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Author(s): Steinar Ekern
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STEINAR EKERN*

ABSTRACT

Building on the stochastic dominance framework, time dominance efficiency analysis provides similar rules for a partial ordering of temporal prospects. Time dominance does not require any quantitative information about temporal preferences for screening decision alternatives according to their net present values. A binary time dominance proposition extends recent sufficient conditions and adds necessity. The paper's main contribution is the development of set time dominance. By eliminating binary undominated projects which no one would choose, set time dominance minimizes time efficient sets without imposing further preference assumptions.

STOCHASTIC DOMINANCE IS BY now a well established approach to choice problems under uncertainty [14]. This paper transfers the methodology to a temporal context, where the consequences of a decision alternative are distributed over time, with capital budgeting being a prime example. Whereas stochastic dominance puts successively stronger restrictions on the utility functions representing risk preferences, in time dominance the discounting functions representing temporal preferences are qualitatively restricted in a similar way. The time dominance approach provides rules for a partial ordering of temporal prospects, yielding an efficient set from which the ultimate choice will be made. Its distinct feature is that no numerical values of the discounting function are needed to identify and eliminate decision alternatives which are inferior according to the net present value (*NPV*) rule.

That property is especially attractive if future temporal preferences cannot be unambiguously quantified on the basis of capital market information. It may be instructive to consider some circumstances where the time dominance procedures may prove useful: a single decisionmaker, being uncertain about future temporal tradeoffs, may be reluctant to quantify the discounting function. A group of decisionmakers with possibly conflicting temporal preferences may have to work out a joint decision. A syndicate organizer may attempt bringing together participants with heterogeneous evaluations of consequences across time. In agency situations the agent may have incomplete knowledge about the principal's temporal preference. The remote client problem may be even more severe in the case

* Norwegian School of Economics and Business Administration, Bergen; and Graduate School of Business Administration, University of California, Berkeley (visiting, 1980/81). The author appreciates the helpful suggestions made by Karl Borch, Øyvind Bøhren, Julian Franks, Frøystein Gjesdal, Lars Mathiesen, Jack Meyer, and Edwin J. Elton. He also thanks the Institute of Business and Economic Research, University of California, Berkeley, for clerical assistance in preparing this manuscript, as well as the Graduate School of Business Administration, University of California, Berkeley, for its hospitality during his stay as a visiting scholar. The research has been supported by grant NAVF 12.14.51.012.

of public cost-benefit analysis. General principles may be shown to remain valid even when restrictive term structure of interest assumptions are somewhat relaxed.

Potential applications of the time dominance methodology include important decisionmaking situations, like comparing competitive technologies, ranking alternative financial management policies, selecting geographic locations, designing marketing strategies, and evaluating public projects.

In Section I the problem is formulated in more detail. A time dominance approach links properties of higher order cumulated (or integrated) consequence values over time to superior *NPV*s for all decisionmakers whose temporal preferences satisfy certain conditions. Generalizing and extending related results of Bøhren and Hansen [1] and of Pratt and Hammond [11], the binary time dominance proposition in Section II depends on pairwise comparisons between individual projects. Motivated by the work of Fishburn [3] and Meyer [9] on stochastic efficiency analysis, Section III uses a convex combination as a basis for comparisons. The resulting set time dominance proposition removes from the efficient set binary undominated projects which no one would choose, without imposing further preference assumptions. Some final remarks in Section IV conclude the main body of the paper. Proofs of the propositions are outlined in appendices.

I. Formulation

Let t denote a point in time, running from one initial point 0 onward into the future until the horizon H. Suppose there is a set S of mutually exclusive decision alternatives available to the decisionmaker(s). Each such alternative is completely characterized by its consequences $x = x(t)$ over time. Temporal preferences are represented by a discounting function $v = v(t)$.

For simplicity, the following exposition will be phrased in a capital budgeting framework. Then the set S is a collection of investment projects. The consequence $x(t)$ may be interpreted as the cash flow at time t. With R_t being the single-period interest rate from $(t - 1)$ to t, and $r(t)$ being the instantaneous spot rate at time t, the discounting function $v(t) = [(1 + R_1)(1 + R_2) \cdots (1 + R_t)]^{-1}$ in discrete time, whereas $v(t) = \exp[-\int_0^t r(s)\, ds]$ in continuous time.

