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Single Factor Duration Models in a Discrete General Equilibrium Framework

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SINGLE FACTOR DURATION models (SFDMs) for valuing debt securities have gained rapid acceptance by bond practitioners in recent years but slower acceptance by academics. In large measure, the slower academic acceptance reflects the lack of a rigorously developed underlying economic or financial theoretic base. Partially as a result, some SFDMs have undesirable behavioral characteristics. Foremost among these is that they appear to be inconsistent with accepted general equilibrium conditions. This paper has two purposes. The first is to construct explicitly the theoretical underpinnings of the model, and the second is to develop the conditions under which the model does satisfy general equilibrium conditions. This is done in sections I and II, respectively.

I.

Duration models did not spring forth full grown from a rigorous and well-defined general equilibrium economic model. Rather, they were developed primarily in piecemeal fashion on an almost ad hoc basis. The term "duration" was coined by Frederick Macaulay in his major review of interest rates and stock prices published in 1938.[1] Macaulay was searching for a more meaningful summary measure of the life of a bond than its term to maturity that incorporated the timing of the bond's flow of coupon payments. The question was how to weight the time periods to each payment (n). After deductively rejecting a weighting of the periods by the future value of each payment, Macaulay chose the present value of each payment. Because "of the insuperable difficulties connected with any attempt to discover the real rates of discount for each half-yearly period in the future," Macaulay chose to use the yield to maturity (i) to discount future payment (S) and defined duration as:[2]

$$D = \frac{\sum_{n=1}^{m} S_n n(1 + i)^{-n}}{\sum_{n=1}^{m} S_n(1 + i)^{-n}}. \tag{1.1}$$

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[1] Frederick R. Macaulay, *The Movements of Interest Rates, Bond Yields and Stock Prices in the United States Since 1856*, (New York: National Bureau of Economic Research, 1938). A good review of the early research on duration appears in Roman L. Weil, "Macaulay's Duration: An Appreciation," *Journal of Business*, October 1973, pp. 589–592.

[2] —, p. 52.

This specification is now referred to as the Macaulay duration. It is important to note that Macaulay developed duration only as a measure that "throws a flood of light on the fluctuations of bond yields on the actual market."[3]

One year after Macaulay, J. R. Hicks independently derived Equation (1.1) by computing the elasticity of capital value with respect to the discount ratio and assuming that the interest rate is the same for all maturities.[4] Like Macaulay, Hicks used duration basically as a volatility measure. In 1945, Paul Samuelson effectively derived Equation (1.1) by examining the effects of changes in interest rates on the present value of net worth of a financial institution and then computing the first derivatives of the inflows and outflows with respect to the average interest rate.[5] Unaware of Macaulay's or Hicks's work, he called the term "average time period," and proceeded to use the statistic to analyze the sensitivity of financial intermediary institutions to interest rate changes. He concluded that "increased interest rates will help any organization whose (weighted) average time period of disbursements is greater than the average time period of its receipts."[6] He did not, however, construct a formal economic model.

In 1952, the British actuary F. M. Redington, apparently unaware of Samuelson's article, used approximately the same methodology to derive Equation (1.1) by solving for "the distribution of the term of the assets in relation to the term of the liabilities . . . (that will) reduce the possibility of loss arising from a change in interest rates."[7] He referred to the statistic as the "mean term." Redington shows that "the existing business is immune to a general change in the rate of interest" when the mean term of the assets is equal to that of the liabilities and coined the term "immunization."[8] Other than assuming "a uniform rate of interest whatever the term, and that all funds are invested on fixed-rate securities which are irredeemable or redeemable at a fixed date" and listing some of the practical complications, Redington did not consider further either the underlying assumptions or implications.[9] It is ironic that Redington's article, published in an actuarial journal not widely read by nonactuarials, has received wider recognition that Samuelson's similar and earlier article in the *American Economic Review.*

The first researchers to explicitly concern themselves with the underlying behavioral assumptions of duration analysis were Lawrence Fisher and Roman Weil in their 1971 article.[10] They explicitly assume the expectations theory of the term structure and that all unexpected interest rate changes—the sole source of uncertainty—will be the same for all terms to maturity, i.e., the changes are additive. Unlike their predecessors, however, they do not assume a flat term

[3] —, p. 50.

[4] J. R. Hicks, *Value and Capital,* 2nd ed., (Oxford University Press, 1939), pp. 185–88.

[5] Paul A. Samuelson, "The Effect of Interest Rate Increases on the Banking System," *American Economic Review,* March 1945, pp. 16–27.

[6] —, p. 19.

[7] F. M. Redington, "Review of the Principles of Life-Office Valuations," *Journal of the Institute of Actuaries,* 1952, p. 289, reprinted in Martin L. Leibowitz, ed., *Pros & Cons of Immunization,* (New York: Solomon Brothers, 1980), pp. 286–340.

