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# On Diversification Given Asymmetry in Returns

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**ABSTRACT**

Complete diversification is the rational investment strategy for a risk averse individual in a homogeneous securities market who considers only the first two moments of return. Observed behavior of market participants, however, demonstrates that the majority of individual investors hold imperfectly diversified portfolios.

The purpose of the present study is to examine one potential cause for this behavior which does not rely on imperfection in the capital market. Specifically, we show that given the existence of, and investor preference for, positive skewness, a rational investor may hold an optimal limited number of homogeneous risk assets.

THE RELATIONSHIP BETWEEN THE number of risk assets in a portfolio and the total risk of the portfolio has been examined extensively both theoretically and empirically [7, 8, 16]. The general conclusion from these analyses (which are set in a mean-variance framework) is that adding randomly selected and equally weighted risk assets to a portfolio will reduce total risk without affecting return. The implication for risk averse investors in a perfect market is that complete diversification is optimal.

Observed behavior of market participants, however, demonstrates that the majority of individual investors hold imperfectly diversified portfolios. In the Wharton Survey of 1975 in [4], the median number of stocks held was found to be fewer than 4, with 34 percent of the investors holding no more than 2 stocks. The two primary studies providing evidence of the undiversified portfolios of individual investors are Blume and Friend [3] and Blume, Friend, and Crockett [5]. The former analysis was based on a sample of individual income tax returns from 1971 and indicated that, considering only dividend paying stocks, 34.15 percent of investors sampled held only one stock; 50.51 percent held no more than 2 stocks; and only 10.72 percent held 10 or more stocks. The latter study which included nondividend paying stocks was based on the Federal Reserve Board's 1962 Survey of the Financial Characteristics of Consumers. The median number of holdings per household was 2, while the average was 3.41. In both [3] and [5] numerous other measures of diversification were employed with the results confirming the conclusion that individual investors are, on average, highly undiversified. The principal theoretical reasons advanced by the authors of these studies relate to the relaxation of assumptions of the standard capital asset

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We are solely responsible for any remaining errors.

pricing model such as those of homogeneous expectations and completely marketable assets.

The purpose of the present study is to examine theoretically one potential cause for this behavior which does not rely on imperfection in the capital market. This is in contrast to earlier explanations which depend upon the existence of transaction costs.[1] Specifically, we show that given the existence of, and investor preference for, positive skewness, a rational investor may hold a limited number of risk assets.

A substantial amount of empirical evidence has been presented which shows that individual securities and portfolios are characterized by nonnormal distributions of returns. One intuitive motivation for anticipating positive skewness is that an investor can never lose more than 100 percent, whereas gains are potentially unlimited. Scott-Horvath [20] recently have shown that investors exhibiting positive marginal utility of wealth for all wealth levels, consistent risk aversion at all wealth levels, and strict consistency of moment preference will have preference for positive skewness. We assume this utility structure throughout.

Anti-diversification may obtain because increasing the number of risk assets in a portfolio not only will reduce variance, a positive consideration, but also will reduce portfolio skewness, a negative consideration. The optimal number is obtained at the point where the marginal increase in expected utility from a decrease in variance is equal to the marginal decrease in expected utility from the reduction in skewness.[2] However, even given this trade-off, under certain market conditions we show that complete diversification is a rational strategy regardless of both the utility function employed and the presence of skewness.

## I. The Optimal Number of Risk Assets in a Randomly Selected Portfolio

In this section we derive the optimal number of risk assets for portfolio inclusion given an arbitrary utility function. Our general model restricts the opportunity

[1] Using a mean-variance framework, Brennan [6], Goldsmith [10], and Subrahmanyam and Patel [22] have shown that an optimal number of securities for portfolio inclusion does exist if allowance is made for fixed transactions costs. Levy [13] has derived an equilibrium asset pricing model when a constraint on the number of securities exists because of transactions costs. By assuming transactions costs and decision costs are independent of the time of revising a portfolio, Goldman [9] recently has shown that the theory-reality discrepancy is consistent with infrequently revised portfolios. If the effect of transactions costs on diversification is significant, one implication is that the appearance of mutual fund shares in investor portfolios should be common. This result has not been entirely confirmed. Friend and Blume [4] indicate that 47.7 percent of households own no mutual fund shares. In addition two recent changes in market structure may have reduced further the impact of transactions costs. Within the past ten years there has been a dramatic increase in the popularity of no-load mutual funds. In May 1975, the elimination of fixed commissions spawned the creation of "discount brokers." Thus, the importance of transactions costs may be overplayed even for investors with little wealth.

