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Flattening of Bond Yield Curves for Long Maturities

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ABSTRACT

The paper presents a theoretical proof that flattening of yield curves for par bonds is inevitable for long maturities. This proof implies that behavioral explanations of flattening are unnecessary. The proof also implies that the use of yields to maturity of coupon-bearing bonds to estimate the true term structure (as well as forward rates) for long maturities has potentially infinite bias, suggesting that a greater effort should be made to directly estimate the true term structure in empirical work.

EMPIRICALLY OBSERVED BOND YIELD to maturity curves have consistently been found to become flat for long maturities, as evidenced by the empirical work of Durand [6], [7], [8], Malkiel [15], and the Treasury Bulletin [25]. A concise summary of this evidence is contained in Malkiel [15]. He presents three-dimensional graphs of corporate bond yield curves from 1900 to 1965 ([15], pp. 8–9) and two-dimensional graphs of yield curves for U.S. government securities for the period 1953–66 ([15], p. 13). Both of these show that flattening of yield curves occurs consistently for long maturities. More recent U.S. government bond yield curves in the Treasury Bulletin exhibit a similar flatness for long maturities.

Malkiel [15] considered flattening of yield curves for long maturities to be an important desiderata that a theory of the term structure must fulfil. He said that a term structure "theory must account for the pervasive tendency of the (yield) curve to level out as term to maturity increases and to develop what is called a 'shoulder,' irrespective of the general shape in the early maturities. Slopes with large absolute values occur only in the early maturities" ([15], p. 16). Malkiel has suggested that the shape of yield curves depends on the sensitivity of bond price to changes in interest rates. According to Malkiel's viewpoint, if long-term bonds of different maturities have essentially the same price sensitivity to interest rate changes, yield curves would tend to be flat for long maturities. In another viewpoint, Lutz [14] has said that the flattening of yield curves is the result of constant long-term forward rates for long maturities. This view would be justified if investors would use the same forecasted rate as the best predictor of all distant rates because of the uncertainties attached to expectations of future rates in different, but distant, future periods.

The major innovative result of our paper is a formal proof that flattening of yield curves for par bonds is inevitable for long maturities. The power of this proof is that no assumptions are required about forward rates, zero coupon

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discount rates, or bond price volatilities. That is, the proof will hold whether or not forward rates rise or fall, whether or not forward rates are very small or very large—even infinity, whether or not investors can differentiate expectations of distant future rates from each other. Our proof that par yield curves must be flat for long maturities has the important implication that the behavioral explanations of flattening of Lutz and Malkiel are unnecessary. Our result formalizes the statement of Schaefer [24] that constant coupon yield curves "are . . . asymptotically horizontal no matter what shape the spot rate [zero coupon rate] curve adopts" ([24] page 64).

The flattening of yield curves for long maturities implies that coupon bias and forward rate bias can be large for long maturities. Coupon bias represents the difference between the yield to maturity for coupon-bearing bonds and the zero coupon discount rates. Since all bonds except Treasury Bills are coupon-bearing, investigators have been forced to make inferences about zero coupon discount rates on the basis of yields for coupon-bearing bonds. Our analysis will show that the difference between zero coupon discount rates and yields for coupon-bearing bonds can be very substantial, indicating that use of coupon-bearing bond yields to make inferences about zero coupon rates involves substantial potential for error. As an illustration, the yield for a coupon-bearing bond will be shown to be a small positive number even when the zero coupon discount rate is infinity.

Our results imply also that marked differences can exist between actual forward rates implicit in zero coupon bonds and forward rates inferred from coupon-bearing bond yields. The examples given in the text will clearly show that forward rates inferred from coupon-bearing bonds convey little information about actual forward rates for long maturities. For example, it is tempting to conclude that forward rates are constant if the yield curve is flat. But we shall show that for long maturities the yield curve for par bonds is flat even when forward rates fluctuate markedly. We shall present an example in which actual forward rates rise monotonically but forward rates inferred from coupon-bearing bonds have a humped pattern.

Most empirical work on the term structure of interest rates assumes a negligible coupon and forward rate bias. In light of the large potential for biases shown in this paper, a greater effort should be made to obtain direct estimates of the term structure in empirical work.

