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Source: *The Journal of Finance*, Vol. 36, No. 4 (Sep., 1981), pp. 769-799
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327547
Accessed: 27/11/2009 08:32

THE JOURNAL OF FINANCE • VOL. XXXVI, NO. 4 • SEPTEMBER 1981

The Journal of FINANCE

VOL. XXXVI SEPTEMBER 1981 NO. 4

A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates

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ABSTRACT

The term structure of interest rates is an important subject to economists, and has a long history of traditions. This paper re-examines many of these traditional hypotheses while employing recent advances in the theory of valuation and contingent claims. We show how the Expectations Hypothesis and the Preferred Habitat Theory must be reformulated if they are to obtain in a continuous-time, rational-expectations equilibrium. We also modify the linear adaptive interest rate forecasting models, which are common to the macro-economic literature, so that they will be consistent in the same framework.

THIS PAPER EXAMINES SOME traditional theories and hypotheses about the term structure of interest rates and discusses the implications of modern portfolio theory and the valuation of contingent claims under diffusion processes in this area of research.

A general equilibrium model of bond pricing which is the basis of further discussion is outlined in Section I. Section II examines the Expectations Hypothesis and concludes that tradition assigns it to three (or four) mutually contradictory "equilibrium" mechanisms. In Section III we prove that only one of these is consistent with a general equilibrium in continuous time. Sections IV and VI discuss the role of risk-neutrality in the Expectations Hypothesis. We show there that the Expectations Hypothesis will not in general describe equilibrium among uniformly risk-neutral investors. Section V looks at Hicksian Liquidity Preference and the Preferred Habitat Theory of Modigliani and Sutch. We find, unlike Hicks, that term premiums may be either positive or negative. Furthermore, we give an interpretation of this result different from that of Modigliani and Sutch. We show that it is not preference for consumption at different times which creates "habitats," but rather the degree of risk aversion. Sections VII and VIII examine the popular discrete-time linear interest rate forecasting models and show how these may be solved in continuous time.

* Stanford University, University of Chicago, and Yale University respectively. We are grateful for the helpful comments and suggestions of Merton Miller and many of our other colleagues, too numerous to mention. This research was supported in part by the Dean Witter Foundation, the Center for Research in Security Prices and the National Science Foundation.

I. Bond Pricing with Diffusion State Processes

In this section we develop a relation linking the expected rates of return on default-free pure discount bonds (single payment debt) of all maturities. Since any default-free coupon-bearing bond may be considered to be a portfolio of pure discount bonds, the derived relation holds for the former as well. We perform this analysis in a continuous-time framework since this would appear to be the only unambiguous choice for the "shortest time interval." Indeed, we shall see that certain characterizations regarding the term structure of interest rates can obtain only on an instantaneous basis.

The following assumptions are taken to characterize the economy:

A1. There is a set of N state variables $\{Y_n\}$ whose current values completely specify all relevant information for investors.

In general the state variables may include both exogenous information like the distribution of returns currently available on physical investment and endogenous information like investors' wealths. The assumption that only current state variable values are relevant does not imply that investors ignore useful historical information. For example, the current value of one or more state variables may be averages of past interest rates if such information is useful in forecasting the future. Since bonds are usually denominated in nominal terms, it may be assumed that the analysis here is carried out using nominal prices and returns. In this case the state variables would include the price(s) of the consumption good(s) or, if relevant, a price index. However, this assumption is only one of convenience, since the results obtained must describe real prices and returns as well for, possibly, a different set of state variables.

A2. The state variables are jointly Markov[1] with movements determined by the system of stochastic differential equations.

$$dY_n(t) = \mu_n(Y, t)\, dt + g_n'(Y, t)\, dz(t) \tag{1}$$

μ_n is the expected change per unit time for the n^{th} state variable. $z(t)$ is a K-dimensional standardized Wiener process.[2] g_n is a K-dimensional vector measuring the response of the n^{th} state variable to each of the $K \leqslant N$ sources of uncertainty in the economy. $g_n' g_m$ is the covariance of changes in the n^{th} and m^{th} state variables. We assume that the variance-covariance matrix $\{g_n' g_m\}$ is positive semidefinite and of rank K (positive definite if $K = N$).[3]

For full generality we admit the possibility that some $(N - K)$ state variables, or linear combinations thereof, may be nonstochastic. This would arise, for example,

[1] The joint Markov assumption is not merely one of convenience, but is rather a logical necessity. If the state variables specify *all* relevant information, then a fortiori they must jointly describe their own future probability distributions.

[2] A K-dimensional standardized Wiener process is a set of K independent normal random variables, each with mean increment zero and unitary variance per unit time increment. See Arnold [1] for a full treatment.

[3] The matrix $\{g_n' g_m\}$ may have rank of less than K at isolated locations in (Y, t). This would occur, for example, at a natural reflecting or absorbing barrier for one of the state variables.

if one variable were a moving average of another. We examine systems of this type in later sections.

A3. Investors are nonsatiated, strictly preferring more wealth to less. Investors are sufficiently risk tolerant that they are willing to hold the risky assets at finite expected rates of return.

This is the only assumption required about investors' preferences. We do not require that they be risk-averse or even that they choose among alternatives on the basis of expected utility maximization. The first part of A3 is sufficient to assure that all assets are priced so that none are dominated (i.e., sell at the same or greater price and offer the same or lesser return under all circumstances as some portfolio made up of other assets). The second part of this assumption is a technical one only and assures that the market for risky assets can reach some equilibrium.

We also assume that the markets for assets are perfect and frictionless.

A4. All investors believe the economy to be as described in (A1) *and* (A2), *and have homogeneous assessments of the parameters.*

A5. All markets are competitive, and each investor acts like a price taker.

A6. There is a market for riskless instantaneous borrowing and lending at the endogenously determined equilibrium interest rate r and markets for longer term borrowing and lending at endogenously set prices.

A7. There are no taxes, transactions costs, or other institutional sources of market friction. Markets are open continuously, and trading takes place only at equilibrium prices.

Let $P(Y, t, T)$ denote the current (time t) price of a pure discount bond promising to pay one dollar at time T. The instantaneously realized percentage price change of this bond may be decomposed into the anticipated equilibrium rate of return and the unexpected variation in return due to the random changes in the underlying state variables.

$$dP(Y, t, T)/P(Y, t, T) = \alpha(Y, t, T)\, dt + \delta'(Y, t, T)\, dz(t) \tag{2}$$

α and $\delta'\delta$ are the expected rate of return and variance of return, respectively.

Using Ito's lemma,[4] the following relations can be established

$$\begin{aligned} \alpha(Y, t, T)P(Y, t, T) &= \textstyle\sum_{n=1}^{N} P_{Y_n}\mu_n + \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} P_{Y_n Y_m} g'_n g_m + P_t \\ &\equiv L[P(Y, t, T)] + P_t \end{aligned} \tag{3a}$$

$$\delta'(Y, t, T)P(Y, t, T) = \textstyle\sum_{n=1}^{N} P_{Y_n} g'_n \tag{3b}$$

where subscripts on P denote partial differentiation. $L[\cdot]$ is the Dynkin operator or differential generator over the state variables. The bond price's response to the k^{th} source of uncertainty is, from (3b), $\delta_k = \sum P_{Y_n} g_{nk}/P$.

If the expected rate of return on bonds were a known function of the state

[4] See Arnold [1] for a statement and formal proof of Ito's lemma. See Merton [32] for a heuristic development.

variables, time, and maturity, then (3a) together with the no-default terminal condition $P(Y, T, T) = 1$ (and other boundary or regularity conditions as required) would form a well-posed problem whose solution completely described the market for default-free bonds, both discount and coupon. For example, if it were known that the expected rates of return on all bonds were equal to the current instantaneous rate of interest, i.e., one characterization of the Expectations Hypothesis held, then expected rates of return would be empirically observable as $\alpha(Y, t, T) = r(Y, t)$. However, if other propositions about expectations, liquidity preferences, or "habitat" behavior were valid, then the expected rate of return might vary with maturity and would require further specification.

As we show below, even when the expected rate of return does vary with maturity, strong restrictions can be placed on the functional form of this dependence. In this regard, consider the potential portfolio strategy[5] consisting of V dollars invested in $K + 1$ discount bonds with distinct (positive) maturities with proportional investment x_i in bond $P(Y, t, T_i)$. The realized rate of return on this portfolio is

$$\frac{dV}{V} = \sum_{i=1}^{K+1} x_i \frac{dP(Y, t, T_i)}{P(Y, t, T_i)} = \sum_{i=1}^{K+1} x_i \alpha(Y, t, T_i)\, dt + \sum_{i=1}^{K+1} x_i \delta'(Y, t, T_i)\, dz(t) \tag{4}$$

If the portfolio weights are selected so that each source of uncertainty vanishes, i.e.,

$$\sum_{i=1}^{K+1} x_i \delta_k(Y, t, T_i) = 0, \text{ all } k \tag{5}$$

then the portfolio will be (locally) riskless. In equilibrium, its expected rate of return must then be the currently prevailing spot rate. Thus, $\sum x_i \alpha(Y, t, T_i) = r(Y, t)$, or

$$\sum_{i=1}^{K+1} x_i[\alpha(Y, t, T_i) - r(Y, t)] = 0 \tag{6}$$

since $\sum x_i = 1$.

Combining (5) and (6), we derive the equilibrium condition

$$\begin{pmatrix} \alpha(Y, t, T_1) - r, \cdots, \alpha(Y, t, T_{K+1}) - r \\ \{\delta(Y, t, T_1)\}, \cdots, \{\delta(Y, t, T_{k+1})\} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_{K+1} \end{pmatrix} = (0) \tag{7}$$

Since a nonzero linear combination of the columns of this $K + 1$ square matrix equals zero, the matrix must be singular. It follows then that the rows are linearly dependent. Furthermore, since the choice of maturities of the bonds was arbitrary, the linear relation among the rows cannot depend upon maturity. Thus, for some vector function $\hat{\lambda}'(Y, t)$ independent of maturity T,

$$\begin{aligned} \alpha(Y, t, T) &= r(Y, t) + \hat{\lambda}'(Y, t)\delta(Y, t, T) = r + \sum_{k=1}^{K} \hat{\lambda}_k \delta_k \\ &= r(Y, t) + \sum_{n=1}^{N} \lambda_n P_{Y_n}(Y, t, T)/P(Y, t, T) \end{aligned} \tag{8}$$

where $\lambda_n = \sum_{k=1}^{K} \hat{\lambda}_k g_{nk}$ is also independent of maturity.

[5] We confine our attention to portfolios which satisfy certain technical restrictions on rebalancing strategies. See Harrison and Kreps [21] or Dothan and Williams [14] for a detailed discussion of these restrictions and their interpretation.