[8] —, p. 289.

[9] —, pp. 289–90.

[10] Lawrence Fisher and Roman L. Weil, "Coping with the Risk of Interest-Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business,* October 1971, pp. 408–31.

structure. Fisher and Weil develop the conditions under which a portfolio of default and option free coupon bonds will realize a return at the end of a given planning period no less than that promised at the time the portfolio was purchased if interest rates change in the assumed pattern, interest is compounded continuously, and investors have single known and fixed investment horizons. The immunization condition involves setting the weighted average of the term to each payment, where the weights are the present values of each payment computed by using one period zero coupon discount rates, equal to the investor's holding period. They term this average duration because of its mathematical similarity to the Macaulay, Hicks, and Redington measures. In a number of subsequent papers, G. O. Bierwag, George G. Kaufman, and Chulsoon Khang assume a number of alternative "reasonable" stochastic processes governing interest rate movements and derive the appropriate statistic, referred to as immunizing duration, that when set equal to the investor's planning period, generates returns for the planning period no less than those promised.[11]

At this point, the models were subjected to a number of criticisms. The earliest and most widespread criticism was that duration is useful only when yield curves are flat. This, of course, was explicitly assumed by everyone prior to Fisher and Weil, but was put to rest with the development of duration measures for a variety of stochastic processes. A second criticism was directed more at the use of SFDMs to achieve immunization than at duration per se. It held that few investors are totally interest rate risk averse and that immunization is only one, and possibly not even a very interesting, point on the risk continuum. However, the emphasis on immunization reflects both computational and practical aspects. Computationally, duration formulas are easiest to derive by solving for the immunization conditions for each assumed stochastic process. Practically, immunization against interest rate risk is of importance to some investors, such as life insurance companies, and to all investors as a means of obtaining the riskless return. Bierwag and Khang develop a formal model to show that immunization is consistent with investors' satisfying minimax conditions.[12] But derived immunizing durations can be used both to value bonds with any degree of interest rate risk exposure and to construct portfolios of coupon bonds that track a desired but not available zero coupon reference bond. Use of duration analysis in active bond portfolio management has been discussed by Bierwag, Kaufman, and Toevs in a number of papers and will be developed more formally later in this paper.[13]

[11] G. O. Bierwag, "Immunization, Duration and the Term Structure of Interest Rates," *Journal of Financial and Quantitative Analysis*, Dec. 1977, pp. 725–42; G. O. Bierwag and George G. Kaufman, "Coping with the Risk of Interest-Rate Fluctuations: A Note," *Journal of Business*, July 1977, pp. 364–70; and Chulsoon Khang, "Bond Immunization when Short-Term Rates Fluctuate More than Long-Term Rates," *Journal of Financial and Quantitative Analysis*, December 1979, pp. 1085–89. See also I. A. Cooper, "Asset Values, Interest-Rate Changes and Duration," *Journal of Financial and Quantitative Analysis*, December 1977, pp. 701–23.

[12] G. O. Bierwag and Chulsoon Khang, "Immunization Strategy is Mini-Max Strategy," *Journal of Finance*, May 1979, pp. 389–414.

[13] G. O. Bierwag, George G. Kaufman, Robert Schweitzer, and Alden Toevs, "The Art of Risk Management on Bond Portfolios," *Journal of Portfolio Management*, Spring 1981, pp. 27–36 and G. O. Bierwag, George G. Kaufman, and Alden Toevs, "Duration Analysis and Active Bond Portfolio Management," (Unpublished paper, University of Oregon), October 1980. See also Myron A. Grove,

(Valuation of risk sources other than interest rate risk will, of course, entail the need for additional factors in the model describing these sources of uncertainty.)

A third criticism argues that the models are completely deterministic and do not contain an error or residual term. This occurred because the earliest developers either did not assume a stochastic process or assumed only one such process known with certainty. However, once uncertainty and multiple stochastic processes were assumed, the need to identify correctly the true process became obvious. If identified incorrectly, the model contains an error term, which is labeled "stochastic process risk."[14] The assumed stochastic process may be obtained either by fitting single factor measures to past observations or by choosing a specific single factor measure on the basis of experience or expectation.

Another criticism concerns the nature of interest rate uncertainty that can be described by a single factor. Changes in all interest rates along the term structure must be perfectly correlated.[15] Thus, stochastic process risk may be incurred even if the stochastic process is identified correctly but the process does not satisfy this condition.