The paper is organized as follows: Section I proves that yield curves for par bonds must be flat for long maturities. This section also develops a stronger bound on the difference in yields for rising term structures. Section II presents examples of term structures, yield curves, and forward rates and discusses coupon bias and forward rate bias.

I. Flattening of Par Bond Yield Curves

In a perfect market, which allows no arbitrage opportunities, the price (P_N) of an N period bond with coupon C, which is default-free, noncallable, and not subject to taxes, must be determined as follows:

$$P_N = \sum_{j=1}^{N} \frac{C}{(1 + R_j)^j} + \frac{1}{(1 + R_N)^N} \tag{1}$$

The R_j's are discount rates of zero coupon bonds (spot rates) of maturity j and represent the true term structure of interest rates. Throughout the rest of this paper, we shall assume that the R_j's are nonnegative.[1] The zero coupon discount rates (the R_j's) in Equation (1) are not directly observable from coupon-bearing bonds.[2]

Bond price may also be expressed as a function of yield to maturity Y_N:

$$P_N = \sum_{j=1}^{n} \frac{C}{(1 + Y_N)^j} + \frac{1}{(1 + Y_N)^N} \tag{2}$$

Since bond prices, coupons, face values, and maturity are observable for marketable bonds, yields to maturity for an N period bond (Y_N) can be computed from (2).

A yield curve will be defined as flat, if the yields to maturity for adjacent maturities are approximately the same, i.e. if the difference $Y_{N+k} - Y_N$, $k \geq 1$, is very close to zero. From Equations (1) and (2), the reader can see that the relationship between bond yields for different maturities is quite complicated in general. This section will prove that the yield to maturity curve will become flat for par bonds.

Let Y_N denote the yield to maturity for a par bond of maturity N. For a par bond, we can set $P_N = 1$ in Equation (1) and note that C equals Y_N. Solving for the yield to maturity for a par bond (Y_N) results in

$$Y_N = \frac{1 - (1 + R_N)^{-N}}{\sum_{j=1}^{N} \frac{1}{(1 + R_j)^j}} = \frac{1 - D_N}{A_N} \tag{3}$$

where $D_N = (1 + R_N)^{-N}$ and $A_N = \sum_{j=1}^{N} \frac{1}{(1 + R_j)^j} = \sum_{j=1}^{N} D_j$

D_N represents the price of a zero coupon bond paying \$1 at time N. A_N is the price of an annuity paying \$1 per period for N periods.

To show that yield curves will become flat for par bonds for long maturities, the following theorems are proved in the Appendix.

THEOREM 1: If $Y_{N+k} > Y_N$, for any $k \geq 1$, then

$$Y_{N+k} - Y_N \leq \frac{1}{N} \tag{4}$$

For rising yields, Theorem 1 shows that the largest possible increase in the yield curve for par bonds, $Y_{N+k} - Y_N$, must be less or equal to $1/N$. For large N, the maximum increase in the yield curve approaches zero and any increase in the yield curve must be close to zero.

An interesting aspect of Theorem 1 is that, once Y_N and N are known, the maximum value for the yields for longer-term par bonds is known. For example, if we know that a 25-year par bond has a 10% yield, we know that all longer-term par bonds must have yields less than or equal to 14%, since $Y_{N+k} - Y_N \leq 4\%$.

[1] We assume the existence of a currency which can be stored at zero cost.

[2] Although not directly observable, the R_j's may be estimated. For example, see McCulloch [17], [18], and Carleton and Cooper [3]. Also see Caks [2].

The previous result can be elucidated by considering the special case when the largest difference of $Y_{N+1} - Y_N$ occurs. This will happen when R_N is finite and R_{N+1} is infinity. When R_{N+1} is infinity, D_{N+1} is zero and Y_{N+1} must equal $1/A_N$. Therefore,

$$Y_{N+1} - Y_N = \frac{D_N}{A_N} \tag{5}$$

The maximum of D_N/A_N will occur when A_N takes on its minimum value. From Proposition 3 in the Appendix, we see that the minimum value of A_N is ND_N and thus the maximum of D_N/A_N equals $1/N$. In general, D_N/A_N will be much smaller than $1/N$ and the maximum increase in the yield curve will be much smaller than $1/N$. Later discussion examines a special type of term structure, the case of rising zero coupon discount rates, where an even tighter bound on the maximum increase in yields on par bonds is shown to exist.