[2] This is precisely the point made by Beedles and Simkowitz [1]. However, note that the addition of risky assets to a portfolio will reduce portfolio skewness only if the first partial derivative of portfolio skewness with respect to $N$ is negative. Unlike the first partial derivative of portfolio variance with respect to $N$, the portfolio skewness derivative is not unambiguously negative. In Appendix A we give the necessary constraints which must hold in order for the derivative to be negative. Empirical support for the reduction of portfolio skewness upon increase in portfolio size is found in Beedles and Simkowitz [1].

set available to investors to risky assets. In Appendix B we extend the opportunity set to include a riskless asset.

We assume a homogeneous security universe; i.e., all risk assets have identical means, identical variances, and identical skewnesses. Further, all combinations of risk assets (i.e., the joint product moments) have identical covariances, curvilinear interactions, and triplicate product moments. These assumptions have the merit of simplicity while allowing for a general clarification of the determinants of the optimum. Under the assumption of a homogeneous security universe, Samuelson [19] has shown that an equal weighting scheme maximizes expected utility.[3] Samuelson's utility structure is completely consistent with that of Scott-Horvath [20].

We assume that the utility function for return $u(R)$, has the properties: $u'(R) > 0$, $u''(R) < 0$, and $u'''(R) > 0$. The Taylor series expansion around the expected return $E(R)$, is given by:

$$u(R) = u[E(R)] + u'[E(R)][R - E(R)] + \frac{u''[E(R)]}{2!}[R - E(R)]^2 + \frac{u'''[E(R)]}{3!}[R - E(R)]^3 + \cdots \tag{1}$$

and for our analysis we assume that the fourth and higher moments are irrelevant since their economic significance remains indeterminate. Taking the expectation of Equation (1) yields:

$$E[u(R)] = u[E(R)] + \frac{u''[E(R)]}{2!}\sigma_R^2 + \frac{u'''[E(R)]}{3!}M_R^3 \tag{2}$$

where $\sigma_R^2$ is the variance of expected return and $M_R^3$ is the third moment about the mean of expected return.

It is a well-known result that the variance of an equally-weighted portfolio of $N$ securities is given by

$$\sigma_R^2 = \frac{1}{N}\bar{\sigma}_i^2 + \frac{(N-1)}{N}\bar{\sigma}_{ij} \tag{3}$$

where $\sigma_R^2$ is the variance of the portfolio, $\bar{\sigma}_i^2$ is the average variance of the assets in the portfolio, and $\bar{\sigma}_{ij}$ is the average covariance of the assets in the portfolio.

The formula of the skewness of an equally-weighted portfolio of $N$ securities is derived in Appendix A and is given by,

$$M_R^3 = \left(\frac{1}{N}\right)^2 \bar{M}_i^3 + \frac{3(N-1)}{N^2}\bar{M}_{iij} + \frac{(N-1)(N-2)}{N^2}\bar{M}_{ijk} \tag{4}$$

where $M_R^3$ is the skewness of the portfolio, $\bar{M}_i^3$ is the average skewness of the assets in the portfolio, $\bar{M}_{iij}$ is the average curvilinear interaction of the assets in the portfolio, and $\bar{M}_{ijk}$ is the average triplicate product moment of the assets in the portfolio.