Both Ingersoll, Skelton and Weil, and Cox, Ingersoll and Ross argue that the stochastic processes assumed in traditional SFDMs are inconsistent with general equilibrium conditions.[16] This occurs because the return functions for immunized portfolios generated by these processes are strictly convex with respect to the magnitude of the interest rate shock so that the larger the shock, the greater the return. It also follows that the larger the coupon rate, the greater the return for a given shock. Thus, opportunities may exist for riskless arbitrage and the highest coupon bonds will dominate all lower coupon bonds. Although this criticism is valid for all of the processes used in the published studies, a general additive stochastic process (GASP), for which the return function is not strictly convex and thus generates implications that are consistent with general equilibrium, is developed in the second part of this paper. For stochastic processes that do not have strict convexity properties, a portfolio of coupon bonds has the same characteristics as a zero coupon bond of the same immunizing duration. Thus, in the absence of transactions costs, the two portfolios are indistinguishable, as are differing compositions of coupon bonds having the same duration.[17] If the desired zero coupon reference bond is not available, portfolios of coupon bonds may be constructed to track the zero coupon reference bond perfectly.

In view of these criticisms, why have traditional single factor duration models been widely adopted, and why can they be expected to continue to receive

"On 'Duration' and the Optimal Maturity Structure of the Balance Sheet," *Bell Journal of Economics and Management Science*, Autumn 1974, pp. 696–709.

[14] G. O. Bierwag, George G. Kaufman, and Alden Toevs, "Bond Portfolio Immunization and Stochastic Process Risk," *Journal of Bank Research*, (Forthcoming).

[15] Ronald Lanstein and William F. Sharpe, "Duration and Security Risk," *Journal of Financial and Quantitative Analysis*, November 1978, pp. 653–68.

[16] Jonathan E. Ingersoll, Jr., Jeffrey Skelton, and Roman L. Weil, "Duration Forty Years Later," *Journal of Financial and Quantitative Analysis*, November 1978, pp. 627–50; and John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross, "Duration and Measurement of Basis Risk," *Journal of Business*, January 1979, pp. 51–61.

[17] G. O. Bierwag and George G. Kaufman, "Bond Portfolio Strategy Simulations: A Critique," *Journal of Financial and Quantitative Analysis*, September 1978, pp. 519–28.

substantial attention? The answer is because they are cheap to use and their empirical results compare favorably with more sophisticated and rigorously developed models. Nonconsol bonds are by their nature more difficult to value than equities. The passage of time automatically changes the returns on the bonds as the term to maturity decreases, except when term structures are flat. Moreover, because of the differing characteristics of bonds, it is unlikely that the passage of time will affect two alike, even if they had equal terms to maturity. Thus, the covariances of returns between any two bonds or between any one bond and a market portfolio, however defined, are unlikely to be intertemporally stable. As Brennan and Schwartz have noted:

> The obstacle to estimation of a variance covariance matrix of bond returns from historical data has been that a structure similar to that for common stocks could not reasonably be assumed, and it is unclear what alternative structure should be assumed.[18]

This is a prime reason why single and multiple factor CAPM-type models have been significantly less successful for individual bonds or bond portfolios than for equity securities.[19] Improved results have been achieved when, following Brennan and Schwartz, individual points on the term structure rather than individual bonds have been used. Whether these models are superior to SFDMs is an empirical question of cost effectiveness. Preliminary evidence suggests that the SFDM performs well relative to two factor models as well as to naive maturity matching models in immunizing default and option free coupon bonds for specific and known planning periods.[20]

Thus, it is worthwhile to develop the general SFD bond valuation model further. In particular, it is important to list explicitly the major economic assumptions underlying its development and use:

1. Investors attempt to maximize expected returns for a given level of risk exposure over a known and certain planning period(s).
2. Only default and option-free debt securities are considered (interest rates are the only source of uncertainty).
3. The true stochastic process is known.
4. Changes in all interest rates on the term structure (all returns) are perfectly correlated.
5. The return function generated by the stochastic process for an immunized portfolio is non-convex.

[18] Michael J. Brennan and Eduardo S. Schwartz, "Duration, Bond Pricing and Portfolio Management" and G. O. Bierwag, George G. Kaufman, and Alden Toevs, "Duration Analysis in Bond Portfolio Management." both in G. O. Bierwag, George G. Kaufman, and Alden Toevs, eds., *Innovations in Bond Portfolio Management: Duration Analysis and Immunization*, (JAI Press, Forthcoming).

[19] Irwin Friend, Randolph Westerfield, and Michael Granito, "New Evidence on the Capital Asset Pricing Model," *Journal of Finance*, June 1978, pp. 903–17; Michael D. Joehnk and James F. Nielsen, "Return and Risk Characteristics of Speculative Grade Bonds," *Quarterly Review of Economics and Business*, Spring 1975, pp. 27–46; John Percival, "Corporate Bonds in a Market Model Context," *Journal of Bank Research*, October 1974, pp. 461-68; and Frank F. Reilly and Michael D. Joehnk, "The Association Between Market Determined Risk Measures for Bonds and Bond Ratings," *Journal of Business Research*, December 1976, pp. 1387–403.

