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Risk Decomposition and Portfolio Diversification When Beta is Nonstationary: A Note

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THE PROCESS OF DIVERSIFICATION and the subsequent reduction in portfolio risk have received considerable attention in the financial literature [5, 7–9, 10, 14–16, 25]. King [9], in his pioneering work on diversification, used multivariate statistical methods to decompose risk on a sample of 63 NYSE firms from six industries over the period 1927 to 1960. From this sample he concluded that on the average about half of a stock's variance could be explained by market behavior. Later, Evans and Archer [5] examined the effect of naive diversification on portfolio risk. Using semi-annual data for the period 1958 to 1967, they examined diversification among 470 securities listed in the Standard and Poor's Index by constructing 60 randomly selected portfolios which varied in size from 1 to 40 stocks. They then calculated the portfolio return's mean standard deviation and estimated the level of systematic or undiversifiable risk by estimating the asymptote to which the mean standard deviation decreased. The results of heir study showed that the majority of risk reduction is accomplished through randomly diversifying among only 10 securities. While these studies served to define the speed and limits of random diversification, other studies [7, 10, 14–16] have shown that systematic attempts to select stocks can result in inefficiencies in diversification.

As the CAPM has gained in popularity, it has become increasingly common to examine the level of diversification and decompose risk through the use of ordinary-least-squares (O.L.S.) regressions in which the variability explained by the market is designated systematic risk and the variability of the error term is used as a measure of unsystematic risk. Inherent in this dichotomization is the assumption that the beta (or relationship between the market's returns and the individual stock's returns) is stationary. In the present study, we shall show that while the use of O.L.S. to partition risk is commonplace, it can lead to incorrect conclusions due to error introduced by the violation of this assumption, that is, the nonstationarity of security betas.

The existence of instability of beta coefficients has been pointed out in studies by Altman, Jacquillat, and Levasseur [1], Baesel [2], Fabozzi and Francis [6], Levitz [11], Levy [12], and Roenfelt, Griepentrog, and Pflaum [19]. Fabozzi and Francis [6], for example, found that many beta coefficients move randomly through time rather than remaining stable as assumed by the O.L.S. model. The result of this nonstationarity is that the O.L.S. technique not only provides an inefficient estimate of systematic risk, but it also allows the nonstationarity of beta to be reflected in the estimate of unsystematic risk. Thus, the O.L.S.

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technique becomes an inappropriate methodological tool with which to decompose risk into its systematic and unsystematic components.

The purpose of this study is to examine the relationship between the process of diversification and the subsequent portfolio risk reduction while allowing for nonstationarity in betas. Specifically, we shall show that when nonstationarity of betas is allowed for, what is identified as unsystematic or residual risk by the O.L.S. technique can be further decomposed into pure residual risk, (that is, residual risk in which the effect of nonstationarity of beta has been extracted), and variability resulting from beta nonstationarity. Empirically, we then examine the relative sizes of these unsystematic risk components and their diversification characteristics. In addition to the obvious importance of these results to the interpretation of systematic risk, these results have further implications concerning the interpretation of O.L.S. estimates of residual returns in residual analysis.

The first section of this paper explores the decomposition of portfolio risk under the assumption of nonstationarity of beta into systematic risk, pure residual risk, and risk due to beta variability. The second section empirically examines the effect of diversification on this relationship, while the third section provides the implications of these results.

I. Beta Nonstationarity and Risk Decomposition

The capital asset pricing model (CAPM) derived by Sharpe [22], Lintner [13], and Mossin [18] is defined as

$$E(r_j) = r_f + \beta_j[E(r_m) - r_f] \tag{1}$$

where E = the mathematical expectation,

r_j = the return on risky asset j,

r_m = the return on the market portfolio,

r_f = the risk-free rate of interest.

