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Portfolio Analysis with Factors and Scenarios

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ABSTRACT

Recently there has been a growing interest in the scenario model of covariance as an alternative to the one-factor or many-factor models. We show how the covariance matrix resulting from the scenario model can easily be made diagonal by adding new variables linearly related to the amounts invested; note the meanings of these new variables; note how portfolio variance divides itself into "within scenario" and "between scenario" variances; and extend the results to models in which scenarios and factors both appear where factor distributions and effects may or may not be scenario sensitive.

MODELING THE COVARIANCES OF intersecurity returns is one of the most important aspects of a portfolio analysis, especially for large numbers of securities. For a universe of 1,000 securities, say, the direct estimation of the roughly half a million covariances is not practicable, let alone the ensuing high cost of computation. The most widely used model today is the multifactor approach, where the source of covariation amongst returns is attributed to a few (say 10) common factors, e.g. [5, 6]. For 1,000 securities, only about 10,000 coefficients have to be estimated, a small fraction of what was originally required. Further, an equally dramatic reduction in the computational effort to compute efficient portfolios can be realized by exploiting the fact that the covariance matrices resulting from these models can be easily diagonalized, e.g. [1, 6].

An alternative approach is the scenario or states of the world model as described, for example, in [2]. This involves selecting, say, 10 future states of the world, assigning probabilities of occurrence and estimating the returns of the securities in each of them. The appeal of this approach is that it frequently seems an ideal vehicle for quantifying one's subjective views of the future. Its drawback, and one of the main reasons why it has not been used for portfolio analyses involving any substantial number of securities, is that with fewer states of the world than securities, the resulting covariance matrix is singular, i.e. riskless portfolios of risky securities can be formed.

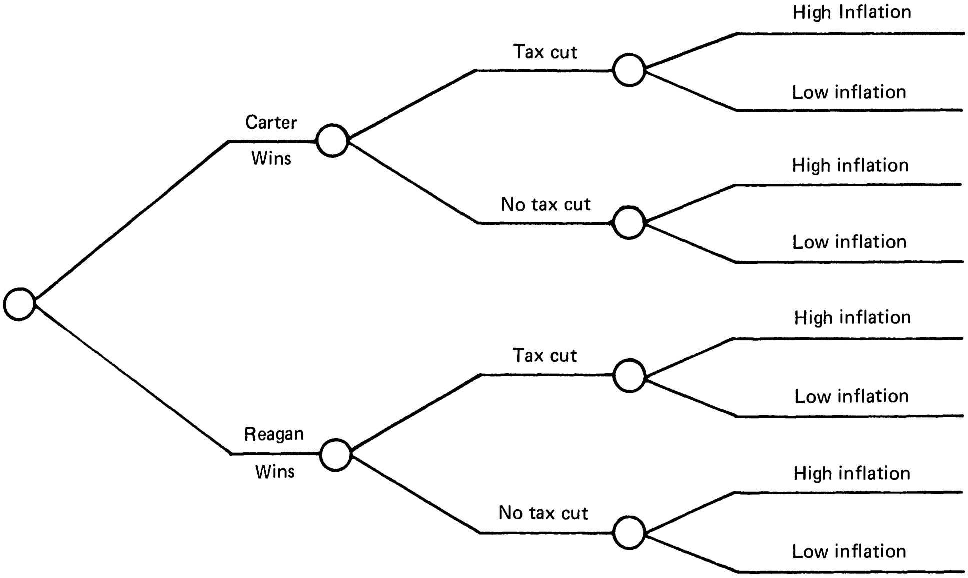
In this paper we show how this drawback can be overcome by explicitly modeling the uncertainty remaining once a scenario has been realized. We then show how the mathematical form of these models is precisely the same as in the multifactor case, and thus, upon diagonalization, able to yield the same savings in computation. The question of computation with this general class of models is pursued in detail in [4].

I. The Model

We begin by partitioning the event space into S mutually exclusive and exhaustive events, to be called scenarios,[1] the sth occurring with probability p_s. Typically,

[1] When a scenario is the resolution of all relevant future uncertainties, we shall call it a "state of the world."

the scenarios might be obtained from an event tree such as the following:



Here, there are $S = 8$ scenarios, and their probabilities are simply the products of the conditional probabilities of the appropriate branches of the tree.

Let there be N securities in the inverse and let r_i be the return on the ith security. Then, without loss of generality, and conditional on scenario s occurring, $s = 1, \cdots, S$, r_i may be written as

$$r_i = \mu_{is} + \epsilon_i \tag{1}$$

where μ_{is} is a constant and ϵ_i satisfies

$$E(\epsilon_i | \text{scenario} = s) = 0 \tag{2}$$

In the following let ν_i denote the expected value of r_i, σ_i^2 the variance of r_i, and σ_{ij} the covariance between r_i and r_j. Note that

$$\nu_i = \textstyle\sum_{s=1}^{S} p_s \mu_{is}. \tag{3}$$

The question now is how to model the conditional distributions of the ϵ's. We consider different assumptions in increasing order of generality.