The major premise of this paper is that the decisionmaker evaluates temporal prospects by the *NPV* criterion. For a given discounting function v, the *NPV* of x, written $NPV(x:v)$, is defined as $\sum_{t=0}^{H} v(t)x(t)$ in the discrete time case, and as $\int_0^H v(t)x(t)dt$ in continuous time. When two projects with cash flows of x_j and x_i are being compared, the former project will be strictly preferred if and only if $NPV(x_j:v) > NPV(x_i:v)$.

There may be considerable flexibility as to what features are to be incorporated into the x or v functions. If x is a deterministic cash flow, then the discounting function v may represent general equilibrium marginal rates of substitution in perfect and complete markets [2, Chap. 2] or subjective marginal rates of substitution in a nonmarket context [7, Chap. 9]. When such *MRS*s can be written as a product of a marginal utility and a pure time preference function [10], x may be the cash flow weighted by the marginal utility and v the pure time preference, or x may be just the cash flow itself with v reflecting both the marginal utility and the pure time preference aspects of discounting. Under uncertainty, x may be expected values of cash flows and v implies risk-adjusted discounting, or x may represent certainty equivalents and v risk-free discounting. Under inflationary conditions, x may be expressed in nominal terms whereas v takes care of both deflating and discounting, or x may be in real terms leaving only time preference adjustments for v. In the presence of taxation, x may be before-tax cash flow with both taxation effects and discounting embodied in v, or x is an after-tax cash flow stream and v reflects only temporal tradeoffs.

The time dominance partial ordering of alternatives depends on mathematical properties of the cash flow distribution x. It uses repeated summations in discrete time and repeated integrations in continuous time. Rewrite $X^0(t) \equiv x(t)$. Then, for $n = 1, 2, \cdots$, the notation $X^n(t)$ is defined as $\sum_{s=0}^{t} X^{n-1}(s)$ in discrete time and as $\int_0^t X^{n-1}(s)\, ds$ in continuous time. Thus, $X^0(t)$ is the cash flow, $X^1(t)$ is the cash balance at zero interest rate, $X^2(t)$ is the cumulative cash balance, etc.

Such repeated cumulations of consequences correspond to repeated cumulations of probabilities in the stochastic dominance approach, although there are some important differences. Unlike probabilities, cash flows may be negative, and different projects' cash balances at the horizon do not necessarily coincide. Hence, $X^k(t)$ may decrease in t, and $X_j^1(H) \neq X_i^1(H)$.

Similar to stochastic dominance using assumptions about utility functions, time dominance calls for curvature restrictions to classify discounting functions. Let $v^0(t) \equiv v(t)$. Then, for any $k = 1, 2, \cdots, n$, let

$$v^k(t) \equiv \begin{cases} dv^{k-1}(t)/dt & \text{in continuous time} \\ v^{k-1}(t+1) - v^{k-1}(t) & \text{in discrete time} \end{cases} \tag{1}$$

Thus, $v^k(t)$ is obtained by differentiating or differencing $v(t)$ k times.

The widest class of discounting functions, denoted V_0, requires simply that at any point in time more is preferred to less. Formally, $V_0 \equiv \{v : v(t) > 0\}$. By adding successive restrictions on $v^k(t)$ as the nonnegative integer k increases, subsets of discounting functions are defined:

$$V_k = \{v : v \in V_{k-1}, \quad \text{and} \quad (-1)^k v^k(t) > 0\} \tag{2}$$

Hence, v belongs to the class V_n if and only if $v^k(t)$ alternates in sign (and starting with plus), as k goes from 0 to n. The domain of $v^k(t)$ contains the interval $[0, H]$ in continuous time and $[0, H - k]$ in discrete time.