Equation (1) describes the equilibrium relationship between the expected return on an asset and its systematic risk. That is, the expected return on asset j, $E(r_j)$, is equal to the risk-free rate of interest, r_f, plus a risk premium which is the product of the systematic risk for asset $j\,(\beta_j)$ and the market risk premium $[E(r_m) - r_f]$. This relationship requires the assumption that the beta coefficient (the measure of systematic risk) is stationary over time. However, if the beta coefficient is not stationary over time as indicated by Blume [3], Levy [12], Fabozzi and Francis [6], and others [1, 2, 19], the time-series equation of a random coefficient CAPM can be written in accordance with Theil and Mennes [24] or Theil [23] as

$$\begin{aligned} &\text{(a)}\quad R_{jt} = \beta_{jt}R_{mt} + \epsilon_{jt} \\ &\text{(b)}\quad \beta_{jt} = \beta_{jo} + u_{jt} \end{aligned} \tag{2}$$

where

$$R_{jt} = r_{jt} - r_f, \qquad R_{mt} = r_{mt} - r_f, \qquad E(\epsilon_{jt}) = 0,$$
$$E(\epsilon_{jt}^2) = \sigma_{\epsilon j}^2, \qquad E(\beta_{jt}) = \beta_{j0}, \qquad E(u_{jt}) = 0$$
$$E(u_{jt}^2) = \sigma_{\mu j}^2, \qquad E(u_{jt}\epsilon_{jt}) = 0, \qquad \text{for all } t,$$
$$E(u_{jt}u_{jt'}) = 0 \quad \text{and} \quad E(\epsilon_{jt}\epsilon_{jt'}) = 0 \qquad \text{for } t \neq t'$$

Then, Equation (2) can be rewritten as

$$R_{jt} = \beta_{jo}R_{mt} + \epsilon_{jt}^* \tag{3}$$

where

$$\epsilon_{jt}^* = (\beta_{jt} - \beta_{jo})R_{mt} + \epsilon_{jt} \tag{4}$$

Equation (3) represents a fixed coefficient model which has been widely used to decompose the total risk of a risky asset and to study portfolio diversification. Note that as the beta coefficient is not stationary over time, the residual error, ϵ_{jt}^*, in Equation (4) contains the effect of beta nonstationarity, $(\beta_{jt} - \beta_{jo})R_{mt}$. In other words, the residual error is equal to the pure residual error, ϵ_{jt}, plus the confounding effect of beta variability and the (excess) market rate of return, $(\beta_{jt} - \beta_{jo})R_{mt}$. As a result, the ordinary least-squares (O.L.S.) estimate of the residual risk, $\mathrm{Var}(\epsilon_{jt}^*)$, will be contaminated with the effect of beta nonstationarity. This results from the O.L.S. method's inability to isolate the beta nonstationarity effect from the residual risk. Thus, if the beta coefficient is nonstationary over time, the identification of the beta nonstationarity effect when decomposing the total risk of a portfolio (or asset) is essential to study portfolio diversification and related areas.

If a portfolio of n securities is broadly diversified using a cross-sectional sample of securities, the total risk (σ_p^2) of the portfolio return (R_{pt}) can be shown to be [using Equations (3) and (4)]:

$$\sigma_p^2 = \beta_{po}^2\sigma_m^2 + \sigma_{\epsilon p}^{*2} + 2\mathrm{Cov}(\beta_{po}R_{mt}, \textstyle\sum_{j=1}^n x_j\epsilon_{jt}^*), \tag{5}$$

where

$$\sigma_p^2 = \mathrm{Var}(R_{pt}),\ R_{pt} = \textstyle\sum_{j=1}^n x_jR_{jt} = \beta_{po}R_{mt} + \sum_{j=1}^n x_j\epsilon_{jt}^*,$$
$$\beta_{po} = \textstyle\sum_{j=1}^n x_j\beta_{jo},\ \sigma_m^2 = \mathrm{Var}(R_{mt}),\ \sigma_{\epsilon p}^{*2} = \mathrm{Var}(\sum_{j=1}^n x_j\epsilon_{jt}^*),$$

and x_j = the investment proportion in security j. The $\sigma_{\epsilon p}^{*2}$ in (5) can be decomposed into

$$\begin{aligned}\sigma_{\epsilon p}^{*2} &= \mathrm{Var}[\textstyle\sum_{j=1}^n x_j(\beta_{jt} - \beta_{jo})R_{mt} + \sum_{j=1}^n x_j\epsilon_{jt}]\\ &= \textstyle\sum_{j=1}^n x_j^2\sigma_{\epsilon j}^2 + \sum_{i=1}^n\sum_{j=1}^n x_ix_j\sigma_{ij}\end{aligned} \tag{6}$$

where

$$\sigma_{ij} = \mathrm{Cov}[(\beta_{it} - \beta_{io})R_{mt}, \quad (\beta_{jt} - \beta_{jo})R_{mt}] \tag{7}$$

