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A Note on Real and Nominal Efficient Sets

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IN A RECENT ARTICLE, Manaster [9] analyzed the difference between the compositions of a real mean-variance-efficient portfolio z and a nominally efficient portfolio x with the same expected return. Following a long-lived tradition, he defined the real return on asset j, $\tilde{r}_j^\circ$, as the difference between the nominal return $\tilde{r}_j$ and the inflation rate $\tilde{\pi}$:

$$\tilde{r}_j^\circ = \tilde{r}_j - \tilde{\pi} \tag{0.1}$$

This appeared to imply that the portfolio weights vector Z of any real efficient portfolio z can always be decomposed as:

—the portfolio weights vector X_j of the nominally efficient portfolio with the same expected real return, and
—a hedging fund h with composition vector H.

That is:

$$Z = X_j + H$$

where

$$X_j = V^{-1}(R, l)A^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} \tag{0.2}$$

V is the $N \times N$ matrix of asset return covariances (in nominal terms)

R is the $N \times 1$ vector of nominal expected returns

l is the $N \times 1$ vector of ones

r_j is the *expected* nominal return of portfolio x_j

$A = \begin{pmatrix} R'V^{-1}R, & R'V^{-1}l \\ R'V^{-1}l, & l'V^{-1}l \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, well known from (Roll [10])

$$H = V^{-1}C - V^{-1}(R, l)A^{-1}(R, l)'V^{-1}C \tag{0.3}$$

He then showed that the real variances on z and x_j were related as

$$\sigma_z^{\circ 2} = \sigma_{x_j}^{\circ 2} - H'VH \tag{0.4}$$

Manaster concluded that the nominal efficient set and the real efficient set are

* Assistant, K. U. Leuven; useful comments, on an earlier version, by Professor Mark Goovaerts and Steven Manaster are gratefully acknowledged. Responsibility for all remaining ambiguities and errors is the the author's. My debt to Steven Manaster will be obvious because of the similarity of his and my mathematics. Both papers merely differ in their starting point, and no mathematical or logical errors on his side are implied.

either coincident ($H = \mathbf{0}$, the zero vector), or strictly disjoint ($H \neq \mathbf{0}$). The case $H = \mathbf{0}$ was shown to imply that the real market risks of the assets depend only on their covariances with inflation—which he rejected as counter-intuitive.

The purpose of the present note is threefold. First, it is shown that the linear approximation (0.1) has implications which are at variance with earlier results by e.g. Fama [3] and Long [7] on the utility of nominal versus real wealth. These contradictions show up most clearly in the special case of unit relative risk-aversion (log-utility of real wealth). However, a quadratic approximation[1] of the form

$$\tilde{r}_k^{\circ} = \tilde{r}_k - \tilde{\pi} - c_k + \sigma^2(\tilde{\pi}) \tag{0.5}$$

where c_k is asset k's covariance with inflation

does not suffer from these drawbacks. This is demonstrated in Section I.

It is obvious that the additional covariance-with-inflation term, which is neglected in the linear approach, will show up when one compares the composition vector Z of a real efficient portfolio z to the composition X of a nominally efficient portfolio x. If one compares z with a portfolio x_i chosen so as to have the same expected *nominal* return, then, as shown in Section II,

$$\begin{cases} Z = X_i + (1 - \lambda/2).H, \qquad H \text{ as in } (0.3) \\ \sigma_z^2 = \sigma_{x_i}^2 + (1 - \lambda/2)^2.H'VH \end{cases} \tag{0.6}$$

where λ is the lagrange multiplier of the real minimum variance problem. One notices that the hedging fund h is the same as the one obtained by Manaster. Clearly, as in his paper, $H = \mathbf{0}$ still implies coincidence of both sets. However, the amount to be invested in h is now a function of relative risk aversion. It will be obvious that $(1 - \lambda/2)$ can itself be rewritten as an expression linear in expected nominal return. The implications are that (a) for $\lambda/2 = 1$ (e.g. log-utility), $Z = X_i$ even when $H \neq 0$: real and nominal efficient sets coincide in *at least* one point; and (b) in the nominal mean-variance plane, the horizontal distance between σ_z^2 and $\sigma_{x_i}^2$ is a *quadratic* function of expected return (with one root) rather than a constant, when $H \neq \mathbf{0}$.