A condition such as (2) may need some interpretation. The positively valued discounting functions of V_0 require nonsatiation, such that (0, 1, 0) is always preferred to (0, 0, 0). Downward sloping discounting functions representing impatience belong to V_1, implying (0, 1, −1, 0) necessarily being preferred to (0, 0, 0, 0). If impatience decreases over time, such that (1, −2, 1) is considered better than (0, 0, 0), then V_2 contains the appropriate convex discounting function. Whenever the decrease in impatience increases over time, (1, −3, 3, −1) is considered superior to (0, 0, 0, 0), the marginal discounting function is concave, and the discounting function itself belongs to V_3. For higher orders of k, suppose

v belongs to the class V_{k-1}. It can then be shown [12] that if $v^k(t)$ does not change sign for any t, then v belongs to V_k as well. Alternatively, if $(-1)^{k-2}v^{k-2}(t)$ is strictly convex, then V_k contains v.

Some readers may prefer putting restrictions on the interest rates rather than directly on the discounting functions. Positive interest rates are equivalent to the class V_1. Corresponding to (1), let $r^k(t)$ be the instantaneous interest rate function $r(t)$ differentiated k times. Then the conditions $r(t) > 0$ and $r^1(t)/r(t) < r(t)$ are equivalent to $v^1(t) < 0$ and $v^2(t) > 0$, i.e., the class V_2. Thus, a positive and nonincreasing interest rate is a sufficient but not necessary condition of v belonging to V_2. Some increasing interest rate functions, such as $r(t) = a \exp[At] + c$, with $a > 0$ and $c > A/2 > 0$, are also in V_2. A positive interest rate and a nonpositive and nondecreasing marginal interest rate constitute sufficient conditions for V_3. Generally, $r(t) > 0$ and $(-1)^k r^k(t) \geq 0$ and for $k = 1, 2, \cdots, n - 1$ are sufficient conditions for the discounting function class V_n, for $n \geq 2$. Note that $r^k(t)$ nonstrictly alternating in sign is not a necessary condition for V_n, whereas (2) yields an exact and complete characterization of each discounting function class V_n. In discrete time, the first through $(n - 1)$th order differences of the single-period interest rates nonstrictly alternating in sign imply V_n.

The time dominance approach is most powerful whenever the discounting functions belong to the class V_n for arbitrary positive integers n. From the previous paragraph, a sufficient condition for this case to occur is that starting with a positive interest rate $r(t)$, successive higher order derivatives $r^k(t)$ alternate between nonpositivity for odd k and nonnegativity for even k. If the nonincreasing interest rate function $r(t)$ itself is either exponential $r(t) = a \exp[-At] + c$, logarithmic $r(t) = -a \ln(A + t) + c$, power $r(t) = -a(A + t)^{1-B} + c$ with $0 \leq B \leq 1$, or power $r(t) = a(A + t)^{1-B} + c$ with $B > 1$, then the sign of $r^k(t)$ alternates between minus and plus as k goes through the positive integers from one upward. The parameters a, A, and c are assumed to be nonnegative, but no numerical parameter values are required for v to be in V_n for any n, as long as $r(t)$ remains positive for all t until the horizon. For the important special case of $a = 0$, by varying c the set of positive constant interest rates $r(t) = c$ is obtained, which will always correspond to proper subsets of V_n for any n.

Whereas a risk-preference utility function is unique up to a positive linear transformation, the temporal preference discounting function v is unique only up to a positive constant. By convention, $v(t)$ will be normalized such that $v(0) = 1.0$.

Using a time dominance methodology, information about particular $X^k(t)$ values for $1 \leq k \leq n$ may suffice to conclude whether some decision alternatives are definitely inferior for all discounting functions in the class V_n. Projects surviving the purging process will form efficient sets, which should be as small as possible to facilitate the final choice.

II. Binary Time Dominance

Starting out with pairwise comparisons of two projects at a time, binary time dominance may be developed.

In continuous time, a project x_j dominates x_i by nth order binary time dominance if and only if

$$X_j^k(H) \geq X_i^k(H) \quad \text{for all } k = 1, 2, \cdots, n-1 \tag{3a}$$

$$X_j^n(t) \geq X_i^n(t) \quad \text{for all } t \text{ in } [0, H] \tag{3b}$$

with (3b) holding as a strong inequality over some subinterval. This definition resembles the corresponding nth order stochastic dominance [6] [13]. It was introduced in a time framework by Bøhren and Hansen [1].

