The Effect of Taxation on Immunization Rules and Duration Estimation
Author(s): Christopher A. Hessel and Lucy Huffman
Source: *The Journal of Finance*, Vol. 36, No. 5 (Dec., 1981), pp. 1127-1142
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327304
Accessed: 27/11/2009 08:38

THE JOURNAL OF FINANCE • VOL. XXXVI, NO. 5 • DECEMBER 1981

The Effect of Taxation on Immunization Rules and Duration Estimation

CHRISTOPHER A. HESSEL and LUCY HUFFMAN

ABSTRACT

Investments in default-free bonds can be insulated from financial loss due to interest rate changes (via additive shock) by a process known as immunization. The literature on this process ignores taxes. This manuscript focuses on three issues. The first is the development of the tax-adjusted immunization process with a comparison to the existing literature. The second issue is the microeconomic effect of a shift in only the individual's tax rates on immunization and investment behavior. The third issue is the macroeconomic effect of an across-the-board shift in tax rates on immunization and investment behavior.

THERE IS GENERAL RECOGNITION among investors that a portfolio of default-free bonds is not entirely riskless. The owner of such a portfolio is subject to risk of financial loss due to interest rate changes occurring over the investment holding period. To attenuate these financial losses the bond holder can hedge or immunize his investment by adjusting the average duration of the aggregate bond portfolio, including reinvested interest and principal payments, to equal exactly the remaining length of his investment holding period. The bond duration equation utilized in the immunization process depends on the nature of the stochastic process generating the changes in interest rates. Therefore, if an investor is to successfully immunize his portfolio, that is, earn income at the end of the period no less than the income promised by the market at the beginning of the period, the investor must correctly identify the stochastic process and then employ the appropriate duration measure to obtain the average duration of his aggregate bond portfolio.

Since in reality each investor's income is subject to taxes, the immunization process should preserve his desired after-tax income. Furthermore, not all investors in the United States are subject to the same tax regulations. Interest and capital gain income earned by institutions with corporate or mutual forms of ownership, such as insurance companies and banks, are taxed at a common fixed rate. However, interest and capital gain income earned by trust institutions, such as investment funds and trusts, are taxed separately with capital gain income taxed at a lower rate than that applied to interest. Further complicating the tax treatment of investment income is the ability of the investor to amortize or not to amortize any tax liabilities or tax shields arising from purchasing a bond at a discount or a premium.

Because of the variable nature of the United States tax structure, several questions arise which are addressed in this paper. How are both the immunization process and the duration estimation altered by the presence of different tax rates applied to interest and capital gains income? Further, how are both duration estimation and investment behavior of an individual altered if his tax rates were the only rates changed? Lastly, how are both duration estimation and investment behavior altered if every investor's tax rates were changed?

The paper is divided into three analytical sections. The first section focuses on the derivation of the duration equation, proof of immunization, and a contrast of the findings with existing literature. The second section addresses the microeconomic issue of the effects of a shift in only the individual's tax rates on his duration estimation and investment behavior. The third section attends to the macroeconomic issue of the effects of an across-the-board shift in tax rates on duration estimation and investment behavior. The findings of these three sections are then summarized and conclusions are drawn.

I. Immunization and Uncertainty

The investigation is centered on the investment decisions of an individual who desires an income stream at the end of a fixed investment period of length "m." The securities comprising his portfolio are default free and therefore all uncertainty arises from a fluctuation in future interest rates. At the time the individual purchases his portfolio of bonds the prevailing structure of interest rate is $g(0)$. Immediately after the acquisition of bonds, the whole structure of future interest rates is disturbed by a small random additive shock, λ, causing the structure of interest rates to be $g(t) + \lambda$ for $t > 0$.[1]

To insulate his investment from this interest rate disturbance, the individual purchases a coupon bearing bond maturing at $T > m$ and reinvests the interest payments in a way which yields the desired income at m. The investment is successfully immunized if its income is at least as much as the income derived from purchasing a zero coupon bond yielding the desired income at maturity.

To prove immunization and to develop the duration measure required to achieve immunization in a market wherein investment income is subject to taxation, the analysis in this paper follows the analysis developed by Bierwag [1]. As in Bierwag's development, the investor purchases a coupon bond subject to continuous compounding, and maturing at $T > m$. However, unlike Bierwag's no-tax treatment, in this analysis interest income is taxed at rate y_1, while capital gain income is taxed at rate y_2. Both rates are constant and known with certainty at the beginning of the investment period.

The analysis considers the two alternative tax treatments of premia and discounts open to the investor.[2] Firstly the duration equation necessary for immunization is derived assuming the investor does *not* amortize any premia or discounts. Then the duration equation is presented assuming the investor *does* amortize. Lastly, inferences are made as to which tax treatments and duration equation would be preferred by investors.

[1] Ingersoll, Skelton, and Weil [6] have shown that an assumption of a noninfinitesimal shift is inconsistent with market equilibrium. They further note that if the infinitesimal additive shock is the only mechanism for interest rate changes, an unrealistic bond equation results. However, following the traditional approach in duration literature, attention will be focused on this type of shock only.