THEOREM 2: If $Y_{N+k} < Y_N$, for any $k \geq 1$, then

$$Y_N - Y_{N+k} \leq Y_N \left[\frac{k}{N+k}\right] \tag{6}$$

For declining yields, Theorem 2 shows that the decline in yields for par bonds must be close to zero for long maturities (i.e. if N is large relative to k).[3] To see how tight the bound is, consider the case when the yield on a 25-year par bond is 10%. We see that the maximum possible decline from maturity 25 to maturity 26 is 0.4% and that the yield on a 26-year bond can never be less than 9.6%.

Taken together, the two theorems show that the rise or the decline in par bond yield curves must be close to zero for long maturities. It follows that par bond yield curves must be flat for long maturities.

The following intuitive interpretation of flattening may be helpful in understanding our results. The yield to maturity for a coupon bearing bond is a complicated weighted average of the zero coupon discount rates. As maturity increases, the weights for the longer-term zero coupon rates tend to become relatively small, because the present value factors for the distant time periods tend to become of lesser relative importance, thereby implying flattening of the yield curve. Another intuitive viewpoint is that a long-term bond (and its yield) tends to become increasingly similar to a perpetual annuity (and its yield) as maturity lengthens.

The results of this section have the important implication that it is unnecessary to explain flattening of yield curves to be the result of some term structure theory. Yield curves will invariably become flat for long maturities.

The previous results showed the flattening of par bond yield curves for long maturities. For rising term structures, we can show the following, stronger result for par bonds. See Appendix.

[3] The special case where the term structure declines by the maximum amount of each maturity can be shown from Proposition 3 in the Appendix to occur when $D_1 = D_2 = \cdots D_N = D$. This would occur if all forward rates were zero for maturity two and beyond. Since $A_N = ND$, $Y_N = (1/N)(1/D - 1) = R_1/N = Y_1/N$. Thus, even in this case of the most steeply declining term structure possible, the yield curve for par bonds will become flat for large N.

PROPOSITION 1: If $R_N \geq Y_N$ and $Y_{N+k} > Y_N$, for $k \geq 1$,

$$Y_{N+k} - Y_N \leq \frac{1}{\frac{1}{Y_N}[(1 + Y_N)^N - 1]} \tag{7}$$

The term structure constraint in the Proposition ($R_N \geq Y_N$) can hold under a wide variety of circumstances, but clearly will occur (a) if the term structure is flat, i.e. $R_1 = R_2 = \cdots = R_N = Y_N$; (b) if the term structure rises monotonically, i.e., $R_1 < R_2 < \cdots\cdot < R_N$; (c) if the N period zero coupon discount rate is greater than every shorter maturity rate, i.e. $R_N > R_j$, for $j < N$; or (d) if R_N is greater than the yield on an N period annuity. (cf. Schaefer [24] p. 62).

We have computed Table I to show how tight a constraint (7) becomes. The reader can see that (a) the bound[4] is always less than $1/N$; (b) the bound gets tighter for longer maturities; and (c) the bound gets tighter for larger Y_N.

The most interesting thing about this bound is that the yield to maturity for an N period par bond (Y_N) sets a bound upon the yield to maturity for any longer-term par bond, assuming $R_N \geq Y_N$. The tightness of this bound for long maturities can be seen easily in Table I. For example, if the yield on a 25-year par bond is 10%, the yield of any longer maturity bond can never be greater than 11.02%.

A second interesting aspect of this bound is that higher levels of interest rates will make the bound tighter. Thus; for rising term structures, higher rates of interest will make the yield curve flatten sooner.

II. Illustration, Coupon Bias, and Forward Rate Bias

Previous results proved that par yield curves must be flat for long maturities. We now present two examples to illustrate flattening.

The first example is shown in Table II. The illustration assumes that zero coupon rates start at 5% for maturity one (i.e. $R_1 = 5\%$) and increase geometrically with maturity at a 5% rate (i.e. $R_{N+1} = 1.05\ R_N$). The reader can easily see in Table II that the yield curve for par bonds (Y_N) becomes very flat by a maturity of 25 periods, even though the zero coupon rates are rising very rapidly at that maturity.