Since nowhere in the derivation of (8) was it required that the assets forming the portfolio be discount bonds, this equilibrium condition applies to default-free coupon-bearing bonds as well. Indeed, Equation (8) describes the expected rates of return on all assets whose values depend only on these N state variables and, in this sense, is equivalent to the valuation equation exposited by Garman [20]. The derivation here, it is hoped, gives a stronger understanding of the economics underlying the equilibrium in outlining the course of action an investor could take by forming a portfolio to realize excess returns if the equilibrium did not hold.

While the derivation above is complete and requires no additional assumptions beyond those stated, the valuation equation is as it stands virtually an empty mathematical shell. A determination of what the relevant state variables are and the form of the factor risk premiums requires an embedding equilibrium. In work on contingent claims analysis, such as option pricing, it is common, and to a first approximation reasonable, to insist only on a partial equilibrium between the prices of the primary and derivative assets. For something as fundamental as the rate of interest, however, a general equilibrium model is to be preferred. Cox, Ingersoll, and Ross [9] provide one such model and in a subsequent paper [10] discuss some problems which can arise when equilibrium modeling is ignored.

The equilibrium condition in (8) may be given the following characterization. The term premium, i.e., excess expected rate of return,[6] on any bond is given by a linear model with K factors, one for each relevant source of uncertainty. Each component in the premium is the product of a measure of the quantity of type-k risk the bond exhibits, δ_k, and the equilibrium compensation for bearing one unit of this risk, $\hat{\lambda}_k$. By choosing K bonds, forming a basis which spans the set of investment opportunities, $\hat{\lambda}$ can be elimintated from the returns relation and (8) can be written as a multifactor CAPM-type model. Define the vector of expected returns on the basis bonds to be $\alpha_B' \equiv (\alpha(Y, t, T_1), \cdots, \alpha(Y, t, T_K))$ and the matrix of risk factors $\Delta' \equiv (\delta(Y, t, T_1), \cdots, \delta(Y, t, T_K))$. Then, solving for $\hat{\lambda} = \Delta^{-1}(\alpha_B - r\mathbf{1})$, we can write (8) for the nonbasis bonds as

$$\alpha(Y, t, T) = r(Y, t) + \delta'(Y, t, T)\Delta^{-1}(\alpha_B - r\mathbf{1}) \tag{9}$$

Returning to the bond valuation problem, we obtain the general bond pricing equation by substituting (8) into (3a)

$$\tfrac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} g_n' g_m P_{Y_n Y_m} + \sum_{n=1}^{N} P_{Y_n}(\mu_n - \lambda_n) - rP + P_t = 0 \tag{10}$$

A comparison of (3a) and (10) shows that the latter would be the valuation equation for an economy in which the expected rate of return on each bond was equal to the prevailing spot rate and investor's expectations of the changes in the state variables were altered to $\mu_n - \lambda_n$.[7] One important implication of this

[6] Throughout this paper, the phrase "term premium" will mean the excess of the expected rate of return over the spot rate of interest, and not the meaning sometimes given to it of the excess of the yield to maturity ofer the spot rate.

[7] This adjustment in both the state variables' drifts and the asset's discount rate is similar to the technique in option pricing of employing an equivalent "risk-neutral" economy. The interpretation here must be different. The equivalent economy is not risk neutral as we show in Section IV. In fact, even though (10) is mathematically identical to (3a), we have not shown that there is any equivalent economy for which the beliefs $\mu_n - \lambda_n$ exhibit rational expectations. In particular, interpretation

equivalency for empirical work is that any test for term premiums conducted by comparing average rates of return is equivalent to some test comparing *properly adjusted* forward rates with actual future rates.[8]

II. The Expectations Hypothesis: Alternate Forms[9]

Many theories have been proposed to explain the relation among the returns on bonds of various maturities. One of the earliest was the Expectations Hypothesis. This theory of the term structure cannot be attributed to any one individual. Rather, it has developed over much of this century. Certainly the proposition that investors' expectations about future spot rates affect the level of current long rates dates back (at least) to Irving Fisher [18]. However, most of the theory underlying the Expectations Hypothesis was not developed until the late 1930s by, notably, Hicks [22] and Lutz [27].

The traditional expectations theory has led these and other authors to set out a number of economic propositions about the relation between short and long rates. For example, Lutz [27, pp. 37, 49] states: "An owner of funds will go into the long market if he thinks the return he can make there over the time for which he has funds available will be above the return he can make in the short market over the same time, and vice versa," and "... a lender who wants to invest for only one year is in principle prepared to buy ... a bond of any other maturity and sell it again after the first year."

The broadest interpretation of these propositions would lead to the conclusion that equilibrium is characterized by an equality among expected holding period returns on all possible default-free bond investment strategies over all holding periods. That is, for any holding period t_0 to t_n, the return on any feasible series of investments must have the same expectation:

$$E\left[\frac{\tilde{P}(t_1, T_1)}{P(t_0, T_1)} \cdot \frac{\tilde{P}(t_2, T_2)}{\tilde{P}(t_1, T_2)} \cdot \ldots \cdot \frac{\tilde{P}(t_n, T_n)}{\tilde{P}(t_{n-1}, T_n)}\right] = \phi(t_n, t_0) \tag{11}$$

where we have suppressed the bonds' dependence on the state variables for convenience of notation. In (11), the expected return ϕ must be independent of the arbitrary reinvestment times t_i ($t_0 < t_1 < \cdots < t_n$) and the bonds selected, as denoted by their maturity dates T_i ($T_i \geqslant t_i$).

It is an easy matter to prove that Equation (11) cannot be generally valid since it demands too much. For example, (11) requires that the expected return on a bond maturing at t_2 over the period (t_0, t_1) must equal the certain return on a

problems may arise if one or more of the state variables has a specified boundary behavior (e.g., the nonnegativity of the nominal interest rate) which is satisfied with the beliefs μ_n but not with $\mu_n - \lambda_n$. There might also be a problem when a state variable dynamics were subject to some control (e.g., wealth) and the beliefs $\mu_n - \lambda_n$ could only be interpreted as suboptimal behavior.

[8] Even with no term premiums an adjustment is required to correct for the difference due to Jensen's inequality. This issue is discussed in the next section.

[9] Some of the results of the next two sections could differ for a discrete-time model. The appendix restates these results. Readers who are not familiar with continuous-time models may prefer to read the appendix before continuing.

bond maturing at t_1

$$\frac{1}{P(t_0, t_1)} = \frac{E[P(t_1, t_2)]}{P(t_0, t_2)} \tag{12}$$

Furthermore, the return expected over (t_0, t_2) on a bond maturing at t_1 rolled over at maturity into a bond maturing at t_2 must equal the guaranteed return on the bond maturing at t_2,

$$\frac{1}{P(t_0, t_2)} = \frac{1}{P(t_0, t_1)} E\left[\frac{1}{P(t_1, t_2)}\right] \tag{13}$$

Together, (12) and (13) imply that

$$E\left[\frac{1}{P(t_1, t_2)}\right] = \frac{P(t_0, t_1)}{P(t_0, t_2)} = \frac{1}{E[P(t_1, t_2)]} \tag{14}$$

But by Jensen's inequality, (14) cannot be true except in the case of certainty. Thus, all expected returns for all holding periods can never be equal in equilibrium.

This contradiction can be avoided if it is postulated that expected holding period returns are equal only for one specific holding period. The natural choice of holding period is the next basic (i.e., "shortest") interval. This interpretation of the Expectations Hypothesis has only recently come into favor, particularly in the field of finance. Under this assumption, equilibrium is characterized by

$$\frac{E[dP(Y, t, T)]}{P(Y, t, T)} = r(t)\, dt \tag{15}$$

Since there are no term premiums in this formulation, many authors have been lured into referring to it as the "Risk-Neutral Expectations Hypothesis." In fact, (15) is not a consequence of universal risk-neutrality, as is demonstrated in Section IV. To avoid possible confusion, we shall refer to this equilibrium as the Local Expectations Hypothesis (L-EH) because (15) asserts equality of local expected rates of return.

Relying on the isomorphic relation between differential and integral equations, (15) can be rewritten as

$$P(Y, t, T) = E\left[\exp\left(-\int_t^T r(s)\, ds\right) \middle| Y(t)\right] \tag{16}$$

where the bond's current price is the expected discounted value of the promised unit payment.[10] Equation (16) states the common sense result that if there is no term premium, then the rate of discount to be used at each instant is the prevailing spot rate.

Traditionally, the equilibrium relation of the Expectations Hypothesis has been stated in a different fashion. It is often assumed, as implied by Lutz's first quote above, that the guaranteed return from holding any discount bond to maturity is

[10] For a proof of the equivalency of (15) and (16) see Cox, Ingersoll, and Ross [9] Lemma 3 or 4.

equal to the return expected from rolling over a series of single-period bonds; i.e.,[11]

$$\frac{1}{P(Y, t, T)} = E\left[\exp\left(\int_t^T r(s)\, ds\right) \middle| Y(t)\right] \tag{17}$$

Some authors make a similar assumption about equality of expected yields. For example, Malkiel [28, p. 20] states that "... all differentials in anticipated holding period yields [are] completely eliminated ..."; i.e.,

$$\frac{-1}{T-t}\ln[P(Y, t, T)] = E\left[\frac{1}{T-t}\int_t^T r(s)\, ds \,|\, Y(t)\right] \tag{18}$$

To distinguish these two equilibrium conditions from (15), we shall refer to (17) and (18) as the Return-to-Maturity Expectations Hypothesis (RTM-EH) and the Yield-to-Maturity Expectations Hypothesis (YTM-EH), respectively.

In other cases it is assumed that forward rates are unbiased estimates of future interest rates. In the words of Malkiel [p. 23]: "Forward rates and expected spot rates are driven to equality." i.e.,

$$\frac{-\partial P(Y, t, T)/\partial T}{P(Y, t, T)} = E[r(T) \,|\, Y(t)] \tag{19}$$

For this final conjecture we employ the traditional name, the Unbiased Expectations Hypothesis (U-EH). Integrating (19) and applying the condition $P(Y, T, T) = 1$ gives

$$-\ln[P(Y, t, T)] = \int_t^T E[r(s) \,|\, Y(t)]\, ds \tag{20}$$

Therefore, in a continuous-time framework these last two conditions, (18 and 19) are tautological,[12] and we need not consider the Unbiased Expectations Hypothesis separately.