[20] See papers in volume cited in footnote 18.

To the extent that either or both assumptions three or four are violated, unexplained return variance (stochastic process risk) results. The model may be written analogous to the one factor market model as follows:

$$R_{jD_sq} = P_q + \alpha_j + \gamma_j(| D_s - q |)r_{K,q} + e_j \tag{1.2}$$

where:

R_{jD_sq} = return relative on bond or portfolio j with duration D_s for period q

s = assumed stochastic process for all interest rates

q = number of years to end of investor's planning period

P_q = risk-free return relative for period q (return on zero coupon bond with maturity q)

D = immunizing duration

$r_{K,q}$ = excess return relative on zero coupon reference bond with maturity $K \neq D$, q for planning period q

α_j = extramarket return relative

γ_j = proportionality factor (not necessarily constant)

e_j = error term

All return variables represent expected values. This equation is derived in basically the same form in the second section of this paper.

If the true stochastic process can be summarized in one factor and is identified correctly, then $e_j = 0$. If the market is efficient, then $\alpha_j = 0$. The degree of interest rate risk the investor wishes to assume is directly related to $|D_s - q|$. If the investor wishes to immunize, then D_s is set equal to q and expected $R_j = P_q$. If the investor wishes to assume interest rate risk, then D_s is set either larger or smaller than q and expected $R_j > P_q$. γ_j depends on the stochastic process and the reference bond selected. If there is no shock—interest rates do not change unexpectedly—the expected return is equal to the riskless return. The model also emphasizes the importance of the investor's planning period and that the expected returns on the same security may differ for different investors having different planning periods.

Equation (1.2) is likely to be difficult to estimate empirically. For example, the model suggests that the market will reward risk only if investors' q's are either homogeneous or distributed unevenly over all trading periods. If q's were distributed evenly, then one investor's risky portfolio would be another's riskless portfolio. Thus, empirical fits of Equation (1.2) may be more difficult for bonds using ex-post data for the market as a whole than it is for equities, where differences in individual planning periods may not be as important.[21] Likewise, the effects on realized returns of misidentifying the stochastic process may not be confined to e_j. Part of the impact may appear in the estimate of γ_j. Nevertheless,

[21] George G. Kaufman, "Duration, Planning Period, and Tests of the Capital Asset Pricing Model," *Journal of Financial Research*, Spring 1980, pp. 1–9.

the equation is useful to investors in providing a conceptual framework for formulating their strategy if they predict interest rate changes for a reference bond.

Equation (1.2) suggests a two-step procedure for bond portfolio management in the "real world." First, the investor must choose the bond that, in a world of complete markets including a continuum of zero coupon bonds, provides the desired degree of risk exposure, proportional to $| D_s - q |$. Second, if the selected zero coupon bond is not available, the investor must construct a portfolio of available bonds that has the same immunizing duration as this reference bond. If the stochastic process has no strict convexity properties and the assumed one factor stochastic process is correct, the composition of the portfolio is irrelevant. The equations for constructing such portfolios for a discrete SFDM are derived in the next section.

II.

In a single factor duration model, there is a single source of interest rate uncertainty that affects the return on each security. As was shown in Equation (1.2), the return on any security can be expressed as a function of the return on any reference security subject only to the same uncertainty. This relationship is technically derived in this section, first for the generally multi-factor case for both single and multiple periods and then specialized to the single factor case. When interest rate uncertainty enters the model in the form of a generalized additive stochastic process, the return relationship is linear and the SFDM has special properties that satisfy general equilibrium conditions and are consistent with a mean-variance framework.

Discount Functions and Future States

Assume that at the beginning of a period of analysis a discount function $P(0, t)$, $t = 1, 2, \cdots$ is observed. This function gives the present value of \$1 to be paid at time t. Each of the flows of an income stream can be evaluated with this discount function. At the beginning of the next period, the discount function will be $P_s(1, t)$, where s is an indicator of the state of the economy and is an integer in $[1, K]$. At time $t = 0$, it is uncertain which of the mutually exclusive future states will occur. It is assumed that trading can occur only at integer dates.

Denote the one period return per dollar on a t-period pure discount bond as

$$R_{ts} = P_s(1, t)/P(0, t),\ t > 1,\ s \in [1, K], \quad \text{and}$$

$$R_{1s} = 1/P(0, 1). \tag{2.1}$$

A matrix of one period returns may be denoted as

$$R = \{R_{ts}\},\ t = 1, 2, \cdots, T;\ s = 1, 2, \cdots, K. \tag{2.2}$$

if T pure discount bonds are available.