And, the last term of (5) can be written as

$$2\mathrm{Cov}(\beta_{po}R_{mt}, \textstyle\sum_{j=1}^{n} x_j\epsilon_{jt}^{*}) = 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma'_{ij} \tag{8}$$

where

$$\sigma'_{ij} = \mathrm{Cov}[\beta_{io}R_{mt},\quad (\beta_{jt} - \beta_{jo})R_{mt}] \tag{9}$$

Substituting (6) and (8) into (5), the total risk of a portfolio of n securities is

$$\sigma_p^2 = \beta_{po}^2\sigma_m^2 + \sum_{j=1}^{n} x_j^2\sigma_{\epsilon j}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}^{*} \tag{10}$$

where

$$\sigma_{ij}^{*} = \sigma_{ij} + 2\sigma'_{ij} \quad \text{and} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}^{*}$$

represents the (total) effect of beta nonstationarity. Equation (10) indicates that if the beta coefficient is not stationary over time, the total portfolio risk can be decomposed into three components: the systematic risk ($\beta_{po}^2\sigma_m^2$), the pure unsystematic risk ($\sum_{j=1}^{n} x_j\sigma_{\epsilon j}^2$), and the risk due to beta nonstationarity ($\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}^{*}$).[1] The implication of this result is that the O.L.S. method of risk decomposition will overestimate the unsystematic risk because it identifies risk other than the systematic risk as the unsystematic risk. Specifically, the O.L.S. estimate of the unsystematic risk is the pure unsystematic risk plus the risk due to beta nonstationarity. Thus, the risk decomposition using the O.L.S. method is an inappropriate method with which to represent the real components of the total risk when the beta coefficient is nonstationary over time. In fact, the O.L.S. method of risk decomposition is appropriate to decompose the total risk only when the beta coefficient is constant over time, then $\beta_{jt} = \beta_{jo}$, and hence $\sigma_{ij}^{*} = \sigma_{ij} = \sigma_{ij} = 0$. Under this circumstance, Equation (10) reduces to

$$\sigma_p^2 = \beta_{po}^2\sigma_m^2 + \sum_{j=1}^{n} x_j^2\sigma_{\epsilon j}^2 \tag{11}$$

Equation (11) represents the traditional (or O.L.S.) risk decomposition in which total risk is equal to the systematic risk plus the unsystematic risk. This risk decomposition is valid only under the condition of beta stationarity.

In the section that follows, an empirical study is used to illustrate risk decomposition and portfolio diversification in a world with beta nonstationarity.

II. Risk Decomposition and Portfolio Diversification: An Empirical Study

The relationship between portfolio diversification and risk decomposition while allowing for time-varying beta coefficients was tested over the 95 month period

[1] The risk decomposition for a single risky asset can be obtained from (10) by letting $n = 1$:

$$\sigma_j^2 = \beta_{jo}^2\sigma_m^2 + \sigma_{\epsilon j}^2 + \sigma_{jj}^{*}$$

where

$$\sigma_{jj}^{*} = \sigma_{jj} + 2\sigma'_{jj},\ \sigma_{jj} = \mathrm{Var}[(\beta_{jt} - \beta_{jo})R_{mt}],$$

$$\sigma'_{jj} = \mathrm{Cov}[\beta_{jo}R_{mt}, (\beta_{jt} - \beta_{jo})R_{mt}], \text{ and } \sigma_j^2 = \mathrm{Var}\,(R_{jt}).$$