[2] Amortization of premium is mandatory where bonds are owned by a corporation but elective where bonds are owned by a taxpayer other than a corporation. See Regulation Paragraph 21.246E1980.

At time m, the end of the investment period, the investor's after-tax income from reinvested coupon payments is:

$$Q(\lambda, y_1) = cA(1 - y_1) \int_0^m e^{h(t,m)(m-t)+\lambda(m-t)}\, dt \tag{1}$$

where the following definitions apply to the notation:

c is the coupon rate of interest

A is the face value of the bond at maturity

V is the market value of the bond at time 0

$h(t, m)$ is the tax adjusted holding period yield for time interval t to m for $t = 0, 1, 2, \cdots, m - 1$.

Equation (1) presumes that throughout the investment period the after-tax income from coupon payments is reinvested in bonds maturing at time m. Since these investments occur after the disturbance, the yield on reinvested income is known with certainty. The investor has a choice of reinvestment vehicles. He may invest in zero coupon bonds, in which case the income at m is taxed at his capital gain rate or he may invest in coupon bonds in which case the interest income is taxed as accrued at his income tax rate. Because the two vehicles are priced by the market, the investor's tax adjusted holding period yield need not be the same. Therefore, the investor will select the security with the greater holding period yield. The holding period yield appears in Equation (1) as $h(t, m)$.

At m the investor liquidates the holdings of the coupon bonds maturing beyond m. The after tax income from the sale is[3]

$$R(\lambda, y_1, y_2) = cA(1 - y_1) \int_m^T e^{-h(m,t)(t-m)-\lambda(t-m)}\, dt + (A + y_2(V - A))e^{-h(m,T)(T-m)-\lambda(T-m)} \tag{2}$$

At time period m the investor's total after-tax income from reinvested coupon payments and sale of bonds is $Q(\lambda) + R(\lambda)$. However, if there were *no* interest rate disturbance, total after-tax income, $Q(0) + R(0)$, would equal $Ve^{h(0,m)m}$, the total income arising from the original interest rate appropriate for investment interval $(0, m)$. The effect of the interest rate disturbance on total income accumulated during the investment period is measured by the ratio $G(\lambda)$, where $G(\lambda)$ is:

$$G(\lambda) = \frac{Q(\lambda) + R(\lambda)}{Q(0) + R(0)} \tag{3}$$

The investor's portfolio is perfectly immunized if $G(\lambda) \geq 1$ for all λ and the minimum occurs at $\lambda = 0$, which implies the following first and second order

[3] Implicit in Equation (2) is the assumption that he sells the bond at a price equal to the discounted value of his future receipts.

conditions for minimization:

$$G'(\lambda) = 0$$

$$G''(\lambda) > 0$$

The first order condition is:

$$\frac{dG(0)}{d\lambda} = \frac{cA(1-y_1)\int_0^m e^{-h(t,m)(t-m)}(m-t)\,dt}{Ve^{h(0,m)m}} + \frac{cA(1-y_1)\int_m^T e^{-h(t,m)(t-m)}(m-t)\,dt}{Ve^{h(0,m)m}} - \frac{(A+y_2(V-A))(T-m)e^{-h(m,T)(T-m)}}{Ve^{h(0,m)m}} = 0 \quad (4)$$

The second order condition is:

$$\frac{d^2G(\lambda)}{d\lambda^2} = \frac{cA(1-y_1)\int_0^m e^{h(t,m)(m-t)+\lambda(m-t)}(m-t)^2\,dt}{Ve^{h(0,m)m}} + \frac{cA(1-y_1)\int_m^T e^{-h(m,t)(t-m)-\lambda(t-m)}(t-m)^2\,dt}{Ve^{h(0,m)m}} + \frac{(T-m)^2(A+y_2V-A))e^{-h(m,T)(T-m)-\lambda(T-m)}}{Ve^{h(0,m)m}} \quad (5)$$

Examination of Equation (5) reveals the second order condition is satisfied. To determine the financial mix required for minimization of income, the first order condition must be solved. Following Bierwag's proof of immunization, the first order condition, Equation (4), simplifies to Equation (6).

$$\frac{dG(0)}{d\lambda} = \frac{cA(1-y_1)\int_0^T e^{-h(0,t)t}(m-t)\,dt}{V} - \frac{(A+y_2(V-A))(T-m)Te^{-h(0,T)T}}{V} \quad (6)$$

For Equation (6) to equal zero, m must equal

$$m = \frac{cA(1-y_1)}{V}\int_0^T te^{-h(0,t)t}\,dt + \frac{A+y_2(V-A)}{V}Te^{-h(0,T)T} \quad (7)$$

or, more simply, m must equal the duration of the portfolio. Therefore, immunization is achieved by selecting securities such that the length of the investment period m exactly equals the tax-adjusted duration of the bond portfolio, Equation (8).