The markets for these securities are complete if the number of pure discount bonds having independent returns is equal to the number of states. Markets are assumed to be complete. In the matrix of one period returns, we need only to consider K independent pure discount bonds.

If the markets are complete, there exist K Arrow-Debreu securities (contingent claims) that can be formed by portfolios of these pure discount bonds. Let $A = \{\alpha_{st}\}$ be a set of K portfolios, where α_{st} is the proportion invested in a t-period pure discount bond in the sth portfolio, and $\sum_{t=1}^{K} \alpha_{st} = 1$. Then

$$AR = \begin{bmatrix} \theta_1, & 0, \dots & 0 \\ 0, & \theta_2 & \vdots \\ \vdots & & \ddots & \\ 0 & & \theta_k \end{bmatrix} = \theta \tag{2.3}$$

is a diagonal matrix giving the returns on the Arrow-Debreu securities. If the markets are complete, the rank of θ is K; that is $\theta_s \neq 0$ for any s.

Assume that every investor observes $P(0, t)$ as the current discount function. If $P(0, t)$ is consistent with competitive conditions in which there are no opportunities for profitable and riskless arbitrage, $0 < \theta_s < \infty$, for all s. The Hicksian forward price for a pure discount bond can be defined as $P(1, t) = P(0, t)/P(0, 1)$. Each of the forward prices is a function of the observed current discount function.[22]

In discrete models of competitive equilibrium in complete markets, a weighted combination of the returns (prices) on a security over the various states can be regarded as an expectation of the future returns (prices) given particular patterns of subjective probabilities and utility functions among investors.[23] This is true in this model. In addition, under risk neutrality and common investor subjective probabilities, the expected future price of a t-period security is equal to the current forward price $P(1, t)$.

If markets are complete the returns on any security can be expressed as a linear combination of the first K reference securities. That is, we can write

$$R_{\tau s} = \sum_{t=1}^{K} \beta_{t\tau} R_{ts}, \quad \text{for} \quad \tau = 1, 2, 3, \cdots \tag{2.4}$$

for any pure discount bond of maturity τ, where $\beta_{t\tau}$ is the proportion invested in the t^{th} reference security. If τ is an integer in $[1, K]$, then $\beta_{\tau\tau} = 1$ and $\beta_{t\tau} = 0$ for $t \neq \tau$. In this last case equation (2.4) is an identity. Equation (2.4), otherwise, is an implication of an arbitrage model; it states that the one-period returns on any security can be expressed relative to the exogeneously determined returns on the

[22] It would seem reasonable that for some probable s_0, $P_{s_0}(1, t) = P(1, t)$, so that the future prices one period hence will be equal to the forward prices currently observed. However, this result can never obtain in the framework developed here if markets are complete and there is a competitive equilibrium. If s_0 is a state such that $P_{s_0}(1, t) = P(1, t)$, then matrix R will contain a row and a column in which each element is $1/P(0, 1)$. It follows that θ_{s_0} will be undefined because the elements of the s_0th row of R^{-1} sum to zero. This implies that $\theta_k(k \neq s_0)$ is undefined, a result inconsistent with complete markets in competitive equilibrium. The discount functions $P_s(1, t)$ consistent with competitive equilibrium must be such as to exclude the possibility that every forward price (rate) can be equal to every future spot price (rate).

[23] J. C. Cox, S. A. Ross, and M. Rubenstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, September 1979, pp. 229–63.

reference securities. Subtracting R_{1s} from both sides of equation (2.4), we can write

$$R_{\tau s} - R_{1s} = \sum_{t=2}^{K} \beta_{t\tau}(R_{ts} - R_{1s}) \tag{2.5}$$

where only $(K - 1)$ terms remain on the right hand side. The expression on the left hand side is the excess one-period return on a τ-period security over the certain return on a one period security.

This analysis can be extended to returns over more than one period. Multiply equation (2.4) by $1/P_s(1, q)$, which is the accumulated return on \$1 for $(q - 1)$ periods if state s occurs one period hence and if all forward prices thereafter are realized.[24] Using equation (2.1), the excess returns on a τ-period security over q periods are:

$$\frac{P_s(1, \tau)}{P_s(1, q)P(0, \tau)} - \frac{1}{P(0, q)} = \sum_{\substack{t=1\\t\neq q}}^{K} \beta_{t\tau}\left[\frac{P_s(1, t)}{P_s(1, q)P(0, t)} - \frac{1}{P(0, q)}\right]. \tag{2.6}$$

where $1/P(0, q)$ is the riskless return over q periods. Note that as τ shifts, the set of coefficients $(\beta_{1\tau}, \beta_{2\tau}, \cdots, \beta_{K\tau})$ also shifts and that as q is varied, (2.6) holds as long as the q^{th} reference security is removed from the right hand side.