Substituting Equations (3) and (4) for $\sigma_R^2$ and $M_R^3$ into Equation (2), and

[3] Mao [15] and Goldsmith [10] present proofs showing equal weighting is optimal in a homogeneous security universe restricted to mean variance space. The Samuelson proof demonstrates the optimality of equal weighting when all moments are considered if the securities are homogeneous.

differentiating the resulting relationship with respect to $N$ in order to determine the optimum portfolio size yields

$$\frac{\partial E[u(R)]}{\partial N}=\frac{u''[E(R)]}{2!}\left[\frac{(-1)\bar{\sigma}_i^2}{N^2}+\bar{\sigma}_{ij}\left(\frac{1}{N^2}\right)\right]$$
$$+\frac{u'''[E(R)]}{3!}\left[\frac{-2}{N^3}\bar{M}_i^3+\frac{3\bar{M}_{iij}}{N^2}\left(\frac{2}{N}-1\right)+\bar{M}_{ijk}\left(\frac{3N-4}{N^3}\right)\right] \quad (5)$$

Setting Equation (5) equal to zero results in a cubic equation in $N$.[4] The only nonzero root for $N$ is:

$$N=\frac{\frac{2}{3}\bar{M}_i^3-2\bar{M}_{iij}+\frac{4}{3}\bar{M}_{ijk}}{\frac{u''[E(R)]}{u'''[E(R)]}(\bar{\sigma}_{ij}-\bar{\sigma}_i^2)-\bar{M}_{iij}+\bar{M}_{ijk}} \quad (6)$$

The condition for (6) to represent a maximization of expected utility is that

$$\frac{\partial^2 E[u(R)]}{\partial N^2}=\frac{u''[E(R)]}{2!}\left[\frac{2}{N^3}(\bar{\sigma}_i^2-\bar{\sigma}_{ij})\right]$$
$$+\frac{u'''[E(R)]}{3!}\left[\frac{6\bar{M}_i^3}{N^4}+3\bar{M}_{iij}\left(\frac{-6}{N^4}+\frac{2}{N^3}\right)+\bar{M}_{ijk}\left(\frac{-6}{N^3}+\frac{12}{N^4}\right)\right]<0 \quad (7)$$

which requires

$$\bar{M}_i^3<3\bar{M}_{iij}-2\bar{M}_{ijk} \quad (8)$$

The only feasible values of $N$ are positive. For there to be a maximum with a positive $N$, Constraint (8) must be met, and $N$ from (6) must be greater than zero. Equation (6) will yield a positive value of $N$ when the following additional constraint is satisfied:

$$\bar{M}_{iij}-\bar{M}_{ijk}>\frac{u''[E(R)]}{u'''[E(R)]}(\bar{\sigma}_{ij}-\bar{\sigma}_i^2) \quad (9)$$

Note that the right-hand side is positive by construction; therefore $\bar{M}_{iij}$ minus $M_{ijk}$ must be positive if the constraint is to be satisfied. Thus, Constraints (8) and (9) must be satisfied in order for the optimal $N$ to be positive.

Table I displays the optimal portfolio size under different conditions. There are four possibilities to consider because the $N$ obtained from optimization may be positive or negative.

Quadrants 1 and 2 depict a situation in which Constraint (8) does not hold. Even given investor preference for positive skewness, the implication is complete diversification. Quadrants 3 and 4 depict a situation in which Constraint (8) is satisfied. In Quadrant 3, Constraint (9) also holds implying both maximization and a positive limited portfolio size. The implication is antidiversification and provides theoretical rationale for the previously mentioned observed behavior of individual investors. For Constraint (9) to hold depends upon a combination of

[4] The discriminant of the cubic equation can be shown to equal zero, thus all roots are real and at least two are equal. In our equation, two of the roots equal zero.

**Table I**

Conditions for Limited Diversification

| | $\partial^2 E[u(R)]/\partial N^2 > 0$ minimization | $\partial^2 E[u(R)]/\partial N^2 < 0$ maximization |
|---|---|---|
| $N > O$ | *Constraints* 1<br>a. (8) does not hold<br>*Implication*<br>Portfolio size is infinite<br>i.e. complete diversification | *Constraints* 3<br>a. (8) holds<br>b. (9) holds<br>*Implication*<br>Portfolio size is limited<br>i.e. antidiversification |
| Optimal $N$ from Equation (6) | | |
| $N < 0$ | *Constraints* 2<br>a. (8) does not hold<br>*Implication*<br>Portfolio size is infinite<br>i.e. complete diversification | *Constraints* 4<br>a. (8) holds<br>b. (9) does not hold<br>*Implication*<br>Portfolio size is zero<br>i.e. anti-diversification |

investor tradeoff of risk aversion for skewness (as given by $u''[E(R)]/u'''[E(R)]$, hereafter labeled speculation ratio) with market parameters.