In discrete time a somewhat different definition is more convenient, building on the work of Pratt and Hammond [11]. Consider the matrix $\{X^k(t)\}$, where $k = 1, 2, \cdots, n$ and $t = 0, 1, \cdots, H$. An admissible nth order connected path starts at the lower left corner of $X^n(0)$, passes through adjoining elements to the right and/or above, and ends in the upper right corner of $X^1(H)$. The project x_j dominates x_i by nth order binary time dominance if and only if

$$X_j^k(t) \geq X_i^k(t) \tag{4}$$

for all (k, t) pairs along some admissible nth order path, with at least one strict inequality.

The proposition below links nth order binary time dominance to NPV superiority for discounting functions in V_n.

PROPOSITION 1: *$NPV(x_j : v) > NPV(x_i : v)$ for all v in V_n, if and only if x_j dominates x_i by nth order binary time dominance.*

Appendix A provides an outline of a proof.

Proposition 1 extends results of Bøhren and Hansen [1] and of Pratt and Hammond [11]. Bøhren and Hansen stated and proved three propositions, which for continuous time models yield the sufficiency parts for $n = 1$, $n = 2$, and the constant interest rate subclass of V_n for any n. However, if the interest rates for two periods were known to be different, and second-order binary time dominance does not hold, the Bøhren and Hansen procedure cannot assist in ranking projects whereas Proposition 1 may still be useful. The Pratt and Hammond work on bounding the number of internal rates of return (IRR) essentially required v to be in V_n for all n. The necessity part of Proposition 1 is new, but it is consistent with the necessary conditions of stochastic dominance theorems [4].

Table I illustrates how Proposition 1 may be applied to the selection of a depreciation method for tax purposes. As depreciation yields a tax shield, rapid depreciation is generally desirable. Two popular methods of accelerated depreciation are SYD = "sum of the years' digits" and DDB = "double declining balance" [8, Chap. 6]. The example assumes an initial outlay of 3,000 to be completely written off in five years with a tax rate of 50 percent. AS DDB has the highest initial saving but SYD has the highest cash balances after three and four years, there exist different discounting functions representing impatience such that either SYD or DDB is the preferred method. However, DDB dominates SYD by a second-order binary time dominance in this case. Therefore, any decision-maker with decreasing impatience will select DDB under the postulated depre-

Table I

Second-Order Binary Time Dominance

Cumulations	Depreciation Method	Time 0	1	2	3	4	5	6
0	DDB	0	600	360	216	129.6	194.4	0
	SYD	0	500	400	300	200	100	0
1	DDB	0	600	960	1176	1305.6	1500	1500
	SYD	0	500	900	1200	1400	1500	1500
2	DDB	0	600	1560	2736	4041.6		
	SYD	0	500	1400	2600	4000		

Table II

Postponing the Horizon from the Project Terminus

Cumulations	Time 0	1	2	3	4	5	6
0	1	−3	2.5	0	0	0	0
	1	1	−2	0.5	0.5	0.5	0.5
	2	1	−1	−0.5	0.0	0.5	0.5
	3	1	0	−0.5	−0.5	0.0	
	4	1	1	0.5	0.0		

ciation period. Note how this conclusion was obtained by a simple "back of the envelope" calculation, without using any tables of compound interest.

The cash flow $x_1 = (1, -3, 2.5)$ is a prototype often used for demonstrating the nonexistence of an internal rate of return (*IRR*) [5, p. 79]. Table II shows that x_1 dominates $x_0 = (0, 0, 0)$ by fourth-order binary time dominance. The sufficiency part of Proposition 1 confirms that x_1 has in fact no *IRR*. Hirshleifer [5] stated that there exist "perfectly respectable" interest patterns for which $-x_1 = (-1, 3, -2.5)$ would be desirable to the null alternative. The proposition restricts these "respectable" interest rates to discounting functions which do not belong to V_n for $n \geq 4$. Incidentally, Hirshleifer's successive one-period interest rates of 100 percent and 200 percent imply $v(1) = \frac{1}{2}$ and $v(2) = \frac{1}{6}$, such that his v does not belong to V_n for $n \geq 3$, as $v(0) - 3v(1) + 3v(2) - v(3) = 1 - \frac{3}{2} + \frac{3}{6} - v(3) = -v(3) < 0$.