Complete competitive markets implies that no profitable riskless opportunities exist. The relationship between $P(0, t)$ and $P_s(1, t)$, for all s and t defines a "stochastic process" that is consistent with complete competitive markets. In a world of uncertainty, the wrong process may be assumed and then an error term must be added to (2.5) and (2.6) to account for the mis-identification.

Duration and 2-State Processes

For two state processes, two securities are specified as the reference securities. Let these be the one and K period pure discount bonds respectively. The matrix of one period returns can be written as

$$R = 1/P(0, 1)\begin{bmatrix} 1 & , \quad 1 \\ \dfrac{P_1(1, K)}{P(1, K)}, & \dfrac{P_2(1, K)}{P(1, K)} \end{bmatrix} = \begin{bmatrix} R_{11}, & R_{12} \\ R_{K1}, & R_{K2} \end{bmatrix} \tag{2.7}$$

The one period return on any t-period pure discount bond $(t \neq 1, t \neq K)$ can then be stated as

$$R_{ts} = (1 - \beta_t)R_{1s} + \beta_t R_{Ks}, \quad \begin{array}{l} s = 1, 2, \\ t \neq 1, \\ t \neq K, \end{array} \tag{2.8}$$

where β_t and $(1 - \beta_t)$ are respectively the proportions invested in K-period and one-period bonds. There is consequently a mapping of the maturity of a pure

[24] The accumulated excess return is used because the analysis is undertaken in a discrete multi-period context.

discount bond into a value of the proportion β_t. Moreover, the returns on any portfolio of bonds in this 2-state model will be equivalent to the returns on some portfolio of the reference securities. Consider, for example, the portfolio mixture consisting of proportion ϕ_t invested in t-period pure discount bonds and a portion $(1 - \phi_t)$ invested in $(t + 1)$-period pure discount bonds. Such a portfolio may be of importance to investors who wish to assume a degree of risk that is between that of a t and a $t + 1$ period pure discount bond. Because in this model bonds are available only with maturities equal to the integer dates on which trading occurs, this degree of risk can be assumed only by constructing portfolios that are mixtures of the available bonds. Using (2.8), one can show that such a portfolio has the same returns as one in which the proportion $\phi_t\beta_t + (1 - \phi_t)\beta_{t+1}$ is invested in K period pure discount bonds and the proportion $\phi_t(1 - \beta_t) + (1 - \phi_t)(1 - \beta_{t+1})$ is invested in one period pure discount bonds. As ϕ_t is varied in the interval [0, 1], then the proportion invested in K period bonds varies between β_{t+1} and β_t. As one considers the returns on all possible portfolios, it is evident that the proportion invested in K period securities can be regarded as a continuous function of t, with the property that $\beta_K = 1$ and $\beta_1 = 0$. If β_t is a monotonic continuous function of t, as is most likely, one can regard t in this mapping as the duration of the portfolio because it has many of the properties of the traditional durations. Every pure discount bond has a duration equal to its maturity. If the duration of a bond portfolio is an integer it behaves like the corresponding pure discount bond. If the duration of the portfolio is not an integer then there corresponds a β that indicates that the portfolio behaves like some mixture of the two closest integer pure discount bonds. In this model a portfolio is completely specified by the stipulation of its duration. Any two portfolios with the same duration will have exactly the same returns.

Subtract R_{1s} from both sides of Equation (2.8). Then,

$$R_{ts} - R_{1s} = \beta_t(R_{Ks} - R_{1s}), \tag{2.9}$$

which is an excess return equation for the 2 state model using the K and one period bonds as reference securities. This is analogous to Equation (2.5) for the multistate model. Any desired portfolio can be constructed by choosing an appropriate duration corresponding to β_t. The excess returns on the selected portfolio will then be proportional to the excess return on the K-period reference bond.

If we replace the one period reference bond with a q period reference bond, the excess return equation for a q-period planning period can be written as

$$R^A_{ts} - R^A_{qs} = \beta_t(q)[R^A_{Ks} - R^A_{qs}] \tag{2.10}$$

where $R^A_{ts} = P_s(1, t)/P(0, t)P_s(1, q)$, $R^A_{qs} = 1/P(0, q)$, and $R^A_{Ks} = P_s(1, K)/P(0, K)P_s(1, q)$. Here, the A superscript denotes the accumulated return for q periods, and $\beta_t(q)$ indicates the proportion invested in a K period bond when the planning period is of length q. We can also write $\beta_t(q) = \gamma_t|\, t - q\,|$ as indicated in Equation (1.2). Thus, $\beta_t(q) = 0$ when duration t equals the planning period length. If γ_t is a constant independent of t, then $\beta_t(q)$ is linear in duration.

The shape of the function β_t depends on the stochastic process. In the next section, we consider a one-parameter additive stochastic process and derive the corresponding β_t.