Many researchers are not precise in distinguishing among (16), (17), and (18), perhaps not realizing that these propositions are (pairwise) incompatible. If we define the random variable $\tilde{X} \equiv \exp(-\int_t^T r(s)ds)$, then (16) through (18) can be rewritten as

$$P = E[\tilde{X}] \tag{16'}$$

$$P^{-1} = E[\tilde{X}^{-1}] \tag{17'}$$

$$\ln(P) = E[\ln \tilde{X}] \tag{18'}$$

[11] For some stochastic processes, the expectation in (17) may not be bounded. This is true even if r has a well-defined (finite variance) steady-state distribution. For example, if the spot rate is governed by the Markov stochastic process utilized by Cox, Ingersoll, and Ross [10], $dr = K(\mu - r)\,dt + \sigma\sqrt{r}\,dz$, then, for $2\sigma^2 > K^2$, $E[\exp(\int_0^T r(t)\,dt)] = ([K\sin(\eta T) + 2\eta\cos(\eta T)]/2\eta)^{-2K\mu/\sigma^2}\exp(K^2\mu T/\sigma^2 + [\eta\cot(\eta T) + K/2]^{-1}r)$ where $\eta \equiv \sqrt{2\sigma^2 - K^2}$. This expectation exists only for $T < (2/\eta)\cot^{-1}(-K/\eta)$ (where the inverse function is interpreted to give a positive angle in the interval $(\pi/2, \pi)$). At this point the two terms in brackets become zero and the expectation is infinite. In cases such as these, (17) obviously cannot serve as a useful equilibrium condition.

[12] In a discrete-time model the equivalent relations are not tautological and, in general, are even inconsistent. See the appendix for details.

By invoking Jensen's inequality, it is clear that if Equation (18) describes equilibrium, then the yield on $T - t$ period bond will be greater than the value given in (16) and less than the value given in (17). On the other hand, if equilibrium condition (16) (condition (17)) is valid, then Equations (17) and (18) (Equations (16) and (18)) give long yields which are too large (small).

In summary, the Expectations Hypothesis has (at least) three distinct forms. Only the local version (16) is characterized by the absence of term premiums in the expected instantaneous rates of return on long bonds.[13] In the other three versions, term premiums are positive.

III. The Expectations Hypotheses and Equilibrium

In the previous section we saw that the Expectations Hypothesis is in fact a set of three distinct propositions. In this section, we examine each version using the equilibrium relation (8) to determine if it demonstrates rational expectations. We prove that only the Local Expectations Hypothesis, expected instantaneous rates of return on all bonds are equal, can be sustained in equilibrium (except under certain interest rates when all three propositions are identical). The other versions are incompatible not only with whatever economy supports the L-EH, but with any continuous-time rational expectations equilibrium whatsoever. In the next section, we show sufficient conditions to support the Local Expectations Hypothesis.

The equilibrium condition (8) is stated in terms of risk (term) premiums. So, to establish the possibility of sustaining equilibria for these expectations theories, the premiums must be extracted from the appropriate partial differential pricing equations. From (15) it is clear that the Local Expectations Hypothesis obtains in an economy if there are no premiums, $\lambda' = O'$. As we saw in the previous section, term premiums under the other traditional forms of the Expectations Hypothesis will be positive. In our continuous-time model we can derive the exact form of these premiums.

Applying Ito's lemma and using the price dynamics in (2), the reciprocal of the bond price evolves according to

$$d(1/P) = (-\alpha + \delta'\delta)(1/P)\, dt - (1/P)\delta'\, dz \tag{21}$$

But under the RTM Expectations Hypothesis the expected change in the reciprocal of the bond price is, from (17), $E[d(1/P)] = -r(1/P)\, dt$; hence

$$\alpha(Y, t, T) = r(Y, t) + \delta'(Y, t, T)\delta(Y, t, T) \tag{22}$$

and the premium of each bond is equal to its variance of returns.

Applying Ito's lemma again, the log of the bond price evolves according to

$$d(\ln P) = (\alpha - \tfrac{1}{2}\delta'\delta)\, dt + \delta'\, dz \tag{23}$$

Under the Unbiased Expectations Hypothesis or the YTM-EH, the expected

[13] The unbiased expectations hypothesis is characterized by an absence of "premiums" in the forward rates and yields-to-maturity. The traditional version has no premiums in holding period returns.

change in the log of the bond price is, from (18) or (20), $E[d(\ln P)] = rdt$; thus

$$\alpha(Y, t, T) = r(Y, t) + \tfrac{1}{2}\delta'(Y, t, T)\delta(Y, t, T) \tag{24}$$

To show that neither of these two hypotheses can be sustained in equilibrium, we compare the expected rates of return under these two hypotheses, as given in (22) and (24), with the general expression for the equilibrium expected rate of return in (8). For expression (22) or (24) to yield equilibrium results, it is necessary that

$$\hat{\lambda}'(Y, t)\delta(Y, t, T) = a\delta'(Y, t, T)\delta(Y, t, T) \tag{25}$$

where $a = 1$ for the RTM Expectations Hypothesis and $a = \tfrac{1}{2}$ for the YTM- (or U-) Expectations Hypothesis.

Now it is certainly possible to construct an example for which (25) holds for a particular state at a given point of time, but we require that this relation be true in *all* states at *all* times. The state variables Y are driven by a set of K independent sources of uncertainty (the Wiener processes). However, to impose (25) at both times t and $t + dt$ requires that there be a functional relation among the K realizations of the Wiener processes. Since by definition they are independent, this is impossible.

To clarify this reasoning, consider the case when there is exactly one relevant source of uncertainty (with one or more state variables); then $\delta(T)$ is a scalar function and from (25), $\delta(T) = \hat{\lambda}/a$. Since $\hat{\lambda}$ and a do not depend upon the maturity of the bond in question, δ must be independent of maturity as well. But maturing bonds are riskless over the last instant of their life, $\delta(Y, t, t) = 0$, so if δ is to be constant, it must be identically zero. In this case, bond prices are nonstochastic, violating the assumption that there was one relevant source of uncertainty. Consequently, if one of the traditional forms of the Expectations Hypothesis held (and bonds were not riskless), excess (nonequilibrium) profits would be possible. To demonstrate, consider two discount bonds, with $\delta(T_1)\delta(T_2) > 0$,[14] combined in a portfolio with weights $x_1 = \delta_2/(\delta_2 - \delta_1)$ and $x_2 = 1 - x_1$. By construction, this portfolio will be riskless and have an expected and therefore guaranteed rate of return of

$$\alpha_P = r + a(x_1\delta_1^2 + x_2\delta_2^2) = r + a\delta_1\delta_2 > r \tag{26}$$

which exceeds the fair (riskless) return.

We have proved that the RTM, YTM, and Unbiased Expectations Hypotheses are not sustainable in a continuous-time rational expectations equilibrium. In the next section we consider the Local Expectations Hypothesis. We show that while it can obtain as an equilibrium condition, it does not do so in the generally cited case of universal risk-neutrality unless interest rates are nonstochastic or other special circumstances obtain.

IV. Risk-Neutrality and the Local Expectations Hypothesis

The development of the expectations theory has relied heavily on earlier models of interest rate behavior under certainty. It is well known that all forms of the

[14] This condition can be imposed with no loss of generality. For any three risky bonds with different values of $\delta(T)$ at least two of them must have $\delta(T)$ of the same sign.

Expectations Hypothesis are equivalent under certainty, and the relations given in the previous sections all must hold identically to ensure that arbitrage possibilities do not exist.

As all students of economics learn, certainty models can, in many cases, be adapted to uncertainty by the judicious use of expectations of the uncertain quantities if investors are risk-neutral. Unfortunately, uncertainty concerning interest rates is not one of these cases. In this section we examine the connection between the Local Expectations Hypothesis and risk neutrality. (Given the analysis of the previous section, only this form of the Expectations Hypothesis need be considered.) We find, despite the assumptions and claims of many,[15] that the Expectations Hypothesis is not a natural consequence of universal risk neutrality, except in the uninteresting case when interest rates are nonstochastic.

To illustrate this point, we consider two distinct market economies. The first is a pure exchange economy of the type studied by Lucas [26] and the second is a production-exchange economy of the Cox, Ingersoll, Ross [9] type. As we shall see, there are some substantive differences between the two types. In the three subsections below we show that: (a) in a risk-neutral pure exchange economy interest rates must be nonstochastic. All forms of the Expectations Hypothesis are therefore sustained but only in the singular sense of a certainty model; (b) in a risk-neutral production-exchange economy interest rates may be either certain or stochastic. In the first case, the Expectation Hypotheses hold (again in the singular sense); in the latter case the Local Expectations Hypothesis does not generally obtain; and (c) there are for each type of economy nontrivial conditions under which the Local Expectations Hypothesis does obtain for risk-averse agents.

A. Risk Neutrality in Pure Exchange Economies

Following Lucas, we assume that the quantity of the consumption good available each period is determined exogenously and if not consumed perishes. Suppose initially that the risk-neutral investors have homogeneous expectations, equal endowments, and the same time preference. For an economy of identical investors, prices will be set as if markets were complete, regardless of their actual scope. Thus the equilibrium allocation is characterized by

$$U_1(C_0, t_0) = \pi(s, t; s_0, t_0)U_1[C(s, t), t]/p(s, t; s_0, t_0) \tag{27}$$

where $\pi(s, t; s_0, t_0)$ is the probability as assessed at t_0 in state s_0 that state s will occur at time t, $p(s, t; s_0, t_0)$ is the price prevailing in state s_0 at time t_0 for one unit of the consumption good numeraire at time t in state s and $U_1(C, t)$ denotes the partial derivative $\partial U/\partial C$.

Using (27) and risk-neutrality, $U(C, t) = f(t)C$, we can write the price of a unit

[15] See, for example, Meiselman [29] and Richard [36]. We consider only single-good, time-additive functions, for which risk neutrality is defined as utility of the form $\sum f(t)C_t$. It could be argued, perhaps, that this description is incomplete and that other utility functions, such as $\prod f(t)C_t$, are also "risk-neutral." Certainly the time-additive form given would fit in virtually any definition of risk neutrality, and, as we are looking for its "natural consequences," it suffices to consider only this generally accepted case. Richard [36, footnote 13] explicitly considers this form, claiming it leads to the Local Expectations Hypothesis.

discount bond maturing at t as

$$P(s_0, t_0, t) = \int_S p(s, t; s_0, t_0)\, ds = \frac{f(t)}{f(t_0)} \int_S \pi(s, t; s_0, t_0)\, ds = \frac{f(t)}{f(t_0)} \tag{28}$$

So bond prices are state independent and hence nonstochastic. If investors display a constant rate of time preference $f(t) = e^{-\rho t}$, then the yield curve is flat as well with a constant interest rate of ρ. In this world the Expectations Hypothesis characterizes the bond market, but it does so simply by arbitrage because there is no interest rate uncertainty.[16, 17]

If investors' initial endowments differ, these results are unchanged because the competitive equilibrium in an economy of risk-neutral investors is independent of the original allocation of wealth if they are otherwise identical. On the other hand, if the investors differ in beliefs *or* time preference, then no unconstrained equilibrium is possible. An investor would willingly supply (purchase) an infinite quantity of any Arrow-Debreu security whose price exceeded (was less than) his utility-discounted personally assessed probability for that state [i.e., $p(s, t; s_0, t_0) > (<) \pi_i(s, t; s_0, t_0) f_i(t)/f_i(t_0)$]. In such a situation it is meaningless to examine the proposed equilibrium relation.