Next, the approximation (0.5) also implies that two portfolios with the same expected nominal return will not in general have the same expected real return. So the portfolio x_j chosen so as to have the same expected *real* return as a real efficient portfolio z will, in general, have a different composition from the portfolio x_i that provides the same expected *nominal* return. Let X_j be the weights vector of the nominally efficient portfolio x_j. In comparing Z with X_j, a new hedging fund g, with composition G, emerges:

$$\begin{cases} Z = X_j + (1 - \lambda/2).G \\ \sigma_z^{\circ 2} = \sigma_{xj}^{\circ 2} + (1 - \lambda/2)^2 G'VG \end{cases} \tag{0.7a}$$

[1] This is the Ito approximation used by e.g. Stanley Fischer [4], or Friend, Landskroner, and Losq [5].

where

$$G = V^{-1}C - V^{-1}(R - C, l)D^{-1}(R - C, l)'V^{-1}C$$

$$D = (R - C, l)'V^{-1}(R - C, l), \tag{0.7b}$$

$$(R - C, l) \text{ assumed to be of full rank.} \tag{0.7c}$$

The conclusions of this analysis (Section III) are: (a) Manaster's hedging fund h makes the difference between a nominal and a real efficient portfolio with the same expected *nominal* return rather than the same expected *real* return; (b) the amount to be invested in the hedging fund g depends on risk aversion and is linear in the expected real return; (c) even when $G \neq \mathbf{0}$ (which is equivalent to $\mathbf{H} \neq 0$), there exists one portfolio which is both real and nominal efficient; and (d) when $\mathbf{G} \neq 0$, the distance between the real variances of both portfolios in the real mean-variance space is a quadratic function of the expected real return, with one root, rather than a constant.

Finally, Section IV shows that $H = \mathbf{0} = G$ implies linear dependence between the vector of asset expected nominal returns and the vector of covariances with inflation: $C = ml + nR$, m and n real constants. As shown in passing in Section I, this trivially implies congruence of real and nominal efficient sets, all information about C already being captured by R and l. Linear dependence in (R, l, C) also immediately implies linear dependence of the real betas on covariances with inflation, an implication already pointed out by Manaster. Here again the linear and the Ito approximation are in agreement. Section V, then, provides a graphical illustration of the results.

I. Utility of Nominal and Real Wealth

Mean-variance theory assumes that the investor maximizes a function

$$E\{U(c_1, \tilde{w}_2^\circ)\} = V(c_1, r_p^\circ, \sigma_p^{\circ 2})$$

with c_1 real consumption at the beginning of the period, $\tilde{w}_2^\circ$ real end-of-period wealth, and r_p°, $\sigma_p^{\circ 2}$ the mean and variance of the real portfolio return. Risk-aversion implies that

$$\frac{\partial V}{\partial r_p^\circ} > 0, \frac{\partial V}{\partial \sigma_p^{\circ 2}} < 0 \tag{1.1}$$

and the mean-variance measure of relative risk-aversion is

$$-2\frac{\partial V/\partial \sigma_p^{\circ 2}}{\partial V/\partial r_p^\circ} = 2\frac{dr_p^\circ}{d\sigma_p^{\circ 2}} = \frac{dr_p^\circ}{d\sigma_p^\circ} \cdot \frac{1}{\sigma_p^\circ} = \eta, \text{ relative risk-aversion} \tag{1.2}$$

This corresponds to $2/\lambda$, λ being the multiplier for the required expected return constraint in the Lagrangean of the "real" Minimum Variance problem.

Fama [3], Long [7], and Fischer [4] treat the question whether or not a risk avert investor will, ceteris paribus:

–dislike price uncertainty $\sigma(\tilde{\pi})^2$, and
–prefer a higher covariance with inflation.

