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The Calculation of Implied Variances from the Black-Scholes Model: A Note

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BLACK AND SCHOLES [2] HAVE derived a model for the equilibrium price of a European Stock Purchase option. According to the Black and Scholes model, equilibrium option prices are a function of the time to maturity of the option, the exercise price, the current price of the underlying stock, the risk free rate of interest, and the instantaneous variance of the stock's rate of return. Of these five variables, only the first four can be directly observed; the instantaneous variance of the stock's return can only be estimated.

Recently, several authors [3, 4, 5, 6, 10, 11], rather than estimating a variance from past data, have attempted to employ the Black-Scholes option pricing formula to derive an "implied variance." An implied variance is the value of the instantaneous variance of the stock's return which, when employed in the Black-Scholes formula, results in a model price equal to the market price.

Unfortunately, the implied variance cannot be calculated explicitly, and previous researchers have used numerical methods such as the Newton-Raphson method and its variants [8]. However for many options, no values of implied variance could be found to justify the observed option prices.

In this note, necessary and sufficient conditions for the existence of a positive implied variance are given. An algorithm is presented which converges monotonically and quadratically to the (unique) implied variance, when it exists. The most valuable feature of this algorithm is that it provides a starting value for the first iteration of the Newton-Raphson procedure. If one applies the Newton-Raphson procedure without the correct starting value, solutions that do exist can be overlooked.

The option pricing formula of Black and Scholes is:

$$w = xN(d_1) - ce^{-rt}N(d_2) \tag{1}$$

with

$$d_1 = \frac{ln\left(\frac{x}{c}\right) + rt}{v\sqrt{t}} + \frac{v\sqrt{t}}{2}$$

$$d_2 = d_1 - v\sqrt{t}$$

and

w = premium for the option;

x = current market price of the underlying stock ($x > 0$ assumed);

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t = remaining life of the option ($t > 0$ assumed);

$N(\cdot)$ = cumulative standard normal density function;

c = exercise price ($c > 0$ assumed);

r = continuous constant known rate of interest; and

v^2 = instantaneous variance of stock returns.

For convenience, Equation (1) is rewritten:

$$w = f(x, c, r, t, v) \tag{2}$$

Using the formula given in (2), the implied standard deviation is denoted by v^*, which for given values of x, c, r, and t, satisfies Equation (3):

$$w = f(x, c, r, t, v^*) \tag{3}$$

Equation (3) has a positive solution, v^*, if and only if the option is rationally (see [7]) priced, so that,

$$\mathrm{Max}(0, x - ce^{-rt}) < w < x \tag{4}$$

To demonstrate the previous statement, observe that for given values of x, r, c, and t, f is a function of v alone and that

$$\lim_{v \to 0^+} f(v) = \mathrm{Max}(0, x, - ce^{-rt}) \tag{5a}$$

and

$$\lim_{v \to \infty} f(v) = x \tag{5b}$$

Since the function $f(\cdot)$ is strictly monotone increasing [1] in v over $(0, \infty)$ the option value, w, must fall between the expressions given on the right-hand sides of (5a) and (5b). If the option price falls within this range, then the monotonicity and continuity of $f(\cdot)$ guarantees that there is a unique solution to Equation (3), v^*.

A common numerical method used for solving nonlinear systems of equations is the Newton-Raphson method (see [8] for example). For Equation (3) this method is

$$v_{n+1} = v_n - \frac{f(v_n) - w}{f'(v_n)} \tag{6}$$

where v_n is the nth estimate of v^*, and f' is the first derivative of $f(v)$ with respect to v. Since $f'(v_n) > 0$ when t,x, and $c > 0$, Equation (2) is well defined over $(0, \infty)$.

A well known result concerning the Newton-Raphson procedure gives that if Equation (3) has a solution, there is an open interval (a, b) about v^* such that if $v_n \in (a, b)$ for any n, then $\{v_n\} \to v^*$ [2] When this is the case, convergence is quadratic.

[1] $f'(v) > 0$ when c, x and $t > 0$ and $v > 0$.

[2] In solving $g(\alpha) = 0$ for the scalar α where $g'(\ ) \neq 0$ and g'' is continuous, there is an open interval about α such that if any of the iterates found using the Newton-Raphson procedure fall in this interval, convergence to α is assured. In solving Equation (3), $f'(v^*) > 0$, and f'' is continuous.

Expression (4) tells us when Equation (3) has a solution. We only need to identify one point of (a, b) to guarantee that (6) will always lead to v^*. We shall show that such a point, v_1, is given by

$$v_1^2 = \left| ln\left(\frac{x}{c}\right) + rt \right| \frac{2}{t}$$

The proof that $v_1 \epsilon (a, b)$ is now given below.[3]

For our choice of v_1, it is readily verified that if $v^* < v_1 (v^* > v_1)$ f' is increasing (decreasing) on $[v^*, v_1]([v_1, v^*])$[4] and hence f is strictly convex (strictly concave) over this interval. It is readily seen then, that $v^* \leqq \cdots v_3 \leqq v_2 \leqq v_1$ (resp. $v^* \geqq \cdots v_3 \geqq v_2 \geqq v_1$) since supports of a strictly convex (strictly concave) function always underestimate (overestimate) the function. Since the sequence is monotone and bounded, it converges. Any limit point must satisfy Equation (3). However, any such solution must be v^* by the uniqueness argument. Thus we have that, starting with the above v_1, $\{v_n\} \to v^*$ monotonically quadratically.[5]

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[3] Using (6) and the mean-value theorem one easily gets

$$\frac{|v_{n+1} - v^*|}{|v_n - v^*|} = \left| 1 - \frac{f'(v^*\lambda + (1 - \lambda)v_n)}{f'(v_n)} \right|$$

for some $\lambda \epsilon (0, 1)$. This motivates us to consider choosing v_1 as the v that maximizes $f'(v)$ so that v_2 will be closer to v^* than v_1. This v can be obtained by maximizing $g(\mathrm{d}_1)$ where $g(\cdot)$ is the standard normal density and d_1 is as defined below (1). First order conditions give that $d_1 d_2 = 0$. This happens if either $d_1 = 0$ in which case

$$v^2 = \frac{-2\left(ln\left(\frac{x}{c}\right) + rt\right)}{t}$$

or $d_2 = 0$ in which case

$$v^2 = \frac{2\left(ln\left(\frac{x}{c}\right) + rt\right)}{t}$$

Checking second order conditions show that $g''(d_1) < 0$ under both cases.

[4] Since $f'' = f'd_1 d_2/v$ and $\left(ln\left(\frac{x}{c}\right) + rt\right)^2 = v_1^4 t^2/4$ we get (substituting for d_1 and d_2)

$$f''(v) = \frac{f't}{4v^3}[v_1^4 - v^4]$$

for any v. Thus for $v^* < v_1 (v^* > v_1)$ we have that $f'' > 0$ $(f'' < 0)$.

[5] In an extended version of this paper [6], we evaluate the sensitivity of the algorithm given in Equation (6) to various adjustment made to accommodate the influence of dividends.

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