B. *Risk Neutrality in Production-Exchange Economies*

We now extend the model to include production by assuming that any of the good unconsumed can be invested to provide for future consumption. If the investment opportunities are stationary or change deterministically over time, then results similar to those in the pure exchange model will obtain. Consequently, we shall assume that the investment opportunities evolve stochastically over time as governed by the state variables Y. As shown by Cox, Ingersoll, and Ross [9], the (representative) investor's consumption-investment decision is characterized by the derived utility of wealth function $J(W, Y, t)$ which is the solution to the partial differential equation

$$U(C, t) + J_t + [\alpha_W(Y, W, t)W - C]J_W + \tfrac{1}{2}\sigma_W^2(Y, W, t)W^2 J_{WW} + \textstyle\sum_{n=1}^N \sigma_{nW}(Y, W, t) W J_{WY_n} + L[J] = 0 \tag{29}$$

[16] Leroy [25] has independently derived the same result. However, he considers only exchange economies. As we demonstrate below, in a risk-neutral production economy interest rates may be stochastic. But in this case the expectations hypothesis generally does not obtain. Neither is it true that the expectations hypothesis is approximately correct for, in Leroy's terminology, "near risk-neutrality."

[17] This result is not unique to risk-neutrality. Under certain assumptions, other preferences can also result in nonstochastic interest rates (and, a fortiori, the expectations hypothesis), even in a risky economy. For example, for identical investors with state-dependent power utility, equilibrium bond prices will be

$$P(s_0, t_0, t) = \frac{f(t)}{f(t_0)} \int_S \pi(s, t; s_0, t_0) \left(\frac{C(s)}{C(s_0)}\right)^{\gamma-1} ds$$

If percentage changes in aggregate consumption are assumed to be independently distributed, then the distribution of $C(s)/C(s_0)$ is independent of the current state, and bond prices and interest rates are again nonstochastic.

where α_W and σ_W^2 are the state dependent expected return and variance of return on optimally invested wealth (the market portfolio), σ_{nW} is the covariance of change in the n^{th} state variable with the return on wealth, and $L[\cdot]$ is the differential generator defined in (3).[18] The equilibrium interest rate in this economy is

$$r(W, Y, t) = \alpha_W - \frac{-WJ_{WW}}{J_W}\sigma_W^2 - \sum_{n=1}^{N}\frac{-J_{WY_n}}{J_W}\sigma_{nW} \tag{30}$$

To solve (29), the first order condition for a maximum, $U_C = J_W$, is generally imposed; however, for risk-neutral investors, $U(C, t) = f(t)C$, utility is not concave and an interior solution is not guaranteed. Under risk neutrality, investors must solve a singular control problem. There are three possibilities to consider: (1) investors are not currently consuming and $U_C = f(t) < J_W$; (2) investors are consuming at an infinite rate, completely exhausting their wealth and $f(t) > J_W$; and (3) investors are consuming at some finite rate with $J_W = f(t)$. Let us assume for the moment that conditions are such that this third case is always valid. Then integrating once gives

$$J(W, Y, t) = f(t)W + g(Y, t) \tag{31}$$

Substituting (31) into (29)

$$f(t)C + f'(t)W + g_t + (\alpha_W W - C)f(t) + L[J] = 0 \tag{32}$$

The third and fifth terms do not depend upon wealth; therefore, we require

$$\alpha_W(Y, W, t) = -f'(t)/f(t) \tag{33}$$

The time preference factor $f(t)$ is exogenous and state-independent, so Equation (33) can only be satisfied if α is deterministic. This will be true only if it does not depend upon the state variables or if one or more of the state variables is subject to sufficient control by investors such that the equality in (33) can be maintained. For example, if there are decreasing returns to scale in physical investment, wealth will then be one state variable and $\partial\alpha/\partial W < 0$. Then if α_W is less than $-f'(t)/f(t)$ at the current level of wealth, investors could consume more until (33) was satisfied.

If, as we have been assuming, the economy is such that this equilibrium is always achieved, then the expected rate of return on wealth is nonstochastic. Furthermore, $J_{WW} = J_{WY} = 0$, so from (30) $r = \alpha_W$ and the interest rate is nonstochastic as well. As in the Lucas-type model above, the Expectations Hypothesis will obtain, but only because there is no interest rate uncertainty.

If α_W does not depend upon wealth but only exogenous state variables or the economy is otherwise such that the equilibrium characterized in Case (3) cannot be achieved, then the interest rate will be stochastic. Analyzing the consumption-investment problem in this case requires a determination of the conditions under which investors will switch from investing their entire wealth to consuming it.

[18] If wealth is one of the relevant state variables included in Y, then it should *not* be included in $L[\cdot]$ here or subsequently in this section, since the dynamics for wealth are explicitly given.

Solving this "optimal stopping time" problem would lead us far astray from our original purposes and little of a general nature would be learned since investor behavior would depend upon the assumed dynamics of the state variables. However, a related problem suffices to show that the Local Expectations Hypothesis will not be generally valid.

Suppose that the risk-neutral investors consume only at a single point of time in the future, τ, i.e.,

$$U(C, t) = 0 \quad t \neq \tau$$
$$U(C, \tau) = C \tag{34}$$

The problem to be solved is formally identical to that in (29) with C and $U(C, t)$ set to zero. At time τ all wealth will be consumed so the appropriate boundary condition is $J(W, Y, \tau) = W$. Solving (30) for α_W and substituting into (29) yields

$$J_t + rWJ_W - \tfrac{1}{2}\,\sigma_W^2(Y, t)W^2J_{WW} + L[J] = 0$$
$$J(W, Y, \tau) = W \tag{35}$$

As a trial solution to (35), take $J(W, Y, t) = WQ(Y, t)$, then

$$Q_t + rQ + L[Q] = 0$$
$$Q(Y, \tau) = 1 \tag{36}$$

A formal solution to (36) is[19]

$$Q(Y, t) = E_t\left[\exp \int_t^\tau r(Y, s)\, ds\right] \tag{37}$$

So at any point of time the marginal utility of wealth is equal to the expected terminal value of one dollar earning the prevailing spot rate of interest.

As we have shown elsewhere [9, Equation (20)], the expected rate of return on a default-free bond in this continuous-time production economy of investors with isoelastic utility is given by

$$\alpha(Y, t, T)$$
$$= r + \textstyle\sum_{n=1}^N (P_{Y_n}/P)\left[-\dfrac{WJ_{WW}}{J_W}\sigma_{Wn} + \sum_{m=1}^N -\dfrac{J_{WY_m}}{J_W}\operatorname{Cov}(Y_m, Y_n)\right] \tag{38}$$

For risk-neutral investors $J_{WW} = 0$; therefore, the covariance of the bond's return with that of the "market" portfolio, $\sum (P_{Y_n}/P)\sigma_{Wn}$, does not contribute to its risk premium. In some models, such as the CAPM in finance, a positive market covariance is the only cause for premium returns; however, such is not the case here. In (38) there also is a premium due to the final sum which measures the extra compensation, positive or negative, for each bond's potential to hedge against shifts in the investment opportunity set. Substituting for $J(\cdot)$ Equation

[19] See Friedman [19, Theorem 5.2]. For some stochastic processes, the integral in (36) may be undefined. See footnote 11.

(38) simplifies to

$$\begin{aligned}\alpha(Y, t, T) &= r - \sum_{n=1}^{N} \sum_{m=1}^{N} (Q_{Y_m}/Q)(P_{Y_n}/P)\mathrm{Cov}(Y_m, Y_n) \\ &= r - \mathrm{Cov}\left(\frac{dQ}{Q}, \frac{dP}{P}\right) \end{aligned} \tag{39}$$

From the last line in (39) and the definition of Q in (37), the premium on a bond can be expressed as a single term which is the negative of the covariance of the bond's return with the percentage change in the expected value of a dollar compounded up to the consumption date τ by the time path of future instantaneous riskless interest rates. Only if this covariance is uniformly zero does the Local Expectations Hypothesis hold. While it might be possible to construct examples for which these terms are uncorrelated, there is no obvious reason for this to be the case unless interest rates are deterministic.

In the particular case when the spot rate of interest is the only relevant state variable, the covariance is negative since $Q_r > 0$ and $P_r < 0$.[20] In this economy term premiums will be positive and an increasing function of maturity.

C. The Local Expectations Hypothesis without Risk Neutrality

Since the Expectations Hypothesis is not a consequence of risk-neutrality, it is natural to ask if there are any conditions for which it does obtain. The answer is yes. There are two distinct cases, outlined below, corresponding to each of the models examined here, and a third case for uncertain inflation discussed in Section VI.

In a single-good pure exchange economy when the dynamics of aggregate consumption are locally certain, a default-free bond will not command a premium if utility of consumption is state-independent. This result follows immediately from the observation by Breeden [4] that at each instant any asset's risk premium is proportional to its instantaneous covariance with aggregate consumption. Clearly all these covariances and, thus, the risk premiums on all assets must be zero if consumption is locally certain. A fortiori there are no term premiums. Nevertheless, in a model of this type, the spot interest rate and hence bond prices can be locally stochastic if investors are risk-averse and changes in the rate of growth of consumption are locally stochastic.

In a production-exchange economy, universal state-independent Bernoulli-logarithmic utility of consumption is the primary requirement for the Local Expectations Hypothesis. For log utility, it is known that the derived utility of wealth function is separable $J(W, Y, t) = f(t)\ln W + q(Y, t)$. Hence, log utility investors' portfolio decisions are completely myopic being made with no regard for hedging against changes in the state variables since $J_{WY} = 0$. From (38), then, an additional condition sufficient for the Local Expectations Hypothesis is that returns on physical capital be uncorrelated with changes in the state variables.

[20] For a Markov interest rate process with a continuous sample path an increase in the current spot rate holding the realizations $dz(s)$ fixed will increase the value of the random variable $\int_t^T \tilde{r}(s)ds$. Thus, $Q_r > 0$, from (37).