In Quadrant 4 where Constraint (9) does not hold, the optimal $N$ is negative even though expected utility is maximized. The implication, since we disallow negative holdings, is a portfolio size of zero. This case is the limit of antidiversification.

Quadrants 3 and 4 imply antidiversification whereas Quadrants 1 and 2 imply complete diversification even given investor preference for skewness.[5] Because Quadrant 3 is the most enlightening for the purpose of this study, consideration of the effect on the optimum with respect to changes in the speculation ratio due to investor specific risk coefficients is of interest.

The partial derivative of the optimal $N$ in Equation (6) with respect to the

[5] In a multiperiod setting our model also provides a rationale for a temporally changing naive investment strategy. A mutable investment strategy derived from our model depends upon a change in market parameters, the speculation ratio, or a combination of both. An individual investor, independent of the speculation ratio, would alter portfolio size temporally between complete diversification and limited diversification (or vice versa) if the change in market parameters were sufficient to reverse the inequality of (8).

Even if market parameters are constant temporally, investors will shift investment strategy if Constraint (8) initially indicates antidiversification, and there is a sufficient shift in the speculation ratio so that the investor now prefers a different limited number of risk assets. Note that this does not signify investors require a move to another quadrant in order to force a shift in investment strategy. For example, an investor initially investing under the conditions of Quadrant 3 may undergo a change in speculation ratio that simply indicates a different limited optimal portfolio size. Thus, the investor still will invest under the conditions of Quadrant 3 but will change portfolio size.

Note also the speculation ratio may change, ceteris paribus, but investment strategy may not. The two cases are: (1) shifts between Quadrants 1 and 2 do not alter portfolio strategy; (2) investors initially operating under conditions of Quadrant 4 will not change portfolio size unless the shift in the speculation ratio is of enough magnitude to cause a change to Quadrant 3.

investor's speculation ratio is always negative in quadrant 3, so that, as the degree of either constant absolute risk aversion (exponential utility) or constant relative risk aversion (power utility) increases, the optimal number of risk assets decreases.

Partial derivatives of the optimal $N$ in Equation (6) with respect to skewness, variance, covariance, and the joint product moments are very difficult to examine because of the association between the measures. Only by specifying a particular distributional form could the effect be ascertained.

## II. Empirical Example

In this section we use actual data to estimate the optimal number of risk assets for portfolio inclusion given that Constraint (8) is satisfied, i.e. Table I, Quadrant 3 or 4. Two random samples are drawn in order to compare the model estimates of the optimum number of risk assets for different sets of inputs. Two independent random samples of monthly returns for fifty stocks were drawn from the Compustat data base from January 1972 to December 1976, and estimates of the expected return, expected variance, and expected skewness for all individual risk assets, and in addition the expected covariance, expected curvilinear interaction, and expected triplicate product moment for all combinations of risk assets were formed by using the grand average of each parameter.

In the original selections, Constraint (8) was satisfied for Sample II, but was not satisfied for Sample I. Naturally, if Sample I were perfectly representative of the market, complete diversification would be optimal even given investor preference for positive skewness as shown in Table I, Quadrants 1 and 2. A ranking by skewness within each sample was performed as suggested by Beedles and Simkowitz [1], and outliers examined. It was evident that outliers existed in both samples. Upon further examination, it appeared the relatively large raw skewnesses (in absolute sense) resulted from presumably correct, but extremely large, monthly total rates of return. As Beedles and Simkowitz [1] have argued: "If these HPR's are viewed as being unrepresentative of the return generating processes that are relevant to most investors, then eliminating them might provide better or more useful insight into the behavior of equity values." This argument also has been referenced by Rosenberg and Houglet [17], who additionally show that errors in Compustat virtually can determine skewness.