Table II also illustrates the somewhat surprising result that it may be necessary to extend the horizon H beyond the occurrence of the last nonzero element of any x. If the project terminus was used as the horizon, such that $H = 2$, no binary time dominance results could be obtained, as $X_1^2(2) = X_1^3(2) < 0$. In contrast, by sufficiently postponing the horizon into the future to $H \geq 4$, binary time dominance can be established. Of course, adding on a zero cash flow from time 3 onward does not influence the real attractiveness of the project, but for purely mathematical reasons it may increase the discriminatory power of the dominance approach. Pratt and Hammond [11] made a related observation with respect to strengthening the bound on the number of internal rates of return, where a bound of zero *IRR*s for a differential project indicates binary time dominance.

III. Set Time Dominance

Repeated applications of Proposition 1 to weed out inferior projects from the set S yield the nth order binary time dominance efficient set S_n^*. It consists of projects which are not dominated by any other project according to nth order binary time domnance. Thus, S_n^* is derived by successive pairwise comparisons. The necessity part of Proposition 1 implies that for any pair of alternatives x_j and x_i in S_n^*, there exist both some v in V_n such that $NPV(x_j{:}v) > NPV(x_i{:}v)$ and also different v in V_n which reverse the order of preference. It is important to note that even if x_j belongs to some nth order binary efficient set S_n^* containing more than two elements, there is no presumption that there actually exists any v in V_n such that x_j is the *most* preferred alternative! If it can be shown that there exists no v in V_n such that x_j has the highest NPV, then the project may be removed from the binary efficient set without restricting the choice opportunities of interest.

This section will show how the discriminatory power of time dominance relations may be sharpened without requiring additional assumptions. Such a further size reduction of efficient sets may be achieved by replacing a pairwise comparison between individual projects by a comparison between any project and a convex combination of other projects. This approach has been stimulated by Fishburn's [3] work on convex stochastic dominance and Meyer's [9] set stochastic dominance exposition.

As an illustration, let $S = \{x_1, x_2, x_3\}$. Suppose no alternative can be eliminated based on pairwise comparisons, that is $S_n^* = S$. Define $x_4 = \lambda x_1 + (1 - \lambda)x_2$, where $0 < \lambda < 1$. The fact that x_4 does not belong to S is unimportant, as it is just an auxiliary construct. Assume x_4 dominates x_3 by nth order binary time dominance. Hence, $NPV(x_4{:}v) > NPV(x_3{:}v)$ for all v belonging to V_n. By the linearity of the NPV criterion,

$$\lambda \mathrm{NPV}(\mathrm{x}_1{:}\mathrm{v}) + (1 - \lambda)\mathrm{NPV}(\mathrm{x}_2{:}\mathrm{v}) > \mathrm{NPV}(\mathrm{x}_3{:}\mathrm{v}) \tag{5}$$

for all v in V_n. But, if (5) holds, and as $\lambda > 0$ and $(1 - \lambda) > 0$, then either $NPV(x_1{:}v) > NPV(x_3{:}v)$ or $NPV(x_2{:}v) > NPV(x_3{:}v)$ or both. In neither case will x_3 be selected by anyone whose discounting function v belongs to the class V_n, despite the fact that the nth order binary efficient set S_n^* contains the project x_3. Therefore, x_3 may be removed, to form an optimal nth order efficient set S_n^{**} containing as few elements as possible.

To formalize these ideas, let $S_J \subseteq S$ be a subset of feasible alternatives, and let J be the index set of alternatives in S_J. Let $\lambda(J)$ be a vector of nonnegative weights summing to unity. Set time dominance may then be defined as follows: the set S_J dominates x_i by nth order set time dominance, if and only if there exists a vector $\lambda(J)$ of nonnegative weights summing to unity, such that the convex combination $(\sum_{j\in J}\lambda_j x_j)$ dominates x_i by nth order binary time dominance.