V. Liquidity Preference and Preferred Habitats

As mentioned previously, the Expectations Hypothesis would appear to be a straightforward application of certainty results to a world of uncertainty. This "naive" approach has incurred it a substantial amount of criticism. The earliest attack was made by Hicks [22] and has developed into what is usually known as the Hicksian Liquidity-Preference Model. Building on the Keynesian notion of "normal backwardation," it was generally argued that the forward spot rate will normally exceed the expected spot rate. Equivalently, the expected rate of return on a long bond must exceed that on a short bond by a premium which compensates the lender for assuming the increased risks of price fluctuation.

Hicks [22, pp. 138–46] made his argument in three parts. First, "to hedge their future supplies of loan capital ⋯ [borrowers] will have a strong propensity to borrow long." Second, because of "the desire to keep one's hands free to meet ⋯ uncertainty ⋯ most people (and institutions) would prefer to lend short." Finally, "speculators" will offset this "constitutional weakness" in the supply of long funds by borrowing short and lending long; however, they must receive a premium return as "compensation for the risk they are incurring."

Culbertson [11] in his market segmentation hypothesis argued that individuals have strong maturity preferences and that bonds of different maturities trade in distinct markets. In their Preferred Habitat Theory, Modigliani and Sutch [34, pp. 183–84] began with the same approach. They defined an investor to have an n-period habitat if "he has funds which he will not need for n periods and which, therefore, he intends to keep in bonds for n periods." However, recognizing the implied inefficiency, they did not go so far as to assume that this investor considers only n-period bonds. Investors can be "tempted out of their natural habitats by the lure of higher expected returns."

In terms of its implications, the Liquidity-Preference Model can be considered a special case of the Preferred Habitat Theory, namely that case in which all investors have a habitat equal to the shortest holding period. This permits us to restrict our attention to the latter theory. Under what conditions, then, will investors' habitats affect the size or, more to the point, the sign of bond premiums?

To simplify the analysis, we shall assume that each investor has a strict τ-period habitat. That is, utility is derived from consumption only at a single point in time currently τ periods in the future and that all bonds are in zero net supply. In this economy, τ-period bonds are the "safest" and the Preferred Habitat Theory predicts that at the same expected return all investors would purchase them exclusively. Alternatively, as there is no net supply of these (or any other) bonds, their equilibrium expected return should be forced below that on all other bonds.

To illustrate this point, consider a simple CIR production economy with a single commodity, identical investors, and one production opportunity. Investment in this opportunity is riskless and yields its output instantaneously. Although production is (locally) riskless, its rate of return changes stochastically over time in a Markov fashion. We choose to examine the case of riskless production to emphasize that the resulting premiums are due only to investors' desires to hedge interest rate changes and are not systematic risk or beta premiums.[21]

In the simple economy just described, the state of nature is completely described by the per capita supply of the single commodity (per capita wealth) W and the currently available rate of return on production. Because production is riskless and must be engaged in, its rate of return will set the interest rate. The dynamics for this economy are

$$dW = rW\,dt \tag{40a}$$

$$dr = \mu(r)\,dt + g(r)\,dz \tag{40b}$$

where μ and g^2 are the drift and variance of the state variable as in (1).

Substituting these dynamics into the equation for the derived utility of wealth function (35) gives

$$J_t + rWJ_W + \tfrac{1}{2} g^2(r)J_{rr} + \mu(r)J_r = 0 \tag{41}$$

Bond risk premiums are from (38)

$$\alpha(r, t, T) - r = -\frac{P_r}{P}\frac{J_{Wr}}{J_W}g^2(r) \tag{42}$$

In the special case when utility is isoelastic $U(W(\tau)) = W^\gamma/\gamma$, the derived utility of wealth function is separable $J(W, r, t) = Q(r, t)W^\gamma/\gamma$ so the ratio of partial derivatives required in (42) is $J_{Wr}/J_W = Q_r/Q$. The function Q is determined as the solution to

$$O = Q_t + \gamma r Q + \mu(r)Q_r + \tfrac{1}{2} g^2(r)Q_{rr} \tag{43}$$

subject to $Q(r, \tau) = 1$. A formal solution to (43) is

$$Q(r, t, \tau) = E_t\left[\exp\left(\gamma \int_t^\tau r(s)\,ds\right)\right] \tag{44}$$

provided the expectation exists.

For a continuous-time economy in which $r(t)$ follows a continuous Markov process, an increase in its current value will increase all future interest rates for a fixed time path of random realizations on dz. Thus, an increase in $r(t)$ unequivocally increases the random variable $\int_t^\tau r(s)\,ds$. From (44), Q_r then has the same sign as γ. Also in this continuous-time economy P_r is negative. Q is clearly positive from (44) and P is also positive. Therefore, term premiums as given in (42) have the same sign as γ which may be positive or negative

$$\operatorname{sgn}[\alpha(T) - r] = \operatorname{sgn}[-(Q_r/Q)g^2(r)(P_r/P)] = \operatorname{sgn}(\gamma) \gtreqless 0 \tag{45}$$

Logarithmic utility ($\gamma = 0$) is the dividing point. Investors whose relative risk aversion is less than unity ($\gamma > 0$) demand positive term premiums. Investors

[21] A positive covariance between returns on physical investment and returns on bonds would contribute positively to term premiums. This feature, although a paradigm in risky asset pricing, has often been ignored in the analysis of "riskless" default-free bonds.

who are more risk-averse require only negative premiums. Furthermore, the more risk-averse the investors, the smaller (in algebraic value) need the premium be.

In an economy of investors who all desired to consume at only one fixed point in the future, it would be expected that the more risk-averse would lend to the less risk-averse in return for a guaranteed payment, since the former perceive a higher cost of bearing risk. The result here certainly suggests this behavior, but it contains an additional surprise. Investors who are sufficiently risk-tolerant (although still globally risk-averse) will at the same expected rate of return actually prefer the risk involved in a series of short-term bonds to the safety of a single bond maturing on the date they desire to consume.

This result may at first seem counterintuitive; however, there is a simple explanation. In portfolio choice, we define a risk premium as the required expected (arithmetic mean) return above the interest rate. This is natural, since, in a portfolio, risk (covariances) add together. However, in a multiperiod investment when the uncertainty is realized period by period, the risk does not add but compounds. Thus, a term premium is the required expected (geometric mean) return above the spot rate. Furthermore, just as linear utility is risk-neutral toward (arithmetic) mean zero gambles, logarithmic utility is risk-neutral toward (geometric) mean unity gambles.[22] Utility functions which are "less concave" than is the logarithmic (e.g., isoelastic with $\gamma > 0$) are, therefore, risk preferring in this context. Consequently, an investor with a τ-period horizon and relative risk aversion less than unity will actually seek the risk involved in rolling over a series of short bonds, unless he is bribed to hold the safe τ-period bond through receiving a higher expected return.

As the example above indicates, a τ-period habitat for consumption, even on the part of all investors, is insufficient to cause a smaller required return on τ-period bonds. Nevertheless, the result presented above is consistent with the Preferred Habitat Theory, provided we reinterpret a "habitat" as a stronger or weaker tendency to hedge against changes in the interest rate as suggested by Merton [31]. To see this interpretation, note that the derived utility and marginal utility of wealth are given by

$$J(W, r, t) = Q(r, t) W^{\gamma}/\gamma$$
$$J_W(W, r, t) = Q(r, t) W^{\gamma-1} \tag{46}$$

where $Q(\cdot)$ is defined in (44). An increase in the interest rate will always increase utility, since $\operatorname{sgn}[\partial Q/\partial r] = \operatorname{sgn}(\gamma)$. The obvious interpretation is that an increase in the physical rate of return will, ceteris paribus, increase future wealth and hence current utility. However, an increase in the interest rate will increase

[22] A unit geometric mean gamble, is one in which the logarithm of the proportion of final wealth to initial wealth has an arithmetic expectation of zero. i.e., $\tilde{W} = W_0 e^{\tilde{x}}$ and $E[\tilde{x}] = 0$. Since $E[\ln(\tilde{W})] = \ln W_0$, a log utility investor would be indifferent about undertaking this gamble. However, individuals with isoelastic utility with $\gamma > 0$ are "risk-prefering" with respect to the same gamble

$$E[\tilde{W}^{\gamma}/\gamma] = (W_0^{\gamma}/\gamma)E[e^{\gamma\tilde{x}}] > W_0^{\gamma}/\gamma(e^{\gamma\bar{x}}) = W_0^{\gamma}/\gamma$$

The second step above follows from Jensen's inequality. For $\gamma < 0$ the inequality would be reversed as utility is negative.

(decrease) ***marginal*** utility of wealth if γ is positive (negative). Therefore, an investor with positive (negative) γ will desire to sell short (purchase) long-term bonds whose price would fall, causing an offsetting change in marginal utility.[23]

VI. Risk Premiums for Nominal Bonds

In deriving the results in the two previous sections it was explicitly assumed that bond payments were riskless in terms of the consumption good. Unfortunately, most bonds are denominated in nominal terms. Indeed, inflation uncertainty comprises the predominant portion of interest rate risk at least for short term bonds.[24] Consequently, in this section we examine term premiums in an economy with uncertain inflation. Through the use of a simple model it is established that uncertain inflation is a third general case which can result in the Expectations Hypothesis with random interest rates. Furthermore, in this one case risk-neutrality does have a somewhat special role.

We add "money" to the previously developed single-good model by assuming that it is used only to denominate the payments on bonds and other nominal contracts. It is not required for transactions purposes or when undertaking real investment; neither does it enter the direct utility function of consumption. To permit concentration on the effects of inflation, we further assume that the real yield curve is flat and nonstochastic. Conditions sufficient for this assumption to obtain are that the real investment opportunity set be constant over time and that the common utility function be time additive with a constant rate of time-preference, and isoelastic. Given these assumptions, it is straightforward to show that any investor's derived utility function is also isoelastic and depends only on his real wealth and time.

Although the real yield curve is flat, the state variables other than wealth do affect the prices of nominal contracts through the distribution of future inflation. The expected real rates of return on nominal contracts will depend upon the remaining N state variables. From (39)

$$\alpha(Y, t, T) = r + \textstyle\sum_{n=1}^{N} (P_{Y_n}/P)(1 - \gamma)\sigma_{Wn} \tag{47}$$

where r is the (constant) real interest rate, σ_{Wn} is the covariance of real returns on optimally invested wealth and changes in the nth state variable, and P is the nominal bond's real price. The valuation equation (10) can then be written as

$$\tfrac{1}{2} \textstyle\sum_1^N \sum_1^N P_{Y_n Y_m} g_n' g_m + \sum_1^N [\mu_n - (1 - \gamma)\sigma_{Wn}] P_{Y_n} + P_t - rP = 0 \tag{48}$$

subject to the condition that at maturity $P(Y, T, T) = 1/\pi(Y, T)$ where π is the price level.