In consideration of the preceding arguments and in light of our purpose to illustrate limited optima, we eliminated outliers from the samples even though Sample II initially satisfied Constraint (8). Additional stocks were selected for

**Table II**

Average Security Market Parameters

| | Sample I | Sample II |
|---|---|---|
| $\bar{R}_i$ | .0076553 | .0065827 |
| $\bar{\sigma}_i^2$ | .0073379 | .0070611 |
| $\bar{M}_i^3$ | .0000526 | .0001335 |
| $\bar{M}_{iij}$ | .0001034 | .0001021 |
| $\bar{M}_{ijk}$ | .0000722 | .0000832 |
| $\bar{\sigma}_{ij}$ | .0020071 | .0024131 |

**Table III**

The Optimal Number of Securities for Different Utility Functions

| Utility Functions | Speculation Ratio | | Sample I | Sample II |
|---|---|---|---|---|
| $U(R) = \ln R$ | $\frac{-R}{2}$ | | 7.00 | 1.19 |
| | | $b$ | | |
| | | .1 | −12.78 | −0.48 |
| $U(R) = R^{1-b}$ | $\frac{-R}{1+b}$ | .3 | −379.86 | −9.92 |
| $(0 < b < 1)$ | | .5 | 18.94 | −2.82 |
| | | .7 | 10.50 | 4.80 |
| | | .9 | 7.77 | 1.53 |
| | | $a$ | | |
| $U(R) = 1 - e^{-aR}$ | $\frac{-1}{a}$ | .001 | −0.00 | −0.00 |
| $(a > 0)$ | | .010 | −0.00 | −0.00 |
| | | .100 | −0.00 | −0.00 |
| | | 1.000 | −0.01 | −0.00 |
| | | 10.000 | −0.15 | −0.01 |