$$D = \int_0^T t w_t \, dt + T w_T \tag{8}$$

where weights w_t and w_T are:

$$w_t = \frac{cA(1 - y_1)}{V} e^{-h(0,t)t} \tag{9a}$$

$$W_T = \frac{A + y_2(V - A)}{V} e^{-h(0,T)T} \tag{9b}$$

Since immunization requires the equating of the portfolio's duration with the length of the investment holding period, the process of immunization is identical for markets wherein investment income is taxed and capital gains or losses are not amortized, and the traditionally assumed market wherein investment income is exempt from taxes. However, the equations used to estimate duration in these two markets differ. If the market permits tax exempt status for investment income, as traditionally assumed, each weight in the duration estimation equation is the ratio of the present value of the income received in that period (discounted at the investor's holding period yield) to the market value of the security. However, if the market attaches a tax liability to investment income, as assumed in this paper, the weights are more complex. The weight attached to each period in which a coupon payment is received, w_t, is the ratio of the present value of the after-tax income received in that period (discounted at the investor's tax adjusted holding period yield) to the market value of the security. The weight attached to the period in which the principal payment is received, w_T, is the ratio of the present value of the principal, adjusted for any tax liability or shield,[4] to the market value of the security.[5] Furthermore, examination of Equation (9b) reveals a strong relationship between the purchase price of the bond and the magnitude of weight w_T. If $V > A$, the bond is purchased at a premium; a tax shield is created causing an increase in the after-tax income received in that period and consequently an increase in the magnitude of the weight applied to that time period. If $V < A$, the bond is purchased at a discount; a tax liability is created causing the after-tax income to decrease in that period. If $V = A$, the bond is purchased at par; there is no capital gains effect on weight w_T.

It is observed that although the immunization process is unaltered by the taxation of investment income with no amortization, the equation used to estimate duration differs from the traditional equation in three ways. Firstly, after-tax income per period replaces coupon income. Secondly, the principal payment is corrected for tax shields or liabilities caused by the purchase of bonds at a premium or discount. Lastly, the discount rate applied to both income streams is the tax-adjusted holding period yield.

[4] The term "tax shield" is used throughout this paper to describe a negative tax liability, i.e., an amount to be subtracted from the investor's tax liability.

[5] The weight w_T also, of course, requires the use of the investor's tax-adjusted holding period yield.

If the investor, however, chooses to amortize premia or discounts, weights (9a) and (9b) change to (9c) and (9d)[6]

$$w_t = \frac{cA(1-y_1) + y_1(V-A)T^{-1}}{V} e^{-h(0,t)t} \tag{9c}$$

$$w_T = \frac{A}{V} e^{-h(0,T)T} \tag{9d}$$

Two differences exist between weights (9a), (9b), and weights (9c), (9d). The first is that due to the amortization of tax shields and liabilities, the effect of purchasing a bond at a premium or a discount is shifted to the coupon weights, w_t, and does not appear in the final payment weight, w_T. The second difference is that the amortized premium or discount is applied against coupon income. Thus the premium or discount is taxed at the income tax rate, y_1, rather than the capital gain tax rate, y_2.

Therefore, it is concluded that the immunization process is unaltered by taxation of income regardless of the tax treatment applied to premia and discounts. However, the appropriate duration measure depends on the tax treatment selected by the investor. The tax treatment selection is made such that the individual's wealth is maximized. Since wealth is maximized by recognizing tax liabilities in the last possible moment and by recognizing tax shields as soon as possible, the investor will choose *not* to amortize tax liabilities from discounts but will choose to amortize tax shields from premia. Consequently, his measure of duration depends on whether he purchases a bond at a discount ($V < A$) or at a premium ($V > A$). The appropriate duration equation for a bond purchased at a discount utilizes the weights given in (9a) and (9b). Conversely, the appropriate duration equation for a bond purchased at premium utilizes the weights given in (9c) and (9d). Lastly, if the bond is purchased at par, the weights given in (9a) and (9b) are identical to those given in (9c) and (9d).

II. Impact of a Change in the Individual's Tax Rates

In this section investigation is made of the impact that a small alteration of the investor's tax rate has on both the measure of duration he applies to his portfolio and the composition of his portfolio. To examine the influence of taxes on duration, an analysis is made of the derivative of the duration equation in terms of each of the investor's tax rates; i.e., the rate applied to interest and the rate applied to capital gains.

To conduct the analysis, three assumptions are made. Firstly, the capital market is perfectly competitive; therefore the individual is incapable of influencing the market price of the bond. Secondly, the investor *alone* experiences a small shift in each of his investment income tax rates. Since he is incapable of influencing market price of bonds, the marginal change in his tax rates forces a

[6] These weights may be obtained by the same immunization procedure as described in Equations (3) through (8), but with the substitution into Equations (1) and (2) of the total after-tax income under the amortization assumption,

$$cA(1-y_1) + y_1(V-A)T^{-1}$$

change in his tax-adjusted holding period yield. Therefore the marginal change in the investor's tax rates influences duration in two ways, via the tax liability on income and via a change in the investor's holding period yield. Thirdly, as a wealth maximizer, the investor will not amortize tax liabilities from bonds purchased at a discount and will amortize tax shields from bonds purchased at a premium.