Although not necessary, it will be convenient for the analysis to assume that

[23] From footnote 21 we see that an increase in wealth to xW ($x > 1$) provides a greater (lesser) gain in utility than a decrease to W/x if γ is positive (negative). Thus, a given infinitesimal increase in the interest rate increases utility by more (less) than the same decrease in r would decrease it. Consequently, investors with $\gamma > 0$ are willing to gamble for interest rate increases, while investors with $\gamma < 0$ will buy long bonds to hedge against interest rate decreases.

[24] See Fama [17].

the price level is the first state variable and that its evolution is locally multiplicative. i.e.,

$$d\pi = \mu_1(Y, t)\, dt + g_1'(Y, t)\, dz$$
$$\equiv \pi\hat{\mu}\, dt + \pi\hat{g}'\, dz \tag{49}$$

where the expected rate and variance of inflation, $\hat{\mu}$ and $\hat{g}'\hat{g}$ may depend upon time and all the state variables except the price level itself.

In this case the nominal valuation equation can be determined from (48) by substitution of the bond's nominal price $P^* = P\pi$. This gives

$$\tfrac{1}{2} \textstyle\sum_2^N \sum_2^N P^*_{Y_n Y_m} g_n' g_m + \sum_2^N [\mu_n - (1-\gamma)\sigma_{Wn} - \hat{g}'g_n] P^*_{Y_n}$$
$$+ P_t^* - (r + \hat{\mu} - (1-\gamma)\hat{\sigma}_{W\pi} - \hat{g}'\hat{g}) P^* = 0 \tag{50}$$

where $\hat{\sigma}_{W\pi} \equiv \sigma_{W\pi}/\pi$ is the covariance of real returns on wealth with the percentage change in the price level. (This valuation equation is to be solved subject to the condition at maturity $P^*(Y, T, T) = 1$.) The last term in (50) is the price multiplied by the nominal rate of interest[25]; therefore, the nominal factor risk premium for the nth state variable is

$$\lambda_n = (1-\gamma)\sigma_{Wn} + \hat{g}'g_n \tag{51}$$

The first term in (51) is relative risk-aversion multiplied by the covariance of real returns on wealth with changes in the state variable. The second term is the covariance of inflation with changes in the state variable.

In the economy described above, the term structure of real interest rates will be nonstochastic. To prevent arbitrage the real returns on index bonds, those with payoffs denominated in the real numeraire, must all be equal and all forms of the Expectations Hypothesis will hold. However, nominal interest rates will be stochastic, and nominal bonds of different maturities may have different expected nominal rates of return.

The Local Expectations Hypothesis, but not the other forms unless inflation is nonstochastic, will obtain for nominal rates only if each factor risk premium is zero. A sufficient, but not necessary, condition for this is that each nominal state variable (i.e., excluding real wealth) be uncorrelated with the real return on aggregate wealth and realized inflation. If investors are risk-neutral, the latter alone is sufficient (since $\gamma = 1$). This, or similar cases, appears to be the only

[25] It is by now well known that the nominal rate of interest is not simply the real rate plus the expected rate of inflation. In a continuous-time-state model these two factors overstate the nominal rate by the variance of inflation and a wealth related risk premium for inflation. To prove that the four term expression in (50) is the nominal rate we write the nominal price as $P^* = \exp[-R(Y, t, T)(T - t)]$. Then

$$P_Y^* = (t - T)R_Y P^*$$
$$P_{YY}^* = (t - T)R_{YY}P^* + (t - T)R_Y R_Y P^*$$
$$P_t^* = [R + (t - T)R_t]P^*$$

Substituting these expressions into (50) and taking the limit as T approaches t gives the nominal spot rate $R(Y, t, t) = r + \hat{\mu} - (1-\gamma)\hat{\sigma}_{W\pi} - \hat{g}'\hat{g}$.

situation in which risk-neutrality is compatible with both the Expectations Hypothesis and stochastic interest rates, and it obtains only because inflation has no real consequences within the economy.

VII. A Continuous-Time Formulation of a Class of Popular Discrete-Time Interest Rate Models

While any observable characteristic of importance in the economy could be selected as a state variable, a large body of literature has relied on models, recently surveyed by Dobson, Sutch, and Vanderford [13], which use only the current and lagged values of interest rates. Typically in these models the assumed equilibrium yield-to-maturity on a pure discount bond of maturity T is

$$R(t, T) = \frac{1}{T - t} \sum_{i=0}^{T-t} E_t[r_{t+i}] + h(t, T) \tag{52}$$

where h captures the effects of any premium. The forecasts required in (52) are generated by

$$E_t[r_{t+1}] = w_0\mu + \sum_{i=1}^{\infty} w_i r_{t+1-i} \text{ [26]} \tag{53}$$

Using the properties of conditional expectations to evaluate (53) recursively permits a reexpression of the long rates in (52) as a linear function of observable past rates

$$R(t, T) = x_0(T - t) + \sum_{i=1}^{\infty} x_i(T - t) r_{t+1-i} \tag{54}$$

Different choices for the forecast weights in (53) can lead to a wide variety of shapes for the yield curve for the same history of interest rates. Nevertheless, all of the resulting models can be easily studied due to the simple linear structure of (54). For these two reasons, numerous studies have employed models of this type in both theoretical and empirical research. Unfortunately, in light of the discussion in the previous sections of this paper, these models must be re-evaluated. The continuous-time equivalent of (52) is very similar to (20), the Unbiased Expectations Hypothesis, which we have demonstrated to be invalid as an equilibrium specification. Consequently, the form of the term premium in (52) must be carefully chosen to ensure a tenable model. Nevertheless, since the simple linear structure of these models makes them so useful, the formulation of a consistent model with this same property is desirable.

The continuous-time equivalent of the forecasting equation (53) is

$$E_t[r(t + dt)] = \left[w_0\mu + w_1 r(t) + \int_0^{\infty} w(s) r(t - s)\, ds \right] dt \tag{55}$$

In general (55) cannot be transformed into a simple diffusion model since the

[26] Typically, strong regularity conditions are imposed on the set of weights w_i, and they are usually assumed to be constant over time. In addition, the constant term $w_0\mu$ is often set to zero. For stationarity of the model it is required that $w_0 \neq 0$, $\sum w_i = 1$, and $\sum w_i^2 < \infty$. In this case μ represents the steady state mean of the spot rate of interest.

implied interest rate model is not Markov but depends upon an uncountable infinity of "state variables," all past levels of the interest rate. Nevertheless, if the weighting function $w(s)$ is sufficiently well behaved, a diffusion representation is possible.

One particularly simple formulation is the elastic random walk or autoregressive model with $w(s) = 0$. This model places no predictive weight on any values of the interest rate other than its current level. That is, the spot rate follows a Markov process. Equation (55) expressed in difference form becomes

$$dr = k(\mu - r)\, dt + \sigma(\cdot)\, dz \tag{56}$$

where $k = w_0 = 1 - w_1$ and σdz is the forecast error. This model of interest rate dynamics dates back, at least in spirit, to the Keynesian [1930] concept of a "return to a normal level" of interest rates, and is characterized by the following behavior. Over the next period, the interest rate when above (below) its usual level is expected to fall (rise) by an amount proportional to the current deviation from normal.

Cox, Ingersoll, and Ross [10] have examined this model and demonstrated that it gives long rates which are a linear function of the spot rate if and only if the uncertainty term is linear in the spot rate, $\sigma^2(r) = a + br$.[27] This finding indicates that models which ignore the inherent uncertainty in predictions are misspecified and can reach erroneous conclusions about the shape of the yield curve. Nevertheless, the model does demonstrate that the desired linear property can be incorporated into a continuous-time equilibrium model. Unfortunately, it does not have the same flexibility to incorporate the variety of yield curves that the general formulation in (54) does. We can, however, increase the scope of the model by an expansion of states.

In this regard, we introduce a state variable $x(t; \beta)$ which is an exponentially smoothed average of past spot rates

$$x(t; \beta) \equiv \beta \int_0^\infty e^{-\beta s} r(t - s)\, ds \tag{57a}$$

$$= \beta e^{\beta t} \int_{-\infty}^{t} e^{\beta t} r(\tau)\, d\tau \tag{57b}$$

From (57b) the dynamics of x can be derived as

$$dx = \beta(r - x)\, dt \tag{58}$$

so local changes in x are nonstochastic and no additional uncertainty is added to the model.[28]

[27] The risk compensation factor, λ, must also be linear in the interest rate. Furthermore if $b \neq 0$ equilibrium requires that $\lambda(r)$ be proportional to the variance $\lambda(r) = \lambda(a + br)$. Richard [35] and Vasicek [37] examined special cases of this model with $a = 0$ and $b = 0$, respectively. Merton [30] and Dothan [14] looked at the pure random walk, $k = 0$, for $\sigma^2(r) = \sigma^2$ and $\sigma^2(r) = \sigma^2 r^2$. In the latter case long rates are not linear in the short rate.

[28] All forecasting models of this type share this important feature. Regardless of the number of state variables (smoothed averages of the interest rate) there is but a single source of uncertainty in each period. An intuitive explanation of this feature is that once "next period's" spot rate is observed, no further information is required to update all the state variables.

A simple model incorporating this state variable is the choice $w_0 = 0$, $w_1 = 1 - b$, $w(s) = b\beta e^{-\beta s}$ which can be written as the two-state variable Markov process

$$dr = b(x - r)\,dt + \sigma(\cdot)\,dz$$
$$dx = \beta(r - x)\,dt \tag{59}$$

Here we have replaced the constant μ of model (56) with the moving average x.

From (57a) we can see that the state variable x is equivalent to a geometric average of discrete observations. Malkiel [28] proposed the discrete time equivalent of the model in (59) by using a moving average[29] as a "normal range" for the interest rate, a substitute for the Keynesian concept of a normal level

$$E_t[r_{t+1}] = (1 - b)r_t + bB \textstyle\sum_{s=0}^{\infty} (1 - \beta)^s r_{t-1-s} \tag{60}$$

Bierwag and Grove [2] and Mincer [33] have suggested generalizations of this model in which the "normal range" is a convex combination of geometric averages. The extension of these models to a continuous-time framework is immediate. The drift of r includes additional terms with additional state variables, and each new state variable evolves in a locally nonstochastic fashion similar to (58).

Duesenberry [16] and others have challenged the hypothesis of an autoregressive evolution of the spot rate. Instead they suggest that the interest rate might be expected to continue to evolve along recent trends. The forecasting structure in (59) can also accommodate this hypothesis if $b < 0$. Now, if the interest rate has moved above (below) its recent average level, it will tend to go even higher (lower). The extrapolative nature of this model becomes clearer if we adopt an alternate state variable description. This description will also prove useful in empirical applications of the model.