Table I

Risk Decomposition and Portfolio Diversification When Beta is Nonstationary

Portfolio Size	Total Risk (σ_p^2)	Systematic Risk ($\beta_p^2\sigma_m^2$)	Pure Residual Risk ($\sigma_{p\epsilon}^2$)	Risk due to Beta Nonstationarity (BN)	$\beta_p^2\sigma_m^2/\sigma_p^2$	$\sigma_{p\epsilon}^2/\sigma_p^2$	BN/σ_p^2
1	0.02693	0.01285	0.00514	0.00894	0.47710	0.19098	0.33191
2	0.02007	0.01309	0.00261	0.00437	0.65230	0.12994	0.21776
3	0.01869	0.01272	0.00303	0.00294	0.68055	0.16218	0.15727
4	0.01639	0.01285	0.00133	0.00220	0.78418	0.08138	0.13444
5	0.01611	0.01306	0.00110	0.00195	0.81061	0.06830	0.12109
6	0.01576	0.01289	0.00091	0.00195	0.81806	0.05799	0.12395
7	0.01496	0.01257	0.00084	0.00155	0.84022	0.05627	0.10351
8	0.01466	0.01245	0.00070	0.00152	0.84915	0.04744	0.10340
9	0.01445	0.01243	0.00067	0.00135	0.86030	0.04662	0.09308
10	0.01424	0.01249	0.00060	0.00114	0.87736	0.04248	0.08016
12	0.01428	0.01281	0.00059	0.00088	0.89711	0.04114	0.06175
14	0.01401	0.01280	0.00050	0.00071	0.91344	0.03556	0.05101
16	0.01379	0.01267	0.00041	0.00071	0.91866	0.02989	0.05145
18	0.01373	0.01266	0.00037	0.00070	0.92156	0.02723	0.05121
20	0.01365	0.01268	0.00032	0.00064	0.92929	0.02356	0.04714
25	0.01365	0.01290	0.00023	0.00052	0.94449	0.01718	0.03833
30	0.01348	0.01274	0.00021	0.00054	0.94473	0.01524	0.04003
40	0.01337	0.01273	0.00018	0.00046	0.95218	0.01366	0.03416
50	0.01324	0.01274	0.00014	0.00037	0.96214	0.01024	0.02761
100	0.01312	0.01277	0.00007	0.00029	0.97277	0.00520	0.02202

Note: Risk due to beta nonstationarity (BN) is estimated using the following equation:

$$BN = \hat{\sigma}_p^2 - \hat{\beta}_{po}^2\hat{\sigma}_m^2 - \sum_{j=1}^{n} x_j^2\hat{\sigma}_{\epsilon j}^2, \quad \text{where} \quad \hat{\sigma}_p^2 = \sum_{t=1}^{T} (R_{pt} - \bar{R}_p)^2/(T-1), \quad \bar{R}_p = \sum_{t=1}^{T} R_{pt}/T, \quad T = 95.$$

from February 1970 through December 1977. Over this period a sample of monthly prices on 811 firms adjusted for cash and stock dividends and stock splits was obtained from the Compustat PDE tape and used to calculate logarithmic price relatives.

Using this sample 80 random portfolios of varying size from one to 100 securities were formed and the variance of each portfolio was calculated and then segmented into systematic risk, pure residual risk, and risk due to beta nonstationarity.[2] Since the stocks were selected randomly with no consideration given to the stock's total market value, the CRISP tape index which is composed of all NYSE stocks equally weighted and adjusted for dividends was used as a proxy for the market. The results of this segmentation are given in Table I.

These findings show that for a single security variability due to beta nonstationarity accounts for approximately one-third of that security's total variability, while pure residual risks accounts for only 19 percent. However, after ten securities have been randomly chosen for a portfolio, the proportionate contribution to total risk of variability due to beta nonstationarity and pure residual risk has fallen to 8 and 4 percent, respectively, and for portfolios of size 30 their respective contributions drop again by one-half. These results indicate that the

[2] The time varying security beta, β_{jt}, is estimated using Chen and Lee's [4] approach.

two components of O.L.S. measured residual risk are randomly diversified away at approximately the same speed.

III. Risk Decomposition and Portfolio Diversification: Implications

The most obvious implication of these results involves the interpretation of unsystematic risk. Allowing for the nonstationarity of beta leads to a reinterpretation of unsystematic risk from returns uncorrelated with the market to returns resulting from structural shifts in the relationship between the market's and security's returns that can be randomly diversified away in addition to returns uncorrelated with the market. Moreover, although both these components of the O.L.S. measure of unsystematic risk are randomly diversified away at approximately the same speed, the initial contribution to the O.L.S. measured unsystematic risk of risk due to beta nonstationarity is approximately 74 percent greater than the contribution of pure residual risk.

An additional implication of these findings involves the use and interpretation of O.L.S. estimates of residual returns in examining time series measures of abnormal performance. While the use of the CAPM has come under considerable criticism recently [20, 21], the use of residual analysis to examine the effect of shocks on the firm's return generating process has avoided such criticisms [17, 20]. However, the results of the present study show that the results of such analysis can in fact be misinterpreted if O.L.S. estimates are used. Thus, while risk due to nonstationarity is actually a form of diversifiable or unpriced risk, its interpretation is critically different from that of pure residual risk.

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