It is interesting to compare the yields for par bonds in Table II with the bounds in Theorem 1 and Proposition 1. At maturity 25, Theorem 1 tells us that for $N > 25$, $Y_N - Y_{25} \leq 4\%$, or $Y_N \leq 14.00\%$. That is, yields for longer maturity par bonds can never exceed 14.00%. Proposition 1 tells us that, for $N > 25$, $Y_N - Y_{25} \leq 1.02\%$, or $Y_N \leq 11.02\%$ or yields for longer maturity par bonds cannot exceed

[4] To see that the bond in Equation (7) is always less or equal to $1/N$, expand $(1 + Y_N)^N$ binominally, subtract 1, and divide by Y_N resulting in

$$Y_{N+k} - Y_N \leq \frac{1}{N + \frac{N(N-1)Y_N}{2!} + \frac{N(N-1)(N-2)Y_N^2}{3!} + \cdots}$$

Clearly, the righthand side must be less than $1/N$.

Table I

Maximum Increase in Yield for Rising Term Structures, Computed from Proposition 1 (Rates in Percent)

	Number of Periods (N) until Maturity				
Rate Y_N	2	4	8	16	32
5.	48.78	23.20	10.47	4.23	1.33
10.	47.62	21.55	8.74	2.78	.50
$1/N$ Bound from Theorem 1	50.00	25.00	12.50	6.25	3.13

The entries in the table are the maximum values of $Y_{N+k} - Y_N$ for given N and $1/N$.

Table II

Example of Geometrically Increasing Zero Coupon Discount Rates (in Percent)

Maturity	Spot Rate R_N	Par Bond Yield Y_N	$Y_N - Y_{N-1}$	Coupon Bias	Forward Rate	Implied Forward Rate	Forward Bias
1	5.00	5.00		−0.00	5.00	5.00	0.00
2	5.25	5.24	0.24	−0.01	5.50	5.49	0.01
3	5.51	5.49	0.25	−0.02	6.04	6.00	0.04
4	5.79	5.75	0.26	−0.04	6.62	6.52	0.10
5	6.03	6.01	0.26	−0.07	7.24	7.06	0.18
6	6.38	6.28	0.27	−0.10	7.91	7.62	0.30
7	6.70	6.55	0.27	−0.15	8.64	8.17	0.46
8	7.04	6.82	0.27	−0.22	9.41	8.73	0.68
9	7.39	7.09	0.27	−0.30	10.24	9.27	0.97
10	7.76	7.35	0.27	−0.40	11.14	9.80	1.34
11	8.14	7.62	0.26	−0.53	12.10	10.30	1.80
12	8.55	7.88	0.26	−0.67	13.13	10.77	2.36
13	8.98	8.13	0.25	−0.85	14.24	11.19	3.05
14	9.43	8.37	0.24	−1.06	15.44	11.56	3.87
15	9.90	8.60	0.23	−1.30	16.71	11.87	4.84
20	12.63	9.52	0.15	−3.11	24.70	12.39	12.31
25	16.13	10.00	0.06	−6.12	36.17	11.55	24.62
30	20.58	10.16	0.02	−10.42	52.77	10.62	42.15
35	26.27	10.19	0.00	−16.08	77.13	10.25	66.87
40	33.52	10.19	0.00	−23.33	113.44	10.19	103.25

11.02%. Notice that the highest observed yield for a par bond in Table II is 10.19%.

The second example is shown in Table III. The zero coupon discount rate is assumed to start at 5%. Every even maturity zero coupon rate declines by the maximum possible amount from the previous rate. (See Proposition 4 in the Appendix.) Every odd maturity zero coupon discount rate increases and equals the previous odd rate times $(1.05)^2$.

From Table III, the reader can see that the zero coupon discount rates (R_N) oscillate around a basic upward geometric trend. Yet, the yield curve for par bonds (Y_N) becomes quite flat by maturity 25.