A. Change in the Interest Income Tax Rate

The analysis begins by considering a change only in the individual's interest income tax rate, y_1. The investor is initially assumed to purchase the bond at a discount or at par. Hence the appropriate duration weights are given in Equations (9a) and (9b). The derivative of duration with respect to y_1, $\frac{dD}{dy_1}$, is the sum of two components; the direct effect, $\frac{\partial D}{\partial y_1}$, plus the indirect effect on holding period yield, $\frac{\partial D}{\partial h(0,t)} \cdot \frac{\partial h(0,t)}{\partial y_1}$

Substituting Equations (9a) and (9b) into Equation (8) and then taking the derivative with respect to y_1 yields:

$$\frac{dD}{dy_1} = \frac{-cA}{V}\int_0^T te^{-h(0,t)t}\,dt - \frac{cA(1-y_1)}{V}\int_0^T t^2 e^{-h(0,t)t}\frac{\partial h(0,t)}{\partial y_1}\,dt - \frac{y_2(V-A)+A}{V}T^2 e^{-h(0,T)T}\frac{\partial h(0,T)}{\partial y_1} \tag{11}$$

To proceed with the analysis, the behavior of the terms $\frac{\partial h(0,t)}{\partial y_1}$ and $\frac{\partial h(0,T)}{\partial y_1}$ must be examined and the entire expression (11) must be simplified. Since the market value of the bond is constant, $\frac{\partial h(0,T)}{\partial y_1}$ can be obtained by taking the derivative of the market value equation with respect to the tax rate y_1 and setting it equal to zero.

The market value can be written as:

$$V = cA(1-y_1)\int_0^T e^{-h(0,t)t}\,dt + (y_2(V-A)+A)e^{-h(0,T)T} \tag{12}$$

As in the analysis of duration, the derivative, $\frac{dV}{dy_1}$, is the sum of two components: $\frac{\partial V}{\partial y_1} + \frac{\partial V}{\partial h(0,t)} \cdot \frac{\partial h(0,t)}{\partial y_1}$. Therefore the derivative of Equation (12) is:

$$\frac{dV}{dy_1} = 0 = -cA\int_0^T e^{-h(0,t)t}\,dt - cA(1-y_1)\int_0^T te^{-h(0,t)t}\frac{\partial h(0,t)}{\partial y_1}\,dt - (y_2(V-A)+A)Te^{-h(0,T)T}\frac{\partial h(0,T)}{\partial y_1} \tag{13}$$

By multiplying Equation (13) by T/V and substituting into Equation (11), the derivative of duration with respect to y_1 simplifies to:

$$\frac{dD}{dy_1} = \frac{cA}{V} \int_0^T (T-t)\left(e^{-h(0,t)t} + te^{-h(0,t)t}(1-y_1)\frac{\partial h(0,t)}{\partial y_1}\right) dt \tag{14}$$

Since T exceeds t the sign of $\frac{dD}{dy_1}$ is contingent on the sign of the expression in brackets in the integrand,

$$\left(e^{-h(0,t)t} + te^{-h(0,t)t}(1-y_1)\frac{\partial h(0,t)}{\partial y_1}\right)$$

The sign of this expression may be determined using two results from manipulation of Equation (13). Analysis reveals the term $\frac{\partial h(0,t)}{\partial y_1}$ is negative.[7] Furthermore, although $\frac{\partial h(0,t)}{\partial y_1} < 0$, the first term in the expression $(e^{-h(0,t)t})$ exceeds the second term, $\left(te^{-h(0,t)t}(1-y_1)\frac{\partial h(0,t)}{\partial y_1}\right)$ causing the expression in brackets to be positive.[8] Therefore the derivative $\frac{dD}{dy_1} > 0$.

Since the derivative of duration with respect to tax rate y_1 is positive, duration lengthens as the investor's interest income tax rate rises. The lengthening of duration arises from the interaction of three factors: declining after-tax interest income, declining tax-adjusted holding period yields, and a constant principal payment. The tax adjusted duration equation is

$$D = \int_0^T w_t t\, dt + w_T T$$

Since the weight applied to the period in which the principal payment is received, w_T, is the product of a constant principal payment and a rising present value factor, w_T rises as the tax rate on interest rises. Conversely, since the weights applied to periods in which coupon payments are received, w_t, are products of a declining after-tax interest income with rising present value factors, w_t falls. Therefore, this rotation in the values of weights in the duration equation toward the principal payment period causes the duration measure to lengthen.

Now it is assumed that the investor purchases the bond at a premium, amortizing the tax shield. In this case the appropriate duration weights are those given in (9c) and (9d). Substituting Equations (9c) and (9d) into Equation (8) and then taking the derivative with respect to y_1 yields the analogue of Equation (11),

$$\frac{dD}{dy_1} = \frac{[-cA + (V-A)T^{-1}]}{V}\int_0^T te^{-h(0,t)t}\, dt - \frac{[cA(1-y_1) + y_1(V-A)T^{-1}]}{V}$$
$$\cdot \int_0^T t^2 e^{-h(0,t)t}\, dt \frac{\partial h(0,t)}{\partial y_1}\, dt - \frac{T^2A}{V}e^{-h(0,T)T}\frac{\partial h(0,T)}{\partial y_1} \tag{11a}$$

[7] See Appendix A.

