Term Structure Modeling Using Exponential Splines: Discussion

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I hope that BKT find more realistic conditions in which their assumptions hold. Their idea is too good not to be applicable under rather general conditions. Indeed, their analysis shows why single-factor immunization can be far better than mere first-order approximations of optimal policy.

REFERENCES

David Durand, "Growth Stocks and the Petersburg Paradox," *Journal of Finance*, Vol. 12, No. 3. (September 1957), pp. 348-363.

Lawrence Fisher and Roman L. Weil, "Coping with the Risk of Interest-Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business*, Vol. 44, No. 4 (October 1971), pp. 408-431.

Gifford Fong and Oldrich Vasicek, "A Risk Minimizing Strategy for Multiple Liability Immunization." Unpublished paper circulated by Gifford Fong Associates, Santa Monica, California, September 15, 1980.

Frederick R. Macaulay, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856.* Publications of the National Bureau of Economic Research, Inc., Number 33 (New York: National Bureau of Economic Research, 1938).

R. M. Redington, "Review of the Principles of Life-Office Valuations," *Journal of the Institute of Actuaries*, Vol. 78, part 3, No. 350 (1952), pp. 285-340.

DISCUSSION

J. V. JORDAN*: This discussion focuses on three aspects of the Vasicek-Fong (VF) paper: (1) the implicit assumption of the pure expectations hypothesis in sections I and II, (2) some technical points about spline functions and (3) the specification of the term structure estimating equation.

That the pure expectations hypothesis underlies their methodology of estimating market forecasts is obvious from several quotes, e.g., "Specifically, the market implicit forecast of the one-period rate is equal to the forward rate for that period . . .". Certainly there is much literature (e.g., [10]) which maintains that forward rates are not equal to expected rates, but that forward rates include a term premium which may be positive or negative depending on many factors, including the preferences of some investors for some maturities over others. If there are term premia of consistent sign, then VF's forecasts will be biased. To be sure, measurement of premia is not an exact science, but the potential problem in ignoring them should be acknowledged.

Spline functions have a long history of use particularly in the physical sciences where their greatest application probably comes in fitting curves to experimental data (e.g., see [1] and [12]). For example, in the context of this paper, if we had T observations on the discount function, D_t, a cubic spline $f(t)$, could be estimated by the regression (see [12, pp. 21-28])

$$\underline{D}_{Tx1} = \underline{W}_{Tx(m+1)}\underline{Y}_{(m+1)x1} + \underline{\epsilon}_{Tx1} \tag{1}$$

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where

$\underline{D} = (D_1, D_2, \cdots D_T)'$;
$\underline{Y}$ = ordinates of the function $f(t)$ at $m + 1 (i = 0, 1, 2, \cdots m)$ preselected knot points ($\underline{Y}$ is estimated in the regression);
$\underline{W}$ = a matrix of cubic spline coefficients.

Derivation of $\underline{W}$ is too lengthy to include in this discussion, but essentially, the cubic spline coefficients are derived from the fact that the second derivative, $S_i(t)$, over an interval $(i - 1, i)$ is linear and so its equation may be written as a linear interpolation in the second derivatives at the endpoints of the interval, $S_i(t_{i-1})$ and $S_i(t_i)$. This equation may be twice integrated, with values of the constants imposed by continuity requirements, to yield a system of $m - 1$ equations in $m + 1$ unknowns (the endpoint second derivatives). Two unknowns are removed by assigning values of the second derivatives at $t = 0$ and $t = T$. The matrix $\underline{W}$ consists of the coefficients derived from the solution to this system.

In the term structure estimation problem, observations on D_t are not available. Instead these are defined by

$$\underline{P}_{nx1} = \underline{X}_{nxT}\underline{D}_{Tx1} + \underline{\mu}_{Tx1} \tag{2}$$

where

$\underline{P}$ = prices on n bonds

$\underline{X}$ = payments on n bonds through time T.