We define the new state variable $u \equiv r - x$. Then, using Ito's lemma, (59) can be written as

$$dr = -bu\,dt + \sigma(\cdot)\,dz \tag{61a}$$

$$du = -\delta u\,dt + \sigma(\cdot)\,dz \tag{61b}$$

where $\delta = b + \beta$. From (61b) we see that u is an exponentially smoothed average of past forecast errors[30]

$$u(t) = \delta \int_0^{\infty} e^{-\delta s}\sigma(\cdot)\,dz(t - s) \tag{62}$$

The empirical advantage of this specification is that with a little formal manip-

[29] In many studies the moving average $\beta' \sum_{s=0}^{\infty} (1 - B')^s r_{t-s}$ is used in place of the one defined in (60). Even though this state variable is stochastic, the statement in footnote 28 remains valid. No additional uncertainty is added to the model since the error term in r and X' are exactly proportional $\Delta X'_{t+1} = B'(r_{t+1} - X'_t) = B'(r_t - x'_t) + B'\Delta r_{t+1}$.

[30] An intuitive explanation of (61a) is that if the interest rate has risen unexpectedly in the recent past so that $u > 0$, there will be a tendency for it to fall for the autoregressive model ($b > 0$) or to rise further for the extrapolative model ($b < 0$).

ulation (61a,b) can be written as[31]

$$dr(t + dt) = (1 - \delta\, dt)\, dr(t) - [1 - (\delta - b)\, dt]\sigma\, dz(t) + \sigma\, dz(t + dt) \tag{63}$$

which will be recognized as the continuous-time equivalent of a Box Jenkins model, specifically an ARMA (1,1) process in the first difference, dr. In this form the empirical techniques developed for Box Jenkins models can now be employed.

Just as the models of Mincer [33] and Bierwag and Grove [2] could be extended to continuous-time by using more than one weighted average, multiple-variable extrapolative models and combined extrapolative-regressive models can also be formulated. In principle, even the completely generalized model in Equation (55) can be written using only the current spot rate and state variables which are exponentially smoothed averages of past spot rates.

Consider the weighted "sum" of state variables $\int \omega(\beta)x(t;\beta)\, d\beta$. Since $x(t;\beta)$ has weight $\beta e^{-\beta s}$ at lag s, we can match the general model (55) by choosing ω to satisfy:

$$w(s) = \int_0^\infty \beta e^{-\beta s}\omega(\beta)\, d\beta \tag{64}$$

The right-hand side of (64) is the Laplace transform of $\beta\omega(\beta)$, so matching can be achieved for all weighting functions $w(s)$ which decay sufficiently for large lags. (For models which have been proposed, this requirement will always be met.) Furthermore, any weighting can be matched to any desired degree of accuracy by the choice of a finite number of state variables which are exponentially smoothed past spot rates by choosing an approximation

$$w(s) \cong \sum_{m=1}^{M} \beta_m e^{-\beta_m s}\hat{\omega}(\beta_m) \tag{65}$$

VIII. Bond Pricing Models with Linearly Related Interest Rates

If the relevant state variables for the term structure consist of only the spot rate and smoothed averages of the spot rate, then for the dynamics introduced in the previous section the bond valuation Equation (10) becomes

$$\tfrac{1}{2}\sigma^2(r, X)P_{rr} + \left[\sum_0^N b_i(x_i - r) - \lambda(r, X)\right]P_r - rP + \sum_1^N \beta_i(r - x_i)P_{x_i} - P_\tau = 0 \tag{66}$$

where X is a vector of the smoothed averages, $\tau \equiv T - t$ is the time to maturity, and x_0 is a degenerate moving average ($\beta_0 = 0$) with a constant value. For the desired result that long-term discount rates be linear in the state variables the present value function must be expressible as

$$P(r, X, \tau) = \exp\left[B(\tau)r + \sum_0^N C_i(\tau)x_i\right] \tag{67}$$

with $B(0) = C_i(0) = 0$.

[31] Using (61b) write $u(t) = (1 - \delta\, dt)u(t - dt) + \sigma\, dz(t - dt)$. Solve (61a) for $u(t - dt) = -[dr(t) - \sigma\, dz(t)]/b\, dt$ and substitute these two expressions into (61a).

Substituting (67) into (66) gives

$$\tfrac{1}{2}\,\sigma^2(\cdot)B^2 + [\textstyle\sum_0^N b_i(x_i - r) - \lambda(\cdot)]B - r + \sum_1^N \beta_i(r - x_i)C_i - rB' - \sum_0^N x_iC_i' = 0 \qquad (68)$$

Equation (68) can be uniformly satisfied if and only if $\sigma^2(\cdot)$ and $\lambda(\cdot)$ are linear functions of the state variables[32]

$$\sigma^2(\cdot) = \sigma_0^2 + \sigma^2 r + \textstyle\sum_1^N \sigma_i^2 x_i$$

$$\lambda(\cdot) = \lambda_0 + \lambda r + \textstyle\sum_1^N \lambda_i x_i \qquad (69)$$

In this case by equating the constant terms and those multiplying r and x_i each to zero, (68) can be rewritten as a set of simultaneous, first-order ordinary differential equations with constant coefficients

$$x_0C_0' = \tfrac{1}{2}\,\sigma_0^2B^2 + (b_0x_0 - \lambda_0)B \qquad (70a)$$

$$B' = \tfrac{1}{2}\,\sigma^2B^2 - (\textstyle\sum b_i + \lambda)B - 1 + \sum \beta_iC_i \qquad (70b)$$

$$C_i' = \tfrac{1}{2}\,\sigma_i^2B^2 + (b_i - \lambda_i)B - \beta_iC_i. \qquad (70c)$$

One primary advantage of postulating linearity is now apparent. The ordinary differential equations in (70) are more tractable than the general partial differential equation of valuation in its original form. If, for example, the interest rate variance is constant, $\sigma = \sigma_i = 0$, then an analytical solution may be easily obtained as outlined below. On the other hand, if numerical techniques are required for solution, it will be much less costly to apply them to (70) than to (66).[33]

As a practical matter, we desire a parsimonious fit which requires relatively few additional state variables. Most past research has assumed that the weighting function $w(s)$ introduced in the previous section was either strictly decreasing (regressive models) or unimodal (extrapolative-regressive models). These char-

[32] The proof of the "if" clause is verified by the constructive solution technique in (70). To prove "only if" consider a regular point r_0 of both $\sigma^2(\cdot)$ and $\lambda(\cdot)$ where the Taylor series expansions

$$\frac{1}{2}\,\sigma^2(\cdot) = a_0 + a_1(r - r_0) + a_2(r - r_0)^2 + \cdots$$

$$\lambda(\cdot) = b_0 + b_1(r - r_0) + b_2(r - r_0)^2 + \cdots$$

exist. Then by equating terms of r^2 in (68) we conclude that $a_2B^2 - b_2B = 0$. This implies that (1) $a_2 = b_2 = 0$; (2) $B(\tau) \equiv 0$; or (3) $B(\tau) \equiv b_2/a_2$. Since a_2 and b_2 do not depend upon the maturity of the bond, condition (3) implies that B is constant and hence zero since $B(0) = 0$. But if $B \equiv 0$ in (3) or (2) then r is not a relevant state variable contrary to assumption. Thus, (1) must hold. But a_2 and b_2 are proportional to the second derivatives of σ^2 and λ at r_0 and only for a linear function is the second derivative zero at each regular point. A similar analysis shows that σ^2 and λ must be linear in each x_i.

[33] Brennan and Schwartz [6, 7] discuss numerical solutions to partial differential equations such as (66). Both the implicit and explicit methods increase in time required exponentially with the number of state variables. Equations (70) should require computation time which is linear in the number of state variables.

acteristics can be captured with only two smoothed averages. We also permit regression towards a constant mean. The forecasting model is[34]

$$dr = [b_0(\mu - r) + b_1(x_1 - r) + b_2(x_2 - r)]\,dt + \sigma(\cdot)\,dz$$
$$dx_i = \beta_i(r - x_i)\,dt,\ i = 1, 2 \tag{71}$$

If $b_1 > 0 > b_2$, then the first factor would promote regression to a moving mean and the second extrapolation. Although not required, typically $\beta_2 > \beta_1$ as extrapolation is generally considered the shorter-term effect. Under these conditions, the weighting function is uniformly positive and strictly decreasing if $\beta_1^2 b_1 + \beta_2^2 b_2 \geqslant 0$.Otherwise there is a single peak at lag

$$s = \frac{1}{\beta_2 - \beta_1} \ln\,[-b_2\beta_2^2/b_1\beta_1^2] \tag{72}$$

The model in (71) can also be written using two smoothed averages of past forecast errors as[35]

$$dr = [\alpha_0(\mu - r) + a_1u_1 + a_2u_2]\,dt + \sigma(\cdot)\,dz$$
$$du_i = -\alpha_i u_i\,dt + \sigma(\cdot)\,dz,\ i = 1, 2 \tag{73}$$

The model in (73) can be solved in closed form for a constant variance under the Local Expectations Hypothesis. The discount bond price is then

$$P(r, u_1, u_2, \tau) = \exp\{-\mu\tau + (r - \mu + s_1u_1 + s_2u_2)\phi(\tau; \alpha_0)$$
$$- s_1u_1\phi(\tau; \alpha_1) - s_2u_2\phi(\tau; \alpha_2) + \tfrac{1}{2}\,\sigma^2 \textstyle\sum_0^2 \sum_0^2 s_is_j(\alpha_i\alpha_j)^{-1}[\tau - \phi(\tau; \alpha_i)$$
$$- \phi(\tau; \alpha_j) + \phi(\tau; \alpha_i + \alpha_j)]\} \tag{74}$$

where

$$\phi(\tau; \alpha) \equiv -(1 - e^{-\alpha\tau})/\alpha$$
$$s_i \equiv a_i/(\alpha_i - \alpha_0) \quad i = 1, 2$$
$$s_0 \equiv 1 - s_1 - s_2$$

[34] Forecasting models of the type presented here are generally used in an empirical context and little concern is placed on finding a consistent embedding equilibrium. The issue of why past interest rates are plausible state variables in a rational expectations equilibrium is ignored. One possible justification arises when investment is not readily reversible so that past "interest rates" are still reflected in the current production function. Changes in the interest rate will then be affected by past rates as these investments disappear or are abandoned. Such a situation might be modeled as in (71). A somewhat less satisfying model also providing this justification can be built upon the economy outlined in Equation (40). If one method of production is locally riskless, as in (40a), but its physical rate of return is changing over time, as modeled in (71) rather than (40b), then the interest rate will also behave as in (40b), provided that this riskless production opportunity is always employed.