[8] See Appendix B.

The analogue to the market value Equation (12) is:

$$V = [cA(1 - y_1) + y_1(V - A)T^{-1}] \int_0^T e^{-h(0,t)t}\, dt + Ae^{-h(0,T)T} \tag{12a}$$

By following the preceding manipulation, it may be determined that the sign of the derivative $\frac{dD}{dy_1}$ is greater than zero in the case that the investor purchases the bond at a premium and amortizes the tax shield. The lengthening of duration arises because the increase in tax shield is insufficient to offset the declining after-tax income.

Therefore, it can be concluded that a marginal change in the income tax rate has a direct (positive) effect on duration. This effect is observed when the bond is purchased at par, when the bond is purchased at a discount (and the tax liability is not amortized), or when the bond is purchased at a premium (and the tax shield is amortized).

B. Change in the Capital Gains Tax Rate

The investigation now focuses on the relationship between duration and a marginal change in the investor's capital gain tax rate y_2. The investor is initially assumed to purchase the bond at a discount or at par. As in the previous section, the effect of a change in y_2 on duration can be written as the sum of two parts, $\frac{\partial D}{\partial y_2}$ and $\frac{\partial D}{\partial h(0, t)} \cdot \frac{\partial h(0, t)}{\partial y_2}$. By substituting Equation (9a) and (9b) into Equation (8) and then taking the derivative with respect to y_2, Equation (15) is obtained.

$$\frac{\partial D}{\partial y_2} = \frac{(V - A)}{V} Te^{-h(0,T)T} - \frac{cA}{V}(1 - y_1) \int_0^T t^2 e^{-h(0,t)t} \frac{\partial h(0, t)}{\partial y_2}\, dt - \frac{y_2(V - A) + A}{V} T^2 e^{-h(0,T)T} \frac{\partial h(0, T)}{\partial y_2}. \tag{15}$$

As before $\frac{\partial h(0, T)}{\partial y_2}$ is obtained by examining the derivative $\frac{dV}{dy_2}$ where V is defined by Equation (12).

$$\frac{dV}{dy_2} = (V - A)e^{-h(0,T)T} - cA(1 - y_1) \int_0^T te^{-h(0,t)t} \frac{\partial h(0, t)}{\partial y_2}\, dt - (y_2(V - A) + A)Te^{-h(0,T)T} \frac{\partial h(0, T)}{\partial y_2} = 0 \tag{16}$$

Rearrangement of Equation (16) and substitution into Equation (15) yields:

$$\frac{dD}{dy_2} = \frac{cA}{V}(1 - y_1) \int_0^T (T - t)Te^{-h(0,t)t} \frac{\partial h(0, t)}{\partial y_2}\, dt \tag{17}$$

Since T exceeds t the sign of the derivative of duration with respect to y_2, $\frac{dD}{dy_2}$, is contingent on the sign of $\frac{\partial h(0, t)}{\partial y_2}$. Analysis of Equation (16)[9] reveals that the sign of the expression $\frac{\partial h(0, t)}{\partial y_2}$ is the same as the sign of the expression $(V - A)$. Therefore, if the bond is purchased at a discount $(V < A)$, $\frac{\partial h(0, t)}{\partial y_2}$ is negative, causing $\frac{dD}{dy_2}$ to be negative. If the bond is purchased at par $(V = A)$, $\frac{\partial h(0, t)}{\partial y_2}$ equals zero causing $\frac{dD}{dy_2}$ to be zero.

The influence of a change in capital gains tax rate y_2 on duration arises from the creation or lack of creation of a tax liability occurring in the final period. If the bond is purchased at a discount, a tax liability directly related to the magnitude of y_2 is created in the final period. As tax rate y_2 rises, the after-tax principal payment is reduced by the tax liability. Consequently the weight for the final payment decreases causing duration to shorten. If the bond is purchased at par no tax shield or liability is created causing duration to be unaffected by a change in tax rate y_2.

If the investor buys the bond at a premium and amortizes the premium, he uses weights (9c) and (9d) to compute the duration of the bond. Since he is consequently not taxed on any capital gain, a change in y_2 does not affect his computation of duration.

Two observations can be made at this point. Firstly, if the investor uses the traditional duration equation, ignoring taxes, his after-tax portfolio income will not be immunized against interest rate changes. Immunization of after-tax income requires the use of the duration equation reflecting taxes.

Secondly, if, as a result of shifts in the tax rate y_1, the investor's tax-adjusted duration estimations of securities lengthen while his investment period remains constant, he is forced to purchase shorter-maturity securities to immunize his portfolio. However, if, as a result of shifts in y_2, his duration estimates shorten, he must purchase longer-maturity securities to maintain immunization over his holding period.