The Generalized Additive Stochastic Process (GASP) for a 2-State Single Factor Model

For a generalized additive process, one can define the price of a t-period discount bond one period hence as

$$P_s(1, t) = P(1, t) + g_s(t, z_s), \qquad s = 1, 2 \tag{2.11}$$

in a two state model. In this expression a random term, $g_s(t, z_s)$ is added to the current forward price in order to determine the price next period. The term z_s is a random variable generated by some probability distribution specific to state s.

One can show that z_1 and z_2 are generated from a single source of uncertainty. If the markets are complete and a competitive equilibrium obtains, then there exist weights w_s independent of the distributions of z_s, such that

$$w_1P_1(1, t) + w_2P_2(1, t) = P(1, t), \tag{2.12}$$

and the weights are positive and sum to unity. To show this, using (2.3) we note that $A^{-1}e = R\theta^{-1}e = e$, where e is a vector of ones. However, $R\theta^{-1}e = e$ implies, using equation (2.1) in a two-state model, that $\sum_{s=1}^{2} P_s(1, t)/\theta_sP(0, t) = 1$ and that $\sum_{s=1}^{2} 1/\theta_sP(0, 1) = 1$. If we define $w_s = 1/\theta_sP(0, 1)$, $(s = 1, 2)$, then $w_1 + w_2 = 1$ and equation (2.12) follows. Using (2.11) and (2.12),

$$w_1g_1(t, z_1) + w_2g_2(t, z_2) = \frac{g_1(t, z_1)}{\theta_1P(0, 1)} + \frac{g_2(t, z_2)}{\theta_2P(0, 1)} = 0 \tag{2.13}$$

for all t and for all generated values of z_1 and z_2. It is now apparent that the distributions of z_1 and z_2 cannot be independent in order that this result hold. Consistency requires that there exist only one source of uncertainty in the determination of the pair (z_1, z_2). Without loss of generality, assume $z_1 = z_2 = z$. It also follows that g_1 and g_2 cannot be zero for all t because then the system reverts to a one-state certainty model.

One can calculate θ_1 and θ_2 in terms of g_1 and g_2; and since θ_1 and θ_2 must be fixed numbers for complete markets, it will follow that g_1 and g_2 must have particular properties. If R is the matrix of one period returns as before, it follows that

$$R^{-1}e = \begin{bmatrix} 1/\theta_1 \\ 1/\theta_2 \end{bmatrix}.$$

Substitution of g_1 and g_2 into R^{-1}, via (2.11), implies that

$$\begin{aligned} w_1 &= g_2(t, z)/[g_2(t, z) - g_1(t, z)], \quad \text{and} \\ w_2 &= -g_1(t, z)/[g_2(t, z) - g_1(t, z)] \end{aligned} \tag{2.14}$$

are fixed constants that must be independent of all t and z ($z_1 = z_2$) if markets are to be complete. From this condition for complete markets, we see that

$$g_2(t, z)/g_1(t, z) = -w_1/w_2 = -\theta_2/\theta_1, \tag{2.15}$$

for all t and z.

Given that the returns from the contingency claims are independent of z, it follows that the proportions invested in the reference securities to produce the returns on any other security are independent of z and s. To show this, we simply substitute (2.11) into (2.8), using the definitions of R_{ts}, R_{1s}, and R_{Ks}, and solve for the proportion β_t as

$$\beta_t = \frac{P(0, K)}{P(0, t)} \cdot \frac{g_s(t, z)}{g_s(K, z)}, \qquad s = 1, 2. \tag{2.16}$$

The results in (2.14) and (2.15) imply that β_t is independent of s and z.

A Special Example of the Generalized Additive Stochastic Process

To illustrate the GASP process, we select a one-parameter example in which it is assumed that $g_1(t, z) = \lambda^t z$ and $g_2(t, z) = -(\theta_2/\theta_1)\lambda^t z$, where λ is a positive parameter and z has a probability distribution such that Prob $(P_s(1, t) \le 0) = 0$. It follows that $g_s(t, z)/g_s(K, z) = \lambda^{t-K}$. Equation (2.16) now implies that

$$\beta_t = \frac{P(0, K)}{P(0, t)} \lambda^{t-K}, \qquad t > 1. \tag{2.17}$$

One can uniquely identify every portfolio by a measure of duration. Consider the income stream with flows $f_1, f_2, \cdots, f_N$. Its present value is $\sum_1^N f_t P(0, t)$. The one-period return is

$$R_s(f_1, f_2, \cdots, f_N) = \sum_{\tau=1}^N \gamma_\tau \frac{P_s(1, \tau)}{P(0, \tau)} \tag{2.18}$$

where $\gamma_\tau = f_\tau P(0, \tau)/\sum_1^N f_t P(0, t)$, and $P_s(1, 1) = 1$. Now,