PROPOSITION 2: *There exists no v in V_n such that $NPV(x_i{:}v) > NPV(x_j{:}v)$ for all x_j in S_J, if and only if S_J dominates x_i by nth order set time dominance.*

The proof is deferred to Appendix B.

Elimination of as many alternatives as possible by Proposition 2 leaves the nth order set time dominance efficient set S_n^{**}. Every element in S_n^{**} has the property that there exists some v in V_n such that this project will indeed be the preferred one. As binary time dominance is a special case of set time dominance, $S_n^{**} \subseteq S_n^*$. Thus, set time dominance yields a minimum size efficient set. The achieved size reduction relative to binary time dominance has been obtained without imposing further restrictions.

Table III provides a numerical example with three projects. The cash flows have been selected in such a way that there cannot be any binary time dominance of any order. An inspection of the cash balances $X^1(t)$ verifies that the nth order binary efficient set $S_n^* = \{x_1, x_2, x_3\}$ for any $n \geq 1$. Now consider the convex combination $\lambda x_1 + (1 - \lambda)x_2$. For $3/8 \leq \lambda \leq 1/2$, the conditions $-2 + 8\lambda \geq 1$, $6 - 8\lambda \geq 2$, and $4 - 2\lambda \geq 3$ are all satisfied, with at least one strong inequality. Hence, $\{x_1, x_2\}$ dominates x_3 by first-order set time dominance. Any impatient decision maker will select either x_1 or x_2, but not the inferior x_3. Although x_3 is the only project which is guaranteed to have a positive NPV, it is also the only one which does not belong to any nth order set efficient set S_n^{**}!

Bøhren and Hansen [1] introduced a concept called normalized time dominance. It involved dividing through the cash flows by the assumed positive respective cash balances at the horizon, yielding, e.g., $\hat{x}_j(t) \equiv x_j(t)/X_j^1(H)$. Letting $x_0 = (0, 0, \cdots, 0)$ be the null alternative, their approach is equivalent to an examination of whether the convex combination $[\lambda x_j + (1 - \lambda)x_0]$ dominates x_i by nth order binary time dominance, subject to

$$X_j^1(H) \geq X_i^1(H) > 0 \tag{6a}$$

$$\lambda = X_i^1(H)/X_j^1(H) \tag{6b}$$

$$x_0 \text{ belongs to } S \tag{6c}$$

Table III

First-Order Set Time Dominance, No Binary, and no Normalized Time Dominance of any Order

Cumulations k	Project i	Time t=0	t=1	t=2
0	x_1	6	−8	4
	x_2	−2	8	−2
	x_3	1	1	1
	$\lambda x_1+(1-\lambda)x_2$	$-2+8\lambda$	$8-16\lambda$	$-2+6\lambda$
1	x_1	6	−2	2
	x_2	−2	6	4
	x_3	1	2	3
	$\lambda x_1+(1-\lambda)x_2$	$-2+8\lambda$	$6-8\lambda$	$4-2\lambda$
1	$\hat{x}_1$	3	−1	1
	$\hat{x}_2$	−1/2	3/2	1
	$\hat{x}_3$	1/3	2/3	1

Table IV

No Elimination of Project Being Inferior at any Constant Interest Rate

Cumulations	Project	Time t=0	t=1	t=2	IRR	NPV $(x:\bar{v})$
0	x_1	0.0	0.0	0.1	∞	0.075
	x_2	−3.0	0.0	4.0	15.5%	0.000
	x_3	−2.0	0.0	2.8	18.3%	0.100
	x_4	−1.0	−4	5.5	8.2%	0.105
0	x_2–x_1	−3.0	0.0	3.9	14.0%	
	x_2–x_3	−1.0	0.0	1.2	9.5%	
	x_2–x_4	−2.0	4.0	−1.5	50.0%	
	x_3–x_1	−2.0	0.0	2.7	16.2%	
1	x_1	0.0	0.0	0.1		
	x_2	−3.0	−3.0	1.0		
	x_3	−2.0	−2.0	0.8		
	x_4	−1.0	−5.0	0.5		

The restrictions (6a) and (6b) are redundant, and in a general setting (6c) does not necessarily hold.