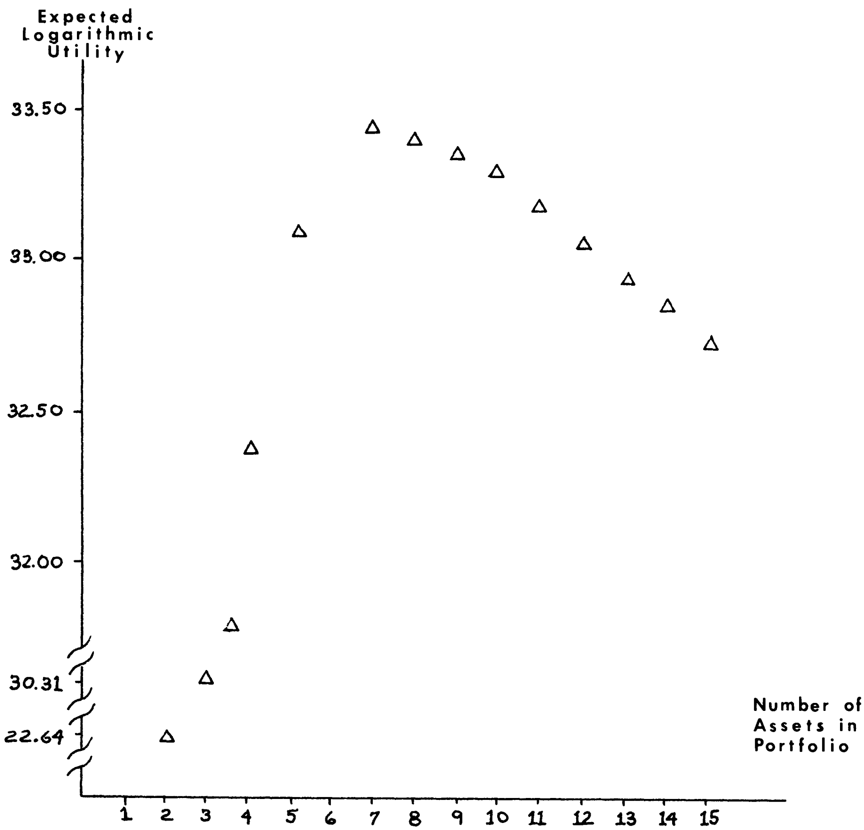


**Figure 1.** Expected Logarithmic Utility Versus Number of Assets in Portfolio for Sample I

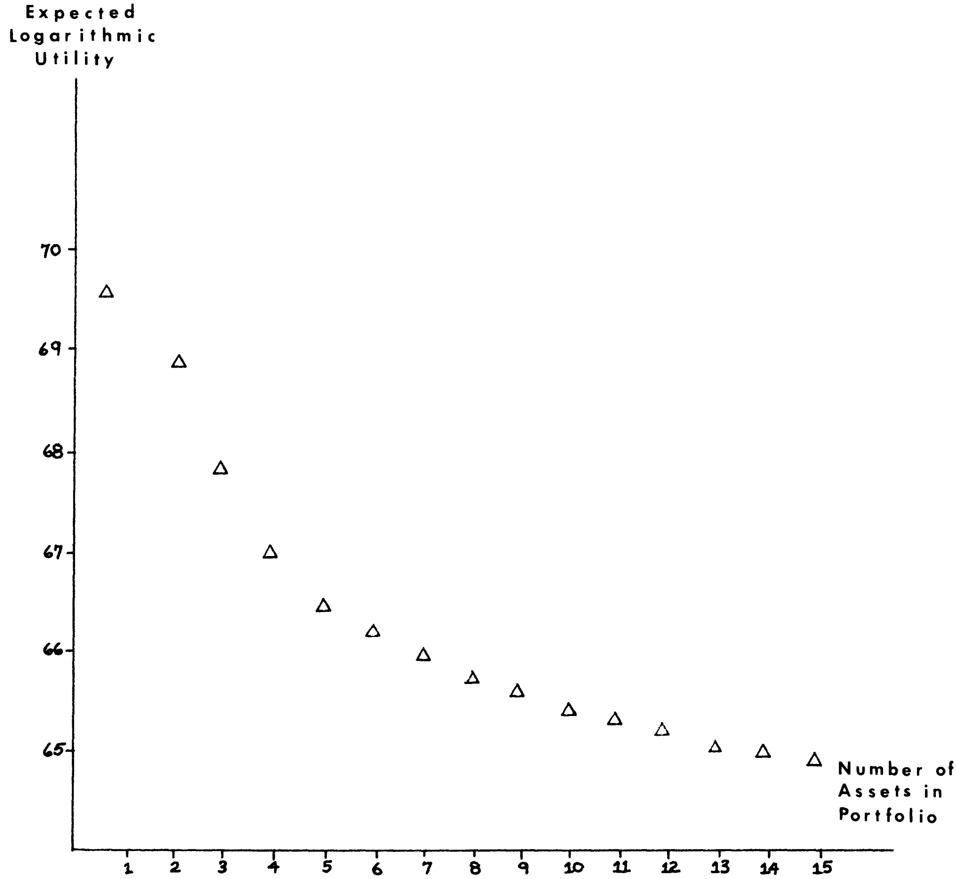


**Figure 2.** Expected Logarithmic Utility Versus Number of Assets in Portfolio for Sample II

replacement in order to maintain sample sizes of fifty. This procedure was continued until the outliers were eliminated from both samples, and Constraint (8) was satisfied. Table II lists the parameter estimates as calculated for both samples.

These parameter estimates are substituted into Equation (6) in order to calculate the optimum $N$. The values of the optimum $N$ for three utility functions are given in Table III. We have suggested that the optimum is a function of the risk aversion coefficient, and our empirical results appear to bear this out for the power function. Interestingly, the optimum is quite small for all combinations from both independent samples.

Figures 1 and 2 show expected utility for the logarithmic utility function as a function of the number of risk assets assuming the return generating process for each asset is consistent with the overall parameters calculated for each sample.[6] Expected utility was computed from the relationship found by substituting

[6] Theoretical support for the logarithmic function as the premier utility model can be found in Rubinstein [18].

Equations (3) and (4) into Equation (2). (The derivative with respect to $N$ of this relationship is given by Equation (5)). The figures depict a rapid increase in utility as risk assets are added to the portfolio. In both figures it can be seen that once the number of risk assets is greater than the optimum, utility continually decreases with the addition of each risk asset to the portfolio. *The graphs are in contrast to comparable graphs derived from mean-variance models which would show utility enhanced as risk assets are added to the portfolio.*

For both samples the optimum portfolio size for investors with power utility functions is sensitive to the risk aversion coefficient, $b$. For $b = 0.1$ or 0.3 in Sample I and for $b = 0.1$, 03, or 0.5 in Sample II, Constraint (9) does not hold, and investors are operating in Quadrant 4 of Table I, so that optimal portfolio size is zero. For other values of $b$ that were examined, investors would be operating in Quadrant 3 of Table I and, depending on the sample, would hold portfolios of either 2, 5, 8, 11, or 19 assets.