III. The Impact of a Change in the Tax Rates on the Market

The marginal analysis developed in Section II also can be applied to a macroeconomic analysis of a change in tax regulations. That is, every investor suffers an identical small alteration in the income and capital gains tax rates. In this situation, then (assuming perfect elasticity of supply), the marginal change in taxes affects the price of bonds rather than the investor's holding period yields. Marginal analysis is applied to this situation to examine the macroeconomic effect of tax rate changes on duration and bond portfolio composition.

A. Changes in the Interest Income Tax Rate

Consider a change in the income tax rate y_1 of every individual in the market. Assume that everyone amortizes any premium.

[9] See Appendix C.

As before, the effect on duration is measured by the derivative $\frac{dD}{dy_1}$ consisting of two parts: the direct effect of the change in y_1, $\frac{\partial D}{\partial y_1}$, plus the indirect effect of the change in bond price $\frac{\partial D}{\partial V} \cdot \frac{\partial V}{\partial y_1}$.

$$\frac{dD}{dy_1} = \frac{\partial D}{\partial y_1} + \frac{\partial D}{\partial V} \cdot \frac{\partial V}{\partial y_1} \tag{18}$$

If the bond is purchased at a discount or at par, the derivative $\frac{dD}{dy_1}$ is obtained from Equation (8);

$$\frac{dD}{dy_1} = \frac{-cA}{V} \int_0^T te^{-h(0,t)t}\, dt - \frac{\left(cA(1-y_1)\displaystyle\int_0^T te^{-h(0,t)t}\, dt + A(1-y_2)Te^{-h(0,T)T}\right)}{V^2} \frac{\partial V}{\partial y_1} \tag{19}$$

The influence that a tax rate change has on price is determined by taking the derivative of bond value Equation (12) with respect to the tax rate y_1.

$$\frac{\partial V}{\partial y_1} = \frac{-cA \displaystyle\int_0^T e^{-h(0,t)t}\, dt}{1 - y_2 e^{-h(0,T)T}} \tag{20}$$

Thus, on substitution of (20) into (19) and rearranging, the derivative $\frac{dD}{dy_1}$ simplifies to:

$$\frac{dD}{dy_1} = \frac{cA^2(1-y_2)e^{-h(0,T)T}}{V^2(1-y_2 e^{-h(0,T)T})} \left(T \int_0^T e^{-h(0,t)t}\, dt - \int_0^T te^{-h(0,t)t}\, dt\right) \tag{21}$$

Since $T > t$, $\frac{dD}{dy_1} > 0$. Thus, an increase in the interest income tax rate increases the duration of the bond.

If the bond is purchased at a premium, the analogues to Equations (19) and (20) are:

$$\frac{dD}{dy_1} = \frac{-cA + (V-A)T^{-1}}{V} \int_0^T te^{-h(0,t)t}\, dt - \left\{\frac{[cA(1-y_1) - y_1 AT^{-1}]}{V^2} \int_0^T te^{-h(0,t)t}\, dt + \frac{AT}{V^2} e^{-h(0,T)T}\right\} \frac{\partial V}{\partial y_1} \tag{19a}$$

$$\frac{\partial V}{\partial y_1} = \frac{[-cA + (V - A)T^{-1}] \int_0^T e^{-h(0,t)t}\, dt}{1 - y_1 T^{-1} \int_0^T e^{-h(0,t)t}\, dt} \tag{20a}$$

Using the same procedure as that applied to (19) and (20), it can be shown that $\frac{dD}{dy_1} > 0$ in the case that all investors purchase the bonds at a premium and amortize the tax shield.

The finding that duration of a bond lengthens as the interest income tax rate increases for all investors agrees with the previous section's finding for the case of a single investor. However, the mechanism in this section is different. The increase in the tax rate reduces the market value V of the bond. Since V appears in the denominator of all duration weights, all weights would appear to increase in value. However, since the after-tax interest income stream from the bond is being reduced by the marginal change in the tax rate, the weights applied to periods in which interest is received are reduced relative to the weight applied to the last period in which the principal is received. Consequently, the duration of the bond lengthens as the interest income tax rate rises.

B. Changes in the Capital Gains Tax Rate

Now consider a marginal change in only the capital gains tax rate y_2 for all investors. The measure of this effect on duration, $\frac{dD}{dy_2}$, is given as the sum of two effects, the direct effect $\frac{\partial D}{\partial y_2}$ plus the indirect effect $\frac{\partial D}{\partial V} \cdot \frac{\partial V}{\partial y_2}$. Assume the investor buys a bond at a discount or at par. The derivative $\frac{dD}{dy_2}$ is obtained by substituting Equations (9a) and (9b) into Equation (8).

$$\frac{dD}{dy_2} = \frac{(V - A)}{V} Te^{-h(0,T)T} - \frac{\left(cA(1 - y_1) \int_0^T te^{-h(0,t)t}\, dt + A(1 - y_2)Te^{-h(0,T)T}\right)}{V^2} \frac{\partial V}{\partial y_2} \tag{22}$$