Table III

Example of Zero Coupon Discount Rates Oscillating around an Upward Geometric Trend (Rates in Percent)

Maturity	Spot Rate R_N	Par Bond Yield Y_N	$Y_N - Y_{N-1}$	Coupon Bias	Forward Rate	Implied Forward Rate	Forward Bias
1	5.00	5.00		−0.00	5.00	5.00	0.00
2	2.47	2.50	−2.50	0.03	−0.00	0.06	−0.06
3	5.51	5.39	2.89	−0.12	11.87	11.43	0.44
4	4.11	4.12	−1.27	0.02	−0.00	0.39	−0.39
5	6.08	5.87	1.75	−0.21	14.34	13.16	1.18
6	5.04	5.01	−0.86	−0.03	−0.00	0.83	−0.83
7	6.70	6.37	1.35	−0.33	17.23	14.87	2.37
8	5.84	5.73	−0.64	−0.11	−0.00	1.39	−1.39
9	7.39	6.87	1.14	−0.52	20.62	16.42	4.20
10	6.62	6.38	−0.49	−0.24	−0.00	2.09	−2.09
11	8.14	7.36	0.98	−0.78	24.59	17.68	6.90
12	7.44	6.99	−0.38	−0.46	−0.00	2.93	−2.93
13	8.98	7.83	0.85	−1.15	29.25	18.54	10.71
14	8.31	7.55	−0.29	−0.77	−0.00	3.88	−3.88
15	9.90	8.27	0.72	−1.63	34.73	18.87	15.86
20	11.40	8.86	−0.10	−2.53	−0.00	6.91	−6.91
25	16.13	9.54	0.17	−6.59	81.82	13.66	68.16
30	18.89	9.66	−0.01	−9.23	−0.00	9.51	−9.51
35	26.27	9.70	−0.01	−16.56	203.71	9.89	193.82
40	31.02	9.71	−0.00	−21.31	−0.00	9.71	−9.71

Coupon Bias. Coupon bias is the difference between the zero coupon discount rate (R_N) and the yield to maturity for a coupon-bearing bond. The issue of coupon bias is important for investigators trying to draw inferences empirically about the term structure (zero coupon rates) from coupon-bearing bonds. Several investigators have argued on the basis of examples using shorter maturity bonds that coupon bias is small.[5] Others have said that coupon bias can be large and have presented evidence of large coupon bias.[6] Some investigators have assumed a small coupon bias and treated coupon-bearing bond yields as essentially identical to zero coupon rates.[7] Other empirical investigators have added a coupon term to a fitted yield curve.[8]

Our examples have several important implications for empirical estimates of the term structure. One of the implications of our examples is that coupon bias can be large. For example, in Table II, consider maturity 25. The zero coupon discount rate is 16.13%; the yield for a par bond is 10%; and the coupon bias exceeds 6%. Similar bias is readily apparent in Table III. A second implication of our examples is that the size of coupon bias is a function of maturity. That is, the size of the coupon bias will be quite different at different maturities. The common

[5] See Wallace [27], and Malkiel [15].

[6] See Conard and Frankena [5], Echols and Elliott [9], [10], Frankena [11], Kaufman and Morgan [12], Pye [22], Schaefer [24].

[7] For example, Nelson [21].

[8] See Cohen, Kramer, and Waugh [4], Conard and Frankena [5], Echols and Elliott, [9] and Frankena [11].

practice of simply adding a coupon term to a fitted yield curve overlooks the fact that the coupon bias changes with maturity. Most importantly, the examples show the need for *direct* estimation of the term structure. Most empirical estimates[9] of the term structure have used yields on coupon-bearing bonds as the equivalent to zero coupon discount rates, with the resulting potential for very large errors.

Forward Rate Bias. An extensive part of the term structure literature is concerned with forward rates for zero coupon bonds. The forward rate from future time $N - 1$ to time N is defined as F_N.

$$F_N = \frac{(1 + R_N)^N}{(1 + R_{N-1})^{N-1}} - 1 = \frac{D_{N-1}}{D_N} - 1 \tag{8}$$

Forward rates have been an important topic because forward rates (at least partially) reflect investors' expectations of future rates. It has also been argued that liquidity premiums affect forward rates (See [1], [3], [9], [10], [13], [15], [16], [19], [20], [21], [27]).