For reasonable values of the coefficient of absolute risk aversion for exponential utility, the optimum is negative for both samples. This implies that although Constraint (8) is met, Constraint (9) is not met. Thus, an investor with an exponential utility function would be operating in Table I, Quadrant 4 for the particular coefficients of absolute risk aversion considered. Constraint (9) could be met by substantially increasing the coefficient of constant absolute risk aversion. The problem, as pointed out by Levy-Markowitz [14], is that the levels required for this coefficient so that Constraint (9) will hold results in very unusual preferences by the individual.

## III. Summary

Complete diversification is the rational investment strategy for a risk averse individual in a homogeneous securities market if he is concerned only with the first two moments of return. Observed market behavior demonstrates that the majority of individual investors hold less than highly diversified portfolios. Our model suggests a rationale for this behavior while maintaining perfect market assumptions. Specifically, we have shown that given the existence of, and preference for positive skewness, a rational investor may hold a limited number of risk assets.

## APPENDIX A

In this appendix we derive the skewness of return for an equally-weighted portfolio. Initially, the result for a three asset portfolio is derived and the result then is generalized to the $N$ asset case. It should be noted it is not possible to derive the $N$ asset case from a two asset portfolio because the triplicate product moments which exist in the $N$-asset case do not exist in the two-asset case.

The third moment about the mean, i.e., skewness, is given by:

$$E[X - \mu_R]^3 = M_R^3 \tag{A.1}$$

Let $X = X_1R_1 + X_2R_2 + X_3R_3$ and $\mu_R = X_1\bar{R}_1 + X_2\bar{R}_2 + X_3\bar{R}_3$ where $X_1$, $X_2$ and $X_3$ are the relative weights, and $R_1$, $R_2$ and $R_3$ are the random returns. Substituting and expanding will yield

$$M_R^3 = X_1^3M_1^3 + X_2^3M_2^3 + X_3^3M_3^3 + 3X_1^2X_2M_{112} + 3X_1^2X_3M_{113} + 3X_1X_2^2M_{122} + 3X_2^2X_3M_{223} + 3X_1X_3^2M_{133} + 3X_2X_3^2M_{233} + 6X_1X_2X_3M_{123} \quad (A.2)$$

where

$$M_i^3 = E[R_i - \bar{R}_i]^3$$
$$M_{iij} = E[(R_i - \bar{R}_i)^2(R_j - \bar{R}_j)]$$
$$M_{ijk} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)(R_k - \bar{R}_k)]$$

Equation (A. 2) can be generalized for an $N$ asset portfolio:

$$M_R^3 = \sum_{i=1}^N X_i^3\bar{M}_i^3 + 3\sum_{i=1,\, i\neq j}^N [\sum_{j=1}^N X_i^2X_j\bar{M}_{iij}] + \sum_{i=1}^N \sum_{j=1,\, i\neq j\neq k}^N \sum_{k=1}^N X_iX_jX_k\bar{M}_{ijk}. \quad (A.3)$$

where $\bar{M}_i^3$ is the average skewness of the assets in the portfolio, $\bar{M}_{iij}$ is the average curvilinear interaction of the assets in the portfolio, and $\bar{M}_{ijk}$ is the average triplicate product moment of the assets in the portfolio. Assuming the portfolio is equally-weighted, i.e., $X_i = X_j = X_k = \dfrac{1}{N}$

$$M_R^3 = \sum_{i=1}^N \left(\frac{1}{N}\right)^3 \bar{M}_i^3 + 3\sum_{i=1,\, i\neq j}^N \left[\sum_{j=1}^N \left(\frac{1}{N}\right)^3 \bar{M}_{iij}\right] + \sum_{i=1}^N \sum_{j=1,\, i\neq j\neq k}^N \sum_{k=1}^N \left(\frac{1}{N}\right)^3 \bar{M}_{ijk} \quad (A.4)$$

Simplifying:

$$M_R^3 = \left(\frac{1}{N}\right)^3 \sum_{i=1}^N \bar{M}_i^3 + 3\left(\frac{1}{N}\right)^3 \sum_{i=1}^N \sum_{j=1,\, i\neq j}^N \bar{M}_{iij} + \left(\frac{1}{N}\right)^3 \sum_{i=1}^N \sum_{j=1,\, i\neq j\neq k}^N \sum_{k=1}^N \bar{M}_{ijk} \quad (A.5)$$