Combining (1) and (2) results in

$$\begin{aligned}\underline{P} &= \underline{X}(\underline{W}\underline{Y} + \underline{\epsilon}) + \underline{\mu} \\ &= \underline{X}\underline{W}\underline{Y} + \underline{\eta}\end{aligned}$$

where[1]

$$\eta = \underline{X}\underline{\epsilon} + \underline{\mu}.$$

McCulloch's [11] adaptation of the cubic spline is different in at least three ways: (i) the value of the ordinate of the discount function at $t = 0$ is one; (ii) the interval 0 to T is broken into $m - 2$ subintervals by $m - 1$ knot points and the spline function is defined in such a manner that Y_i, $i = 1, 2, \cdots m - 1$, are interpretable as the second derivatives; (iii) another parameter, Y_m, is estimated which is the first derivative of the discount function at $t = 0$. It appears that estimation of this mth parameter from a specification with $m - 1$ knot points is directly related to the tendency, noted by VF, of the cubic spline not to "tail off."[2] Algebraically, VF's ingenious transformation of the argument does produce the asymptotic property, because $\lim F(t) = \alpha$ as t approaches ∞. But it is not clear how this algebraic property would interact with McCulloch's particular

[1] Note that this formulation is an errors-in-variables and equations model. This point has not been considered in the literature.

[2] See [15] for other details including the development of cubic spline estimation as restricted least squares.

realization of the cubic spline or if VF have otherwise modified that method. Also, potential users will want visual evidence that the method produces believable spot and forward rate curves.

Finally, it is to be hoped that more rigorous evaluation of the term structure estimating equation will be prompted by advances in the methodology such as this. From the work of Malkiel [9, pp. 40–49], Buse [4], Schaefer [13] and Bierwag [2, pp. 727–30], it is clear that in a no-tax world the term structure of spot rates lies above (below) the yield curve when the term structure is upward (downward) sloping. Indeed, this systematic bias is a primary reason for the emergence of term structure estimation. However, beyond simple simulations (see [4, pp. 47–49] and [9, pp. 813]), there is scant empirical confirmation of this bias using actual bond data. Schaefer [13, p. 5] shows a yield bias of 10% at 25 years. However, his method is also a type of spline estimation, using higher order polynomials, so it is possible that some of this bias is due more to estimation methodology than to the coupon effect. Potential users have cause for hesitancy in adopting a complex (compared to yield-curve methods such as [3] or [5]) and little empirically tested methodology. Tax effects (see [7], [8] and [14] vastly complicate the issue. It is safe to say that there is no generally accepted way to incorporate tax effects into the equation, but, as indicated by the first paper in this session [6], they are probably significant.

REFERENCES

1. Ahlberg, J. H., *et al. The Theory of Splines and Their Applications.* New York: Academic Press, 1967.
2. Bierwag, G. O. "Immunization, Duration and the Term Structure of Interest Rates," *Journal of Financial and Quantitative Analysis*, December 1977.
3. Bradley, S. and Crane, D. "Management of Commercial Bank Government Security Portfolios: An Optimization Approach Under Uncertainty," *Journal of Bank Research*, Spring 1973.
4. Buse, A. "Expectations, Prices, Coupon and Yields," *Journal of Finance*, September 1970.
5. Echols, E. and Elliott, J. "A Quantitative Yield Curve Model for Estimating the Term Structure of Interest Rates," *Journal of Financial and Quantitative Analysis*, March 1976.
6. Ingersoll, J. and Constantinides, G. "Tax Effects and Bond Prices," working paper, University of Chicago, 1981.
7. Jordan, J. V. "Tax Effects in Term Structure Estimation," working paper, Georgia Institute of Technology, August 1981.
8. Livingston, M. "Taxation and Bond Market Equilibrium in a World of Uncertain Future Interest Rates", *Journal of Financial and Quantitative Analysis*, March 1979.
9. Malkiel, B. G. *The Term Structure of Interest Rates.* Princeton: Princeton University Press, 1966.
10. McCulloch, J. H. "An Estimate of the Liquidity Premium," *Journal of Political Economy*, February 1975.
11. ——. "The Tax-Adjusted Yield Curve," *Journal of Finance*, June 1975.
12. Poirer, D. *The Economics of Structural Change.* New York: North Holland, 1976.
13. Schaefer, S. "The Problem with Redemption Yields," *Financial Analysts Journal*, July/August 1977.
14. ——. "Taxes and Security Market Equilibrium," *Economic Journal*, June 1981.
15. Shea, G. "Some Practical Considerations In *Ad Hoc* Smoothing of Interest-Rate Term Structure Data," working paper, University of Washington, January 1982.