Empirical investigators[10] have quite frequently derived implied forward rates from coupon-bearing bonds. The implied forward rate is I_N:

$$I_N = \frac{(1 + Y_N)^N}{(1 + Y_{N-1})^{N-1}} - 1 \tag{9}$$

If the yield curve for par bonds is flat, Y_N will be approximately the same as Y_{N-1}. I_N will then also equal $Y_{N-1} = Y_N$. An empirical investigator using implied forward rates (I_N) could draw the erroneous conclusion that forward rates are constant and equal to Y_N and that there are no liquidity premiums. For example, it is often postulated[11] that the market's expectations of inflation affect interest rates. Under this view, an expectation of snowballing future rates of inflation would result in forward rates that increase exponentially as maturity increases. Yet our analysis indicates that yield curves for par bonds would rapidly become flat, and forward rates derived from these yield curves by using Equation (9) would represent poor estimates of actual forward rates.

The enormous potential for forward rate bias can be seen in our examples. In Tables II and III, actual forward rates for long maturities are immense, but the yield curve for par bonds is flat, as are implied forward rates. In fact, it can be proved[12] that implied forward rates are flat when actual forward rates are infinity.

[9] McCulloch [17], [18] and Carleton and Cooper [3] have used direct estimation techniques. The authors mentioned in Footnote 8 have used various approaches to correct for coupon bias. Most empirical studies do not make any correction for coupon bias, for example references [13], [15], [19], [20].

[10] Many authors have used implied forward rates from Equation (9) to estimate the forward rates. See Bierwag and Grove [1], Meiselman [16], Nelson [21]. There are many others as well.

[11] See Van Horne [26], pp. 124–34, for a summary of the literature.

[12] To see this, note that $F_{N-1} = \infty$ implies that $D_{N-1} = 0$ and $R_{N-1} = \infty$. If $D_{N-1} = 0$, $D_{N-1+k} = 0$, $k \geq 1$, since $D_{N-1} \geq D_N \geq D_{N+1} \geq \cdots$. It follows that

$$Y_{N-1} = Y_N = Y_{N+1} = \cdots = Y_{N+k} = 1/A_{N-2}$$

This means that the yield curve for par bonds will be flat and equal to $1/A_{N-2}$ for maturities of $N - 1$ and greater. The implied forward rates (I) will be $1/A_{N-2}$, although the actual forward rates (F) are astronomically large. The difference between the actual forward rate and the implied forward rate ($F_N - I_N$) will be infinity.

Table II provides another oddity. The implied forward rates follow a humped pattern—with a peak at maturity 19—while the actual forward rates rise monotonically. The hump in foward rates results from the flattening of the yield curve for par bonds and conveys no information about actual forward rates.

The forward rate bias in Table III is illuminating. The actual forward rates for even maturities are zero, but the implied forward rates (I) are positive. For long maturities, the forward rates (F) oscillate from zero for even maturities to an immense number for odd maturities. Yet the implied forward rates (I) are relatively constant.

III. Conclusion

This paper has presented a formal proof that par bond yield curves will become flat for long maturities by showing that the slope of the yield curve is constrained by bounds which become very tight for long maturities. This result was shown to imply that empirical estimates of zero coupon discount rates and forward rates derived from yield curves for par bonds have substantial possibility of error.

These results for par bonds can be extended to nonpar bonds, since it can be shown that, for a Taylor expansion evaluated at par, yield to maturity for nonpar bonds is, within a reasonable interval around par, approximately equal to yield to maturity for par bonds for long maturities. Schaefer [24] has presented an alternate line of argument, namely that all constant coupon yield curves are asymptotic to the perpetuity yield and therefore asymptotically horizontal.