A. The Usual States of the World Model

In this case (e.g. [2]) it is assumed that sufficiently many states have been chosen so that it is reasonable to regard the ϵ's as being identically zero. Since the resulting covariance between r_i and r_j is then

$$\sigma_{ij} = \textstyle\sum_{s=1}^{S} p_s(\mu_{is} - \nu_i)(\mu_{js} - \nu_j)$$

the covariance matrix of returns, C, has the form

$$C = GPG' \tag{4}$$

where G is an $N \times S$ matrix with (i, s)th entry given by

$$g_{is} = \mu_{is} - \nu_i \tag{5}$$

and P is an $S \times S$ diagonal matrix with sth diagonal entry p_s. While this will follow by specializing the result to be established next, note that C in (4) has exactly the same form as a covariance matrix formed from historical observations, where μ_{is} can be interpreted as the return on security i in period s and all states are equally likely, i.e. $p_s = 1/S$ for all s.

B. Modeling the ϵ's as being Conditionally Uncorrelated

The first improvement over the above model is to choose the scenarios in such a way that the ϵ's can be assumed to satisfy

$$E(\epsilon_i \epsilon_j | \text{scenario} = s) = 0 \qquad i \neq j \tag{6}$$

and

$$E(\epsilon_i^2 | \text{scenario} = s) = \sigma_{is}^2 \tag{7}$$

In other words, while there is some remaining uncertainty in the return of a security once the scenario becomes known, it is uncorrelated with the remaining uncertainty in the return of any other security. Here the covariance matrix has the form

$$C = D + GPG' \tag{8}$$

where G and P are as in Section A and D is an $N \times N$ diagonal matrix with ith diagonal entry

$$\textstyle\sum_{s=1}^{S} p_s \sigma_{is}^2$$

To see this, observe that

$$\sigma_{ij} = \textstyle\sum_{s=1}^{S} p_s E\{(r_i - \nu_i)(r_j - \nu_j) | \text{scenario} = s\} \tag{9}$$

From (1) and (5) it follows that, given scenario s,

$$r_i - \nu_i = \epsilon_i + g_{is}$$

Thus, given scenario s,

$$(r_i - \nu_i)(r_j - \nu_j) = g_{is} g_{js} + \epsilon_i \epsilon_j + \epsilon_i g_{js} + \epsilon_j g_{is}$$

By (2), (6), and (7), the right-hand side of this equation has conditional expectation

$$g_{is} g_{js} \qquad i \neq j$$

and

$$g_{is}^2 + \sigma_{is}^2 \qquad i = j$$

From this and (9), the form of (8) follows immediately. Clearly, when $\sigma_{is}^2 = 0$ for all i and s, (4) obtains.

This model is much more realistic than the States of the World Model in A, especially when there are only a few scenarios, and presents little additional difficulty as regards implementation. The only addition is that of σ_{is}^2, the uncertainty in one's best guess at the return of security i in scenario s.

C. Combining Scenarios and Factors

If the zero correlation assumption (6) is unreasonable, the next step might be to model the covariances of the ϵ's using common factors. In the following, covariances amongst factors will all be assumed to be scenario dependent; however, we shall distinguish between factors whose sensitivities (betas) are scenario independent and factors whose betas are scenario dependent. Let K be the number of factors of the first type and L the number of factors of the second type. The following results specialize to various simpler cases, e.g. when K or L equals zero or the factors are orthogonal.

In place of (6) and (7), and conditional on the occurrence of scenario s, the model now becomes

$$\epsilon_i = \sum_{k=1}^{K} \beta_{ik} F_k + \sum_{l=1}^{L} \gamma_{ils} G_l + \delta_i \tag{10}$$

where the factors F_k, G_l, and disturbances δ_i all have conditional expectation zero. Further the δ's are assumed conditionally uncorrelated with each other and the factors. Let

$$\begin{aligned} E(\delta_i^2 | \text{scenario} = s) &= \sigma_{is}^2 \\ E(F_k F_r | \text{scenario} = s) &= u_{krs} \\ E(G_l G_t | \text{scenario} = s) &= v_{lts} \end{aligned} \tag{11}$$

and

$$E(F_k G_l | \text{scenario } s) = w_{kls}$$

where $s = 1, \cdots, S$, $i = 1, \cdots, N$, $k, r = 1, \cdots, K$, and $l, t = 1, \cdots, L$.

A manipulation similar to that presented in Section B yields the following expressions for the variances and covariances of the returns:

$$\begin{aligned} \sigma_i^2 = \sum_{s=1}^{S} p_s \{ \sigma_{is}^2 + g_{is}^2 + \sum_{k=1}^{K} \sum_{r=1}^{K} \beta_{ik}\beta_{ir} u_{krs} \\ + \sum_{l=1}^{L} \sum_{t=1}^{L} \gamma_{ils}\gamma_{its} v_{lts} + 2 \sum_{k=1}^{K} \sum_{l=1}^{L} \beta_{ik}\gamma_{ils} w_{kls} \} \end{aligned} \tag{12}$$

$$\begin{aligned} \sigma_{ij} = \sum_{s=1}^{S} p_s \{ g_{is} g_{js} + \sum_{k=1}^{K} \sum_{r=1}^{K} \beta_{ik}\beta_{jr} u_{krs} \\ + \sum_{l=1}^{L} \sum_{t=1}^{L} \gamma_{ils}\gamma_{jts} v_{lts} \\ + \sum_{k=1}^{K} \sum_{l=1}^{L} (\beta_{ik}\gamma_{jls} + \beta_{jk}\gamma_{ils}) w_{kls} \} \end{aligned} \tag{13}$$

Interpreting (13), for example, we see that the covariance σ_{ij} is derived from the four terms in { }: respectively, these are covariation due to scenarios; covariation due to the F factors; covariation due to the G factors; and covariation due to interrelationships between the F's and G's.