From Table III it was seen how binary time dominance failed to eliminate an obviously inferior alternative, whereas first-order set time dominance succeeded. Inspection of the bottom part of Table III will show also that normalized time dominance of any order will fail to obtain.

If attention is limited to a proper subclass of V_n, then nth order set time dominance is no longer a necessary condition for inferiority. Table IV presents an example. Assume a constant but unspecified interest rate R, such that $v(t) = (1 + R)^{-t}$, which is contained in V_n for any n. By plotting the *NPV* profiles, or from inspection of the *IRRs* of the four projects themselves, as well as of the differential projects, it may be concluded that x_2 is selected if $0\% \leq R \leq 9.5\%$, x_3 is best for $9.5\% < R \leq 16.2\%$, whereas x_1 is superior for $R > 16.2\%$. Hence, x_4 is never a preferred project under the stated constant interest rate assumption. Still, x_4 cannot be eliminated on the basis of nth order set time dominance for any n. To see why, let λ_1, λ_2, and λ_3 be nonnegative weights associated with x_1, x_2, and x_3. The initial cash flow condition is

$$0.0\lambda_1 - 3.0\lambda_2 - 2.0\lambda_3 \geq -1 \tag{7a}$$

The final cash balance condition requires

$$0.1\lambda_1 + 1.0\lambda_2 + 0.8\lambda_3 \geq 0.5 \tag{7b}$$

Finally, the weights should sum to unity:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1.0 \tag{7c}$$

As (7a), (7b), and (7c) cannot all be simultaneously satisfied, set time dominance of any order is precluded.

On the other hand, assume $\bar{v}(0) = 1.000$, $\bar{v}(1) = 0.755$, and $\bar{v}(t) = 0.755(0.750/$

$0.755)^{t-1}$ for $t > 1$. This discounting function belongs to V_n for any n, as it corresponds to a shift from a high single-period interest rate to a lower one after one period. Out of the four projects whose cash flows are given in Table IV, project x_4 does, in fact, have the highest *NPV* for this latter discounting function $\bar{v}$.

IV. Conclusions

Building on the stochastic dominance framework, the time dominance efficiency analysis provides similar rules for a partial ordering of decision alternatives whose consequences are distributed over time. Without requiring any quantitative information about temporal preferences, qualitative restrictions on discounting functions may suffice for screening projects according to their *NPV*s. Two propositions permit elimination of projects which will be considered inferior by any decisionmaker whose temporal preferences exhibit particular characteristics. Properties of repeatedly cumulated (or integrated) consequences over time are crucial for such purging to be feasible.

Based on pairwise comparisons between individual temporal prospects, the binary time dominance proposition extends and supplements recent related results [1] [11]. The paper's main contribution is the development of set time dominance, drawing on recent advances in set (or convex) stochastic dominance [3] [9]. Set time dominance compares individual projects to sets of alternative ones to exclude binary undominated projects which will not be the most preferred by any decisionmaker. It builds on the simple observation that if the *NPV* of a convex combination of projects exceeds the *NPV* of a single alternative, then at least one individual project in the combination has a higher *NPV* than the alternative one. The set time dominance proposition may reduce the size of the efficient set of prospects without imposing further preference restrictions. Time dominance, in general, and set time dominance, in particular, provide convenient tools for reducing the number of candidate decision alternatives requiring a detailed temporal preference evaluation.

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APPENDIX A

Outline of Proof of Proposition 1

Using cash balances $X^1(t)$ rather than cash flows $x(t)$, the NPV can be written as $NPV(x:v) = \int_0^H v(t)dX^1(t)$ in continuous time. From integration by parts n times, assuming $X^k(0) = 0$, the NPV difference is given by

$$NPV(x_j:v) - NPV(x_i:v)$$

$$= \sum_{k=1}^{n} [(-1)^{k-1}v^{k-1}(H)][X_j^k(H) - X_i^k(H)]$$

$$+ \int_0^H [(-1)^n v^n(t)][X_j^n(t) - X_i^n(t)]\, dt \tag{8}$$

Combining (2), (3a), and (3b) shows that the difference is positive. Hence, if x_j dominates x_i by nth order binary time dominance, then $NPV(x_j:v) > NPV(x_i:v)$ for all v in V_n.