APPENDIX

Proof of Theorems 1 *and* 2.[13] Consider an N period par bond with annual coupon Y_N and an $N + k$ period par bond with annual coupon of Y_{N+k}. The cash inflows to holders of these bonds will be, assuming a $1 par value,

Year	1	2	3	⋯	N	$N+1$	⋯	$N+k$
N Period Bond	Y_N	Y_N	Y_N	⋯	$1 + Y_N$			
$N + k$ Period Bond	Y_{N+k}	Y_{N+k}	Y_{N+k}	⋯	Y_{N+k}	Y_{N+k}	⋯	$1 + Y_{N+k}$

Throughout the paper we have assumed zero coupon discount rates (spot rates) to be nonnegative. Consider the case when $Y_{N+k} > Y_N$. If $NY_{N+k} > 1 + NY_N$, an arbitrager could shortsell at time zero the N period bond for its price of par and buy the $N + k$ period bond for its price of par. The net investment would be zero. After N periods, the arbitrager would have a risk-free profit. It follows that in order for the $N + k$ period bond not to dominate the N period bond, the total payments on the $N + k$ period bond up to time N cannot exceed those on the N period bond up to time N. That is,

$$NY_{N+k} \leq 1 + NY_N \tag{A.1}$$

[13] Theorems 1 and 2 and Proposition 1 can be proved algebraically by using the definitions of Y_{N+k} and Y_N in Equation (3).

Rearranging this leads to Theorem 1.

Consider the case when $Y_{N+k} < Y_N$. If $1 + NY_N > 1 + (N + k)Y_{N+k}$, an arbitrager could at time zero shortsell the $N + k$ period bond and buy the N period bond for no net investment. He would make a risk-free profit after $N + k$ periods. It follows that the total payments on the N period bond up to time $N + k$ cannot exceed those on the $N + k$ period bond. That is,

$$1 + NY_N \leq 1 + (N + k)Y_{N+k} \tag{A.2}$$

Rearranging results in Theorem 2.

Proof of Proposition 1. Under our perfect market assumptions, securities are perfectly divisible. Consequently, if $Y_{N+k} > Y_N$, an investor can shortsell the first N coupons on the $N + k$ period par bond at a price of $Y_{N+k}A_N$ (Recall that A_N is the present value of an N period annuity). The investor can simultaneously buy an N period par bond for a price of $Y_NA_N + D_N$. If $Y_{N+k}A_N > Y_NA_N + D_N$, there will be an arbitrage profit. Therefore, to avoid dominance:

$$Y_{N+k}A_N \leq Y_NA_N + D_N \tag{A.3}$$

$$Y_{N+k} - Y_N \leq \frac{D_N}{A_N} \tag{A.4}$$

Using the definition of Y_N, we know that $D_N/A_N = Y_ND_N/(1 - D_N) = Y_N/((1 + R_N)^N - 1)$. It follows that

$$Y_{N+k} - Y_N \leq \frac{D_N}{A_N} \leq \frac{1}{\frac{1}{Y_N}[(1 + R_N)^N - 1]} \leq \frac{1}{\frac{1}{Y_N}[(1 + Y_N)^N - 1]}. \tag{A.5}$$

as long as $R_N \geq Y_N$.

Proposition 2[14]: For $k \geq 1$, $D_j \geq D_{j+k}$.

Proof: Unless this condition holds, investors will have arbitrage opportunities to buy the j period zero coupon and simultaneously shortsell the $j + k$ period bond for an immediate risk-free profit.

Proposition 3: The minimum value of A_N is $N \cdot D_N$. That is $A_N \geq N \cdot D_N$.

Proof: From Proposition 2, we know that $D_j \geq D_N$ for $j = 1, \cdots, N - 1$. It follows that

$$\sum_{j=1}^{N} D_j \geq \sum_{j=1}^{N} D_N \tag{A.6}$$

Since $\sum_j D_j = A_N$ and $\sum_j D_N = N \cdot D_N$, the proposition is proved.

Proposition 4: The maximum decline in zero coupon rates from maturity N to $N + k$, $k \geq 1$, occurs when $D_N = D_{N+k}$ or equivalently when $\ln(1 + R_{N+k}) = (N/N + k) \ln(1 + R_N)$.

Proof: From Proposition 2, we know that $D_N \geq D_{N+k}$. The largest decline from R_N to R_{N+k} will occur when $(1 + R_{N+k})^{N+k}$ is as small as possible or equivalently when $1/(1 + R_{N+k})^{N+k}$ is as large as possible. But this will occur when $D_N = D_{N+k}$.

[14] This Proposition has been previously proved by Roll [23] and probably others.

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