[35] The models in (71) and (73) are related as follows

$$2\nu_1 \equiv \alpha_2 + \alpha_1 + a_2 + a_1 \qquad \nu_2 \equiv (\nu_1^2 - \alpha_1\alpha_2 - \alpha_1a_2 - \alpha_2a_1)^{1/2}$$
$$\beta_1 \equiv \nu_1 + \nu_2 \qquad \beta_2 \equiv \nu_1 - \nu_2$$
$$b_0 \equiv \alpha_0\alpha_1\alpha_2/\beta_1\beta_2$$
$$b_1 \equiv -(\alpha_2 - \alpha_1)^2a_1a_2(\alpha_0 - \beta_1)[2\nu_2\beta_1(\alpha_2 - \beta_2)(\alpha_1 - \beta_2)]^{-1}$$
$$b_2 \equiv (\alpha_2 - \alpha_1)^2a_1a_2(\alpha_0 - \beta_2)[2\nu_2\beta_2(\alpha_2 - \beta_1)(\alpha_1 - \beta_1)]^{-1}$$

If the Local Expectations Hypothesis does not hold, but factor risk premiums are linear in the state variables, then a similar solution with modified parameters is obtained.[36]

IX. Conclusion

The Term Structure of Interest Rates has a long and checkered history in the field of financial economics. In this paper, many of the traditional propositions and hypotheses in this area are reexamined by employing recent advances in the theory of dynamic equilibrium valuation and contingent-claims pricing.

In discussing the Expectations Hypothesis we show that it must be considered to be a theory of several mutually incompatible propositions. We prove further that only one of these propositions can obtain in a continuous-time, rational-expectations equilibrium. This version, which we term the Local Expectations Hypothesis, holds that the instantaneous expected rates of return on all bonds are equal to the prevailing spot rate of interest. Since, by assumption, bonds earn no risk premiums in this equilibrium, tradition has long assigned a special role to risk neutrality as a potential justification. We demonstrate that this connection does not exist. In general universal risk neutrality results in nonzero bond premiums. In fact, the one type of utility which seems to be most strongly related to the Expectations Hypothesis and zero risk premiums is logarithmic utility.

This finding also serves as a basis for analysis of the Preferred Habitat Theory of Modigliani and Sutch. In Section V we provide a partial corroboration for this theory; however, we give a decidedly different interpretation to habitats. It is an investor's risk aversion and not his rate of time preference or desire to consume at a particular point of time which is the determinant.

In the final two sections we consider the linear adaptive interest rate forecasting models found throughout the macroeconomic literature. We show that these models are incomplete because the functional form of the forecast error, which is traditionally unspecified, influences the form of the resulting equilibrium. We complete and adapt these models to a continuous-time framework in such a way that they are readily tractable and retain the desired linearity properties.

Appendix

The Expectations Hypotheses in Discrete Time

In a discrete-time model, the four forms of the expectations hypothesis are in general distinct. The Local Expectations Hypothesis, which holds that expected single period returns are equal for all bonds, can be written as

$$\frac{E_t[P(t+1, T)]}{P(t, T)} = 1 + R_t \tag{A.1}$$

Here we have used R to denote the effective single period rate to distinguish it

[36] For example, if term premiums are linear in the interest rate, $\alpha(r, X, \tau) = r + (\lambda_0 + \lambda r)\delta(r, X, \tau)$ then in the modified dynamics for (71) $b_0' \equiv b_0 + \lambda\sigma$ and $\mu' \equiv (b_0\mu - \lambda_0\sigma)/b_0'$. In (73), the same modified μ' is used and $\alpha_0' \equiv b_0'\beta_1\beta_2/\alpha_1\alpha_2$ replaces α_0.

from the continuously-compounded rate r used throughout the text. Evaluating (A.1) recursively leads to the statement of equilibrium

$$P(t, T) = E_t[[(1 + R_t)(1 + \tilde{R}_{t+1}) \cdots (1 + \tilde{R}_{T-1})]^{-1}] \tag{A.2}$$

Under the Return-to-Maturity Expectations Hypothesis, the guaranteed return from holding any discount bond to maturity is equal to the return expected from a series of single-period bonds

$$1/P(t, T) = E_t[(1 + R_t)(1 + \tilde{R}_{t+1}) \cdots (1 + \tilde{R}_{T-1})] \tag{A.3}$$

If this equality is to be valid instead for the expected yields from these two strategies (the Yield-to-Maturity Expectations Hypothesis), then

$$[P(t, T)]^{-1/(T-t)} = E_t[[(1 + R_t)(1 + \tilde{R}_{t+1}) \cdots (1 + \tilde{R}_{T-1})]^{1/(T-t)}] \tag{A.4}$$

As in the continuous-time models each of these propositions is incompatible with the other two. Defining $\tilde{X} \equiv [(1 + R_t)(1 + \tilde{R}_{t+1}) \cdots (1 + R_{T-1})]^{-1}$ Equations (A.2) through (A.4) can be rewritten as

$$P = E[\tilde{X}]$$

$$P^{-1} = E[\tilde{X}^{-1}]$$

$$P^{-1/(T-t)} = E[\tilde{X}^{-1/(T-t)}] \tag{A.5}$$

Jensen's inequality assures us that at most one of these relations can be valid.

For the Unbiased Expectations Hypothesis, forward rates are equal to expected future spot rates.

$$P(t, T)/P(t, T + 1) = E_t[1 + R_T] \tag{A.6}$$

A recursive evaluation of (A.6) gives the equilibrium condition

$$1/P(t, T) = (1 + R_t)E_t(1 + R_{t+1}) \cdots E_t(1 + R_{T-1}) \tag{A.7}$$

In the continuous-time model the Unbiased Expectations Hypothesis is equivalent to the YTM-Expectations Hypothesis (A.4). Here, as we show below, it is not.[37] It is equivalent to the RTM-EH (A.3) if the *levels* of future interest rates are uncorrelated. However, this is at odds with the empirical evidence, which indicates that changes in the spot rate are close to uncorrelated. If interest rate levels are positively autocorrelated and (A.3) describes equilibrium, then forward rates are biased high.

To see that the Unbiased Expectations Hypothesis (A.7) is incompatible with the other two forms (A.2) and (A.4), note that for a two-period bond the former implies both

$$P(t, t + 2) = [(1 + R_t)E_t(1 + R_{t+1})]^{-1} < E_t[(1 + R_t)^{-1}(1 + R_{t+1})^{-1}] \tag{A.8a}$$

$$[P(t, t + 2)]^{-1/2} = [(1 + R_t)E_t(1 + R_{t+1})]^{1/2}$$

$$> E_t[(1 + R_t)^{1/2}(1 + r_{t+1})^{1/2}] \tag{A.8b}$$

[37] If yields and forward rates are re-expressed as continuously compounded interest rates over a single period, then the Unbiased and the Yield-to-Maturity Expectations Hypotheses are equivalent as in continuous-time models.

In each case the inequality follows by Jensen's inequality. (A.8a) conflicts with (A.2) while (A.8b) contradicts (A.4).

In the body of this paper we proved that only the Local Expectations Hypothesis could obtain in a continuous-time, continuous-state equilibrium. The same types of models, e.g., logarithmic utility, will also support this form of the expectations hypothesis in discrete time. However, it is also possible for the Return-to-Maturity and Unbiased Expectations Hypotheses to obtain in discrete time as we now outline.

Consider a complete-markets pure-exchange economy with identical investors. The quantity of the consumption good available each period is a random variable independent of all past realizations. As investors are identical, individual consumption will also be independent period to period. From the investor's first-order conditions for an optimal portfolio we can derive the state prices

$$p(s_T, s_t) = \pi(s_T, s_t)U_1[C(s_T), T]/U_1[C(s_t), t] \tag{A.9}$$

where π is the probability of state s_T and $U_1(\cdot)$ is marginal utility of consumption.

We denote $U_1[C(\tilde{s}_t), t]$ by $\tilde{u}_t$. From the assumed intertemporal independence of consumption, $\tilde{u}_t$ are also independent of past state realizations, so

$$E[\tilde{u}_T \mid s_t] = \bar{u}_T \quad \text{all} \quad t < T \tag{A.10}$$

Discount bond prices are given by

$$P(s_t, t, T) = \int_S p(s_T, s_t)\, ds_T = \bar{u}_T/u_t \tag{A.11}$$

Future interest rates are

$$1 + R(s_t, t, t+1) = u_t/\bar{u}_{t+1} \tag{A.12}$$

which are also intertemporally independent.

Using (A.10) through (A.12), it is a simple matter to verify that the Unbiased Expectations Hypothesis (A.7) obtains. Substituting (A.12) into (A.7), the definition of the Unbiased Expectations Hypothesis, and using (A.10) gives

$$\begin{aligned} 1/P(t, T) &= (1 + R_t)E_t(1 + R_{t+1}) \cdots E_t(1 + R_{T-1}) \\ &= (u_t/\bar{u}_{t+1})E_t(\bar{u}_{t+1/\bar{u}t}+2)\cdots E_t(\bar{u}_T) \\ &= (u_t/\bar{u}_{t+1})(\bar{u}_{t+1}/\bar{u}_{t+2}) \cdots (\bar{u}_{T-1}/\bar{u}_T) \\ &= u_t/\bar{u}_T \end{aligned} \tag{A.13}$$

which is (A.11), the previously determined equilibrium. Since interest rates are intertemporally uncorrelated, the Returns-to-Maturity Expectations Hypothesis is valid as well.

The differences between discrete-time and continuous-time models are generally a matter of form rather than substance. Indeed, most results in either type of model can be mimicked in the other. Therefore, the demonstration above of a RTM-EH equilibrium might be disturbing, for we proved in Section III that no such equilibrium could obtain in a continuous-time, continuous-state economy. Fortunately, when we examine the continuous-time limit of the model presented here, this discrepancy is reconciled.

Using (A.11) and (A.12) and defining the length of a single period to be Δt, discount bond prices can be rewritten as

$$P(s_t, t, T) = [1 + R(s_t, t, t + \Delta t)]^{-1}\bar{u}_T/\bar{u}_{t+\Delta t} \tag{A.14}$$

But $R(\cdot)$ is the effective single-period interest rate expressed on a per-period basis. In the limit, then, as $\Delta t \to 0$, $R(\cdot) \to 0$ as well, and the first factor in (A.14) does not affect bond prices. This is the case in all continuous-time models. However, here the prevailing interest rate does not affect the quotient $\bar{u}_T/\bar{u}_{t+\Delta t}$ either. Therefore, each realization of the interest rate can have only a vanishingly small effect on bond prices, and they are, in the limit, certain. Thus, in the continuous-time limit of the model presented above, the RTM-EH holds, but only in the singular sense of a certainty model.

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