To obtain the form of C in matrix notation as was done in Sections A and B, let B be the $N \times K$ matrix of coefficients (β_{ik}), and Γ_s the $N \times L$ matrix of coefficients (γ_{ils}), $s = 1, \cdots, S$. Let U, V_s, and W_s respectively be the $K \times K$, $L \times L$ and $K \times L$ matrices of coefficients $(\sum_{s=1}^{S} p_s u_{krs})$, (v_{lts}) and (w_{kls}), and D, G, and P be as before. Then, as can be confirmed by multiplying out HRH' and comparing the result with (13), C may be written as

$$C = D + HRH' \tag{14}$$

where H is the $N \times (S + K + SL)$ matrix

$$H = (G, B, \Gamma_1, \cdots, \Gamma_S)$$

and R is the $(S + K + SL) \times (S + K + SL)$ matrix.

$$R = \begin{bmatrix} \begin{matrix} p_1 & & \\ & \ddots & \\ & & p_S \end{matrix} & \bigcirc \\ \bigcirc & \begin{matrix} U & p_1 W_1 \cdots p_S W_S & \\ p_1 W_1' & p_1 V_1 & \bigcirc \\ \vdots & & \ddots \\ p_S W_S' & \bigcirc & p_S V_S \end{matrix} \end{bmatrix}$$

The need to consider separately the factors whose sensitivities are scenario independent should now be clear: each such factor contributes one to the dimension of R, whereas each factor whose sensitivities are scenario dependent contributes S. Some factors, such as "investor preference for small companies," fall naturally into the former category since there the sensitivity of the ith return is simply the size of the ith company. Others might have sensitivities which vary only slightly from one scenario to the next, so that scenario independence can be imposed with little loss.

The choice of a particular factor model to capture the covariation amongst the ϵ's will be largely dependent on the degree to which the scenarios themselves explain the covariation amongst returns. In some cases, a single factor such as the market (perhaps with scenario dependent betas) might be deemed sufficient, so limiting the additional estimation requirements to a market beta for each security in each scenario.

In concluding this secion, we remark that (14) is also the well-known form [5] of the conventional multifactor model in which R is the covariance matrix of the factors and H is the matrix of sensitivities.

II. Diagonalization

We have seen that the covariance matrices resulting from these models all have the form (14) where D is diagonal, H is $N \times M$, and R is $M \times M$. In the scenario model, $M = S$, the number of scenarios; in the multifactor model $M = L$, the number of factors; and in the combined model, $M = S + K + SL$ where K and L are respectively the number of factors with scenario independent and dependent sensitivities. In all cases we assume M to be small relative to N, at most perhaps

50 and often more like 10. This is a reasonable assumption in practice since the modeling effort simply becomes too great otherwise.

The variance of a portfolio of holdings $x = (x_1, \cdots, x_N)$ is given by the quadratic form

$$x'Cx$$

Using (14), this is equal to

$$x'Dx + x'HRH'x$$

which becomes

$$x'Dx + y'Ry \tag{15}$$

provided

$$Hx = y \tag{16}$$

Since (16) is linear in the "old" variables x and the "new" variables y, it may be introduced as M additional constraint equations in a general portfolio selection model [3]. The resulting quadratic form (15) is then diagonal in all except possibly the y variables.

The only off-diagonal entries (in R) are due to nonzero covariances amongst factors. In the combined scenario and factor case R can be left as is and treated as a sparse matrix, i.e. a record kept of nonzero entries only. In the case of a multifactor model where all or most factors are correlated, R can be written as the product

$$R = TT'$$

where T is a triangular matrix. Then (15) becomes fully diagonalized by substituting the identity matrix for R, and the product HT for H. In Rosenberg's multifactor model [4], where $M = L = 47$, this is not desirable since H is very sparse (i.e. it has a high percentage of zeros), and forming the product HT would make it 100% dense. In this case, however, R itself has the form (14), so that a second level of additional variables and constraints can be defined to achieve further diagonalization.

That these models can be diagonalized in the above manner is highly significant: the cost of computing efficient portfolios can be made to depend directly on the size of the input i.e. NM, rather than the a priori quantity N^2. See e.g. [4].

III. Conclusion

This paper has shown (1) how the scenario model can be extended to yield more meaningful estimates of covariance amongst security returns; and (2) how the well-known computational advantages of the multifactor model can also be realized for a scenario, or mixed scenario and factor, model. These developments should greatly enhance the practicability of the scenario approach for large scale portfolio analysis.

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