$$\frac{P_s(1, \tau)}{P(0, \tau)} = (1 - \beta_\tau) \frac{1}{P(0, 1)} + \beta_\tau \frac{P_s(1, K)}{P(0, K)} \tag{2.19}$$

where $\beta_\tau = 0$ when $\tau = 1$, and $\beta_\tau = 1$ when $\tau = K$. Substitution of (2.19) into (2.18) gives

$$R_s(f_1, f_2, \cdots, f_N) = [\textstyle\sum_1^N \gamma_\tau (1 - \beta_\tau)](1/P(0, 1)) + [\textstyle\sum_1^N \gamma_\tau \beta_\tau] \dfrac{P_s(1, K)}{P(0, K)}. \tag{2.20}$$

Thus, this bond will behave like a bond in which $\sum_1^N \gamma_\tau \beta_\tau$ is invested in the K-period and $\sum_1^K \gamma_\tau(1 - \beta_\tau)$ is invested in the 1-period bond. From (2.17)

$$\sum_1^N \gamma_\tau \beta_\tau = \sum_2^N \gamma_\tau \frac{P(0, K)}{P(0, \tau)} \lambda^{\tau - K}. \tag{2.21}$$

If this bond with flows $(f_1, f_2, \cdots, f_N)$ is equivalent to a pure discount bond of maturity D, then (2.17) with $t = D$ is the proportion on the left hand side of (2.21), and therefore,

$$\frac{\lambda^D}{P(0, D)} = \sum_2^N \gamma_\tau \lambda^\tau / P(0, \tau). \tag{2.22}$$

Thus, Equation (2.22) implicitly defines the duration for this special additive process. If D is not an integer, but $G < D < G + 1$, one can show that the bond with the income stream $(f_1, f_2, \cdots, f_N)$ behaves as a mixture of a G and a $G + 1$ maturity pure discount bond.

Dividing Equation (2.11) by $P(0, t)$ gives us the return equation for every security. If the stochastic process is described by a single parameter and the mean and variance of the distribution of the factor (z) are known, then values of duration correspond uniquely to the mean and variance characteristics of any portfolio. This can be shown as follows. Let the one period return per dollar invested in a t-period pure discount bond $(t \neq 1)$ be

$$R_{ts} = \begin{cases} \dfrac{1}{P(0, 1)} + \dfrac{\lambda^t z}{P(0, t)}, & s = 1 \\[2ex] \dfrac{1}{P(0, 1)} - \dfrac{(\theta_2/\theta_1)\lambda^t z}{P(0, t)}, & s = 2. \end{cases} \tag{2.23}$$

Let p_1 and p_2 be the respective probabilities of states one and two. Assume $E(z) = \mu$ and $\text{Var}(z) = \sigma^2$. Then

$$E(R_{ts}) = \frac{1}{P(0, 1)} + \frac{\lambda^t \mu}{P(0, t)} M, \quad t \neq 1, \tag{2.24}$$

where $M = [p_1 - p_2(\theta_2/\theta_1)]$. The variance and covariances of the one period returns per dollar can be calculated as

$$\text{var}(R_{ts}) = \frac{\lambda^{2t}}{[P(0, t)]^2} L, \quad \text{and} \tag{2.25}$$

$$\text{cov}(R_{ts}, R_{\tau s}) = \frac{\lambda^{t+\tau}}{P(0, t)P(0, \tau)} L, \tag{2.26}$$

where

$$L = \left[1 + \left(\frac{\theta_2}{\theta_1}\right)^2\right]\sigma^2 + \mu^2\left[(1 - M)^2 + \left(M + \frac{\theta_2}{\theta_1}\right)^2\right].$$

The correlation between the returns per dollar is equal to unity, as would be expected in a 2 state model with only one source of uncertainty. The relationship between the mean and the standard deviation of the return is linear. Taking the square root of the variance, solving for λ and substituting the result into Equation (2.24) renders

$$E(R_{ts}) = \frac{1}{P(0, 1)} + \frac{\mu M}{\sqrt{L}} \sqrt{\text{var}(R_{ts})}, \tag{2.27}$$

where $1/P(0, 1)$ is the one period risk-free return and $\mu M/\sqrt{L}$ is the slope of the efficiency frontier.

Both the mean and variance are determined by duration D. Thus, each point on the efficiency locus corresponds to a specific duration value. From Equation (2.24) the expected return of any bond portfolio of duration D is

$$\frac{1}{P(0, 1)} + \frac{\lambda^D \mu M}{P(0, D)},$$

and its variance is

$$\left[\frac{\lambda^D}{P(0, D)}\right]^2 L$$

This model may be extended to multiperiods, but the expressions for means and variances become considerably more complex. Thus, the risk-return frontier is more difficult to derive than for single periods.