4	−.15
867068	−.23	.00	−2.235	35.6	−3.01	2040	−.54	−.12	.131	31.7	.74
963320	−.21	.00	−2.403	30.0	−2.91	171583	−.23	−.09	.076	16.1	.91
31897	−.13	.03	−.057	3.7	.69	580135	−.18	.24	−.014	1.8	1.03
989399	−.32	−.03	−.482	107.5	−.78	391442	−.23	−.00	−.842	16.1	−.63
620076	−.20	−.06	.045	6.9	.80	81721	−.18	.00	−2.594	61.7	−3.73
345370	−.20	.00	−4.593	57.6	−6.45	441812	−.23	.00	−2.063	43.2	−2.94
370442	−.51	.74	.048	63.2	−.67	25816	−.55	−.27	.012	50.3	.56
278058	−.35	.99	.019	12.2	.15	432848	−.15	.07	.279	46.1	.42
774347	−.01	.00	−4.365	38.5	−6.64	435081	−.06	1.01	.015	6.5	.82
157177	−.04	.00	−3.812	37.1	−5.18	53015	−.17	.12	−.051	5.7	.88
580169	−.08	−.01	−.192	13.1	.28	264830	−.19	.03	.133	9.2	.42
800	−.11	−.02	−.077	13.4	−.46	783549	−.43	1.00	.017	38.2	1.12
714041	−.05	−.09	.032	3.6	.81	460470	−.16	.00	−3.297	48.2	−4.59
351604	−.14	.00	−.571	10.2	−.20						
370838	−.23	.04	.025	11.4	.61						
26681	−.15	.00	−.417	7.3	.22						
71892	−.11	−.11	.009	1.1	.95						
277461	−.47	−.23	.014	30.8	.54						
604059	−.39	−.11	.056	19.1	.61						

A 4-period moving average process was used for smoothing the dividend stream of each stock to minimize the effects of dividend payment patterns as follows:

$$d_t = \frac{d_t + d_{t-1} + d_{t-2} + d_{t-3}}{4} \tag{22}$$

Because of the recursive nature of the time series model in (4), an initial forecast is needed to start the sequence. The historical quarterly data for the years 1961 to 1971 were used to find the values of the following parameters:

$$\bar{R}_f = 0.0104; \qquad \hat{R}_f = 0.00856; \qquad \hat{\gamma}_f = +0.865$$

$$\bar{R}_m = 0.02842; \qquad \hat{R}_m = 0.024334; \qquad \hat{\gamma}_m = +0.156$$

Each pair of γ_1, γ_2 implies a set of $\hat{R}_{f_t}$, $\hat{R}_{m_t}$. Then, given θ, r can be computed from the function: $r = \hat{R}_{f_t} + \theta(\hat{R}_{m_t} - \hat{R}_{f_t})$. Given γ_f, γ_m and θ, we evaluate the g_t stream and calculate SSE. The values that minimize the SSE become the desired estimates for g_t as shown in Table 3.

An examination of the estimation results in Table 3 shows that many of the estimates of θ are negative, contrary to our hypothesized positive relationship between the rate of return of a share of stock and the return of the market portfolio. Investors would normally demand a higher rate of return for their shares of stocks as the expected return of the general market portfolio rises. There are, of course, stocks having naturally negative relationships with the market portfolio.

An explanation for the negative θ estimates observed is the absence of an explicit function in the model relating the expected rate of growth, g, and the return of the market portfolio, R_m. The expected growth rate for the majority of stocks would normally be positively related to the expected returns of the market portfolio, since economic conditions that give rise to expectations for higher future rates of growth, also give rise to expectations of higher returns for the market portfolio. High R_m values imply larger than usual changes in the prices of a very large number of stocks, and the increase in price is more likely to reflect investors' expectations of increase in the future rate of growth, than an increase in dividends or a decrease in the rate of discount.

VI. Conclusion

It has been shown that duration and price elasticities are useful measures of risk in the price movement of a share of common stock. A simple measure of the duration of a share of common stock was derived, which is one plus the price/dividend ratio. The simplicity of this measure and the ease of computation could make it rather convenient for investors and portfolio managers to use it.

Further work on the model that we have undertaken include market wide estimates for γ_f and γ_m, the separation of the relationships of the discount rate and growth rate with R_m, and the expansion of the model to include a direct relationship between the growth rate (g) and the market portfolio return (R_m).

REFERENCES

1. Bierwag, G. O. and Kaufman, George. "Coping with the Risk of Interest Rate Fluctuations: A Note" *Journal of Business*, July 1977.
2. Boquist, J. A., Racette, G. A., and Schlarbaum, "Duration and Risk Assessment for Bonds and Common Stock," *Journal of Finance*, 1975.
3. Carr, J. L., Halpern, P. J. and McCallum, J. S., "Correcting the Yield Curve: A Re-interpretation of the Duration Problem," *Journal of Finance*, 1974.
4. Durand, David, "Payout Period, Time Spread, and Duration: Aids to Judgment in Capital Budgeting," *Journal of Bank Research*, Spring 1974.
5. Fisher, Lawrence, "An Algorithm for Finding Exact Rates of Return," *Journal of Business*, January, 1966, pp. 111-112.
6. Fisher, Lawrence, and Weil, Romas, L., "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business*, October, 1971.
7. Haugen, Robert A., and Wichern, Dean W., "The Elasticity of Financial Assets," *Journal of Finance*, September 1974.
8. Hicks, J. R. *Value & Capital*, Oxford, Columbia University Press, 1938.
9. Hopewell, Michael, and Kaufman, George, "Bond Price Volatility and Term to Maturity: A Generalized Respecification," *The American Economic Review*, September 1973.
10. Lintner, John, "Optical Dividends and Corporate Growth Under Uncertainty," *Quarterly Journal of Economics*, LXXVII, February, 1964, p. 49.
11. Macaulay, F. R., *Some Theoretical Problems Suggested by the Measurement of Interest Rates, Bond Yields and Stock Prices in the United States Since 1856.* National Bureau of Economic Research, New York; Columbia, University Press, 1938.
12. Malkiel, Burton, G.: "Equity Yields, Growth, and the Structure of Share Prices," *American Economic Review*, Vol. LIII, No. 5, December, 1963, pp. 1004-1031.
13. Malkiel, Burton G., and John G. Cragg, "Expectations and the Structure of Share Prices," *The American Economic Review*, September, 1970, pp. 601-617.
14. Reilly, Frank K., and Rupinder S. Sidhu: "The Many Uses of Bond Duration," *Financial Analysts Journal*, July/August, 1980, pp. 58-72.
15. Samuelson, Paul A., "The Effects of Interest Rate Increases on the Banking System," *American Economic Review*, March 1945.
16. Wallace, Anise: "Is Beta Dead?," *Institutional Investor*, July 1980.
17. Weston, J. Fred, and Eugene Brigham, *Managerial Finance*, The Dayton Press, Holt Rinehart & Winston, 1975.