As before the effect on V, $\frac{\partial V}{\partial y_2}$, is calculated from (12):

$$\frac{\partial V}{\partial y_2} = \frac{(V - A)e^{-h(0,T)T}}{1 - y_2 e^{-h(0,T)T}} \tag{23}$$

Substitution and rearrangement yield

$$\frac{dD}{dy_2} = \frac{(V - A)e^{-h(0,T)T}}{V^2(1 - y_2 e^{-h(0,T)T})} cA(1 - y_1)\left(T \int_0^T e^{-h(0,t)t}\, dt - \int_0^T te^{-h(0,t)t}\, dt\right) \tag{24}$$

As before, $T > t$; thus the term in brackets is positive. Thus, if $V - A$ is less than zero, $\frac{dD}{dy_2} > 0$; if $V = A$, D is unaffected by a change in y_2. This result is obtained because, if $V < A$, the after-tax principal repayment is reduced by the tax liability. Since V also falls, the effect is to shift the duration weights away from the last period. If $V = A$, alteration of y_2 has no effect on duration.

If the investor purchases the bond at a premium, he uses weights (9c) and (9d) to calculate duration. Since y_2 does not appear as a variable in these weights, duration is unaffected by a change in y_2.

The conclusion that may be drawn from these results follows from the fact that, to maintain immunization, the market must readjust its estimation of duration as the tax rates are altered. If shifts in the tax rates cause durations of given instruments to lengthen, the market moves toward shorter-maturity investments to maintain an immunized portfolio over the same holding period. Thus, a rise in the income tax rate causes the market to shift to shorter-maturity instruments. Also, a rise in the capital gains tax rate causes the market to move toward longer maturities in bonds selling at a discount. It should be noted that if supply is not perfectly elastic, as assumed in this analysis, this movement by the market toward bonds of a particular maturity will be offset by price changes which are the consequence of supply and demand equilibration.

IV. Summary and Conclusion

This investigation was focused on the impact of taxes on portfolio immunization and duration estimation. It has been observed that the immunization process is unaltered by taxes; i.e., the investor must match the portfolio's tax-adjusted duration with the length of his investment holding period. However, the investor must estimate the duration on an after-tax basis using his after-tax income stream and his after-tax holding period yield to compute the weights employed in the duration equation. The weights used in the duration equation depend on the tax treatment of the premium or discount. Therefore, the investor must select his tax treatment of premia or discounts prior to calculation of tax-adjusted duration.

The tax-adjusted duration of the security has been shown to be directly related to a marginal change in the interest income tax rate. This conclusion holds for changes in the tax rates of both the individual and the market in its entirety. However, in the case of the individual, the effect on duration arises out of a change in his holding period yields, whereas in the case of the entire market, the effect arises out of a change in the security's current value.

Unlike the result for interest income tax rates, the influence of marginal changes in the capital gain tax rate is contingent on both the tax regulation in effect and the purchase price of the bond. Under present tax regulations in the United States, which allow for amortization of tax shields, the influence of a change in the capital gains tax rate on the tax-adjusted duration estimation depends on whether the bond is purchased at par or at a discount. If the bond is purchased at par or at a premium and the premium is amortized, a change in the capital gain tax rate has no influence on the duration estimate. However, if the bond is purchased at a discount, the tax-adjusted duration is inversely related to the change in the capital gain tax rate.

When a tax rate change occurs, it causes an adjustment of the maturity composition of portfolios to maintain immunization over a given holding period. In particular, if interest income tax rates were to rise, either for the individual or the market, the required adjustment is to choose coupon bonds with after-tax income such that the duration of the portfolio equals the investment horizon. If zero coupon bonds are in the portfolio, the adjustment is simply a shortening of the maturities. If, on the other hand, the capital gain tax rates increase, either for the individual or the market, the only effect is on the duration of bonds bought at a discount. The required adjustment is a lengthening of the maturities.

Appendix A

From Equation (13)

$$0 = -cA \int_0^T e^{-h(0,t)t}\, dt - ca(1 - y_1) \int_0^T te^{-h(0,t)t} \frac{h(0, t)}{y_1}\, dt - (y_2(V - A) + A)Te^{-h(0,T)T} \frac{\partial h(0, T)}{\partial y_1}$$

Rearranging yields

$$cA \int_0^T e^{-h(0,t)t} dt = -cA(1 - y_1) \int_0^T te^{-h(0,t)t} \frac{\partial h(0, t)}{\partial y_1}\, dt - (y_2(V - A) + A)T\, e^{-h(0,T)T} \frac{\partial h(0, T)}{\partial y_1}$$

Since the left-hand side is positive, and the terms $cA(1 - y_1)$ and $(y_2(V - A) + A)Te^{-h(0,T)T}$ are also positive as well as the expression inside the integral, $te^{-h(0,t)t}$, the partial derivatives $\frac{\partial h(0, t)}{\partial y_1}$ and $\frac{\partial h(0, T)}{\partial y_1}$ must be negative. Note that these two derivatives must be of the same sign to maintain the equilibrium relationship among holding period yields[10]

$$h(0, t')t' + h(t', t)(t - t') = h(0, t)t.$$

Appendix B

From Equation (13)

$$0 = -cA \int_0^T e^{-h(0,t)t}\, dt - cA(1 - y_1) \int_0^T te^{-h(0,t)t} \frac{\partial h(0, t)}{\partial y_1}\, dt - (y_2(V - A) + A)Te^{-h(0,T)T} \frac{\partial h(0, T)}{\partial y_1}$$

[10] See Bierwag [1].