To show necessity, one proceeds by example, building on ideas from the stochastic dominance theorems [4, pp. 72–77].

Consider the discounting function

$$v(t) = H^{1-k}(H - t)^{k-1}[(k - 1)!]^{-1} + pe^{-pt} \tag{9}$$

which belongs to V_k for any $1 \le k \le n$. Hence, $NPV(x:v) = H^{1-k} \int_0^H (H - t)^{k-1}[(k - 1)!]^{-1}dX^1(t) + p \int_0^H e^{-pt}dX^1(t)$. As $X^k(H) = \int_0^H X^{k-1}(t)\, dt$, performing $(k - 1)$ integrations by parts of the left integration term shows that $NPV(x:v) = H^{1-k}X^k(H) + pNPV(x:e^{-pt})$. Consequently,

$$NPV(x_j:v) - NPV(x_i:v) = H^{1-k}[X_j^k(H) - X_i^k(H)] + o(p) \tag{10}$$

Therefore, as $p \to 0$, $X_j^k(H) - X_i^k(H) < 0 \Rightarrow NPV(x_j:v) < NPV(x_i:v)$. Hence (3a) is a necessary condition for $NPV(x_j:v) > NPV(x_i:v)$ for all v in V_n.

Proving the necessity of (3b) is done similarly. Assume $X_j^n(\tau) < X_i^n(\tau)$ for some τ satisfying $0 < \tau < H$. Let the discounting function be $v(t) = w(t) + pe^{-pt}$, where

$$w(t) = \begin{cases} \tau^{1-n}(\tau - t)^{n-1}[(n - 1)!]^{-1} & \text{if } 0 \le t \le \tau \\ 0 & \text{if } \tau < t \le H \end{cases} \tag{11}$$

Thus, $NPV(x:v) = \tau^{1-n} \int_0^\tau (\tau - t)^{n-1}[(n - 1)!]^{-1}dX^1(t) + p \int_0^H e^{-pt}dX^1(t)$. After $(n - 1)$ partial integrations of the $w(t)dX^1(t)$ term, $NPV(x:v) = \tau^{1-n}X^n(\tau) + pNPV(x:$

e^{-pt}). As before, $pNPV(x{:}e^{-pt})$ approaches zero when p goes to zero, and

$$NPV(x_j{:}v) - NPV(x_i{:}v) = \tau^{1-n}[X_j^n(\tau) - X_i^n(\tau)] + o(p) \tag{12}$$

For sufficiently small p, $X_j^n(\tau) < X_i^n(\tau) \Rightarrow NPV(x_j{:}v) < NPV(x_i{:}v)$, showing the necessity of (3b).

In a discrete time framework, integrations by parts are replaced by summations by parts, also known as Abel's summation identity [4, p. 85]. In the necessary part, the function $w(t)$ of (11) is then redefined as

$$w(t) = \begin{cases} \begin{pmatrix} \tau + n - 1 \\ \tau \end{pmatrix}^{-1} \begin{pmatrix} \tau - t + n - 1 \\ \tau - t \end{pmatrix} & \text{if } t \leq \tau \\ 0 & \text{if } \tau < t \leq H \end{cases} \tag{13}$$

with similar changes in (9).

APPENDIX B

Outline of Proof of Proposition 2

If S_j dominates x_i by nth order set time dominance, then by definition there is some convex combination $(\sum_{j\in J}\lambda_j x_j)$ which dominantes x_i by nth order binary time dominance. Proposition 1 yields $NPV(\sum_{j\in J}\lambda_j x_j{:}v) > NPV(x_i{:}v)$ for all v in V_n. As the NPV of a convex combination equals the convex combination of the NPVs,

$$\textstyle\sum_{j\in J}\lambda_j NPV(x_j{:}v) > NPV(x_i{:}v) \quad \text{for all } v \text{ in } V_n \tag{14}$$

Because the weights λ_j are nonnegative and summing to unity, for any particular v in V_n there must be at least one x_j in S_J for which $NPV(x_j{:}v) > NPV(x_i{:}v)$, which proves sufficiency. Necessity is immediate from Proposition 1.