Summing:

$$M_R^3 = \left(\frac{1}{N}\right)^2 \bar{M}_i^3 + \frac{3(N-1)}{N^2}\bar{M}_{iij} + \frac{(N-1)(N-2)}{N^2}\bar{M}_{ijk} \quad (A.6)$$

If $N \rightarrow \infty$, $M_R^3 \rightarrow \bar{M}_{ijk}$ (i.e., as equally-weighted assets are added, the skewness of a portfolio approaches the average triplicate product moment) as discussed in Beedles [1]. We now present the partial derivative of portfolio skewness with respect to $N$:

$$\frac{\partial M_R^3}{\partial N} = \frac{-2}{N^3}\bar{M}_i^3 + \frac{3\bar{M}_{iij}}{N^2}\left(\frac{2}{N} - 1\right) + \bar{M}_{ijk}\left(\frac{3N-4)}{N^3}\right) \quad (A.7)$$

Partial derivative (A.7) is negative when:

$$N > \frac{(2/3)\bar{M}_i^3 - 2\bar{M}_{iij} + (4/3)\bar{M}_{ijk}}{\bar{M}_{ijk} - \bar{M}_{iij}} (\bar{M}_{ijk} \neq \bar{M}_{iij})$$

## APPENDIX B

In this appendix we expand the opportunity set, available to investors, to include a riskless asset. The development is similar to Brennan [6] and Goldsmith [10]. The normative framework for the optimization of utility is presented. Though theoretically rigorous, it is seen that a general solution for the optimum (such as (6)) can only be determined by specification of a particular utility structure.

Let the percentage invested in the risk assets be equal to $X$ and the percentage invested in the riskless asset $(1 - X)$. We then have

$$E(R_p) = XR + (1 - X)R_F \tag{B.1}$$

where $R$ is the identical return of all risk assets and $R_F$ is the return on the riskless asset. The variance and skewness of return on the portfolio are then given by

$$\sigma_{R_p}^2 = X^2[\sigma_R^2] \tag{B.2}$$

and

$$M_{R_p}^3 = X^3[M_R^3] \tag{B.3}$$

Note $\sigma_R^2$ and $M_R^3$ are given by (3) and (4) respectively.

The expected utility of return (given by (2)) can be expressed as

$$E[u(R)] = u(XR + (1 - X)R_F) + \frac{u''(XR + (1 - X)R_F)}{2!} X^2[\sigma_R^2] + \frac{u'''(XR + (1 - X)R_F)}{3!} X^3[M_R^3] \tag{B.4}$$

The first order conditions for the optimal portfolio composition are found by differentiating with respect to $X$ and $N$,

$$\frac{\partial E[u(R)]}{\partial X} = \frac{\partial u(XR + (1 - X)R_F)}{\partial X} + \frac{[\sigma_R^2]}{2!}\left\{\frac{\partial u''(XR + (1 - X)R_F)}{\partial X} X^2 + 2[u''(XR + (1 - X)R_F)]X\right\} + \frac{[M_R^3]}{3!}\left\{\frac{\partial u'''(XR + (1 - X)R_F)}{\partial X} X^3 + 3[u'''(XR + (1 - X)R_F)]X^2\right\} = 0 \tag{B.5}$$

$$\frac{\partial E[u(R)]}{\partial N} = \frac{X^2 u''(XR + (1 - X)R_F)}{2!}\left[\frac{\partial \sigma_R^2}{\partial N}\right] + \frac{X^3 U'''(XR + (1 - X)R_F)}{3!}\left[\frac{\partial M_R^3}{\partial N}\right] = 0 \tag{B.6}$$

The partials of $\sigma_R^2$ and $M_R^3$ with respect to $N$ can be found in (5). Second order conditions of maximization would require that the Hessian matrix be negative definite. Presumably, this would allow determination of constraints similar to (8) and (9).

With specification of a particular utility function the optimum could be determined by simultaneous solutions of (B.5) and (B.6). The optimum would then be a function that included the percentage allocation to the riskless asset.

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