Rearrangement and division by V yields

$$\frac{-cA}{V}\int_0^T te^{-h(0,t)t}(1-y_1)\frac{\partial h(0,t)}{\partial y_1}\,dt$$

$$=\frac{cA}{V}\int_0^T e^{-h(0,t)t}\,dt+\frac{(y_2(V-A)+A)}{V}Te^{-h(0,T)T}\frac{\partial h(0,T)}{\partial y_1}$$

Since $\frac{\partial h(0,t)}{\partial y_1}<0, \frac{\partial h(0,T)}{\partial y_1}<0$ from Appendix A, substitution of the absolute values of these terms yields

$$\frac{cA}{V}\int_0^T te^{-h(0,t)t}(1-y_1)\left|\frac{\partial h(0,t)}{\partial y_1}\right|dt=\frac{cA}{V}\int_0^T e^{-h(0,t)t}\,dt$$

$$-\frac{(y_2(V-A)+A)}{V}Te^{-h(0,T)T}\left|\frac{\partial h(0,T)}{\partial y_1}\right|$$

Then

$$\frac{cA}{V}\int_0^T te^{-h(0,t)t}(1-y_1)\left|\frac{\partial h(0,t)}{\partial y_1}\right|dt<\frac{cA}{V}\int_0^T e^{-h(0,t)t}\,dt$$

since

$$\frac{(y_2(V-A)+A)}{V}Te^{h(0,T)T}\left|\frac{\partial h(0,T)}{\partial y_1}\right|>0$$

Since T was an arbitrarily chosen maturity for the bond, this result must hold for any maturity. Thus the integrands must satisfy

$$te^{-h(0,t)t}(1-y_1)\left|\frac{\partial h(0,t)}{\partial y_1}\right|<e^{-h(0,t)t}$$

or

$$e^{-h(0,t)t}+te^{-h(0,t)t}(1-y_1)\frac{\partial h(0,t)}{\partial y_1}>0$$

Appendix C

From Equation (16)

$$0=(V-A)e^{-h(0,T)T}-cA(1-y_1)\int_0^T te^{-h(0,t)t}\frac{\partial h(0,t)}{\partial y_2}\,dt$$

$$-(y_2(V-A)+A)Te^{-h(0,T)T}\frac{\partial h(0,T)}{\partial y_2}$$

Rearranging yields

$$cA(1-y_1)\int_0^T te^{-h(0,t)t}\frac{\partial h(0,t)}{\partial y_2}\,dt + (y_2(V-A)+A)Te^{-h(0,T)T}\frac{\partial h(0,T)}{\partial y_2} = (V-A)e^{-h(0,T)T}$$

Since $e^{-h(0,T)T}$, $cA(1-y_1)$ and $(y_2(V-A)+A)Te^{-h(0,T)T}$ are positive, as well as the integrand $te^{-h(0,t)t}$, the sign of the terms $\frac{\partial h(0,t)}{\partial y_2}$ and $\frac{\partial h(0,T)}{\partial y_2}$ depends entirely on the sign of $(V-A)$. Note that as in Appendix A, the terms $\frac{\partial h(0,t)}{\partial y_2}$ and $\frac{\partial h(0,T)}{\partial y_2}$ must have the same sign to maintain holding-period-yield equilibrium.

REFERENCES

1. G. O. Bierwag. "Immunization, Duration and the Term Structure of Interest Rates." *Journal of Financial and Quantitative Analysis* 12 (December 1977).
2. G. O. Bierwag, George G. Kaufman, and Chulsoon Khang. "Duration and Bond Portfolio Analysis." *Journal of Financial and Quantitative Analysis* 13 (November 1978).
3. John C. Cox, Jonathan Ingersoll, and Stephen Ross. "Duration and the Measurement of Basis Risk." *Journal of Business* 52 (January 1979).
4. Lawrence Fisher and Roman L. Weil. "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies." *Journal of Business* 44 (October 1971).
5. Michael Hopewell and George Kaufman. "Bond Price Volatility and Term to Maturity: A Generalized Respecification." *American Economic Review* 63 (September 1973).
6. Jonathan Ingersoll, Jeffrey Skelton, and Roman L. Weil. "Duration Forty Years Later." *Journal of Financial and Quantitative Analysis* 13 (November 1978).
7. Miles Livingston and John Caks. "A 'Duration' Fallacy." *Journal of Finance* 32 (March 1977).