At first sight, the answers are yes in both cases. Let's consider the familiar linear approximation

$$\begin{cases} r_r^\circ = r_p - \pi \\ \sigma_p^{\circ 2} = \sigma_p^2 - 2c_p + \sigma(\tilde{\pi})^2 \end{cases} \tag{1.3}$$

where r_p and σ_p^2 refer to the nominal returns and $c_p = \mathrm{cov}(\tilde{r}_p, \tilde{\pi})$. With (1.3) the answers to the questions are

$$\frac{\partial V}{\partial \sigma^2(\tilde{\pi})} = \frac{\partial V}{\partial \sigma_p^{\circ 2}} \cdot \frac{\partial \sigma_p^{\circ 2}}{\partial \sigma(\tilde{\pi})^2} = \frac{\partial V}{\partial \sigma_p^{\circ 2}} < 0 \tag{1.4}$$

and

$$\frac{\partial V}{\partial c_p} = \frac{\partial V}{\partial \sigma_p^{\circ 2}} (-2) > 0 \tag{1.5}$$

That is, according to the linear approximation (1.3) all risk-avert investors ceteris paribus dislike price uncertainty because it increases the real variance without affecting the expected real return; and they all prefer a higher covariance with inflation because it decreases the real variance without affecting the real expected return.

These statements have been shown to be erroneous by Fama [3], Long [7], and Fischer [4]. Fama shows that an investor will require a higher nominal return with rising price uncertainty only if his relative risk-aversion exceeds two, and the reason, thereof, was to lie in the asymmetric effect of equal percentage price increases or decreases on the real return.[2] Long mentions that because of the same reason, some investors will ceteris paribus dislike covariance with inflation; and his statements can be expressed along similar lines as Fama's, by the condition that relative risk-aversion has to exceed unity before the investor actually prefers positive covariance with inflation.[3] Fischer [4] also obtains this last result.

With log-utility, (or in any other case of unit relative risk aversion), the investor

[2] Because

$$\frac{\partial^2 U(W/I)}{\partial I^2} = \frac{\partial U(W/I)}{\partial (W/I)} WI^{-3}(2 - \eta), \ \eta = \text{relative risk-aversion.}$$

Fama's example is a gamble with as outcomes either a 20 percent price rise or a 20 percent price decrease, with equal probability. This results not only in a positive variance (0.047) but also in a positive expected return (4 percent). So not all risk-avert investors would decline the gamble.

[3] Because

$$\frac{\partial^2 U(W/I)}{\partial W \partial I} = \frac{\partial U(W/I)}{\partial (W/I)} I^{-2}(\eta - 1)$$

An example may be two gambles. The first one offers, with equal probability, (20 percent nominal return and a 20 percent price decrease) or (−20 percent nominal return and a 20 percent price increase). This results in an expected real return of 8.3 percent and a .174 variance. The second gamble offers (20 percent nominal return and 20 percent inflation) or (−20 percent nominal return and a −20 percent inflation), resulting in a riskless zero real return. So (1) the covariance with inflation affects not only the risk but also the expected real return; and (2) not all investors will necessarily opt for the positive covariance gamble (the second one), although both gambles offer the same means and variances of nominal return and of inflation.

would even entirely disregard covariance with inflation. This property reflects the well-known separation of nominal wealth $\tilde{W}$ and the price level $\tilde{I}$ in the log function:

$$E\{\log(\tilde{W}/\tilde{I})\} = E\{\log(\tilde{W})\} - E\{\log(\tilde{I})\} \tag{1.6}$$

With the standard assumption that the individual's consumption-investment decision does not affect the distribution of the price level, (1.6) implies that the log-utility investor, though maximizing the expected utility of real wealth, acts as if he maximizes the expected utility of nominal wealth. With normality[4], this implies that his portfolio should not only be real efficient but also nominal efficient. However, as Manaster rigorously shows, the linear approximation would suggest that no real efficient portfolio can also be nominal efficient—except in the degenerate case where both sets coincide entirely.

These contradictions with the theory disappear when the quadratic (Ito) approximation to the real return is used. A quadratic expansion of the real return yields

$$\tilde{r}_p^\circ \equiv \frac{1 + \tilde{r}_p}{1 + \tilde{\pi}} - 1 \simeq \tilde{r}_p - \tilde{\pi} - \tilde{\pi}\tilde{r}_p + \tilde{\pi}^2 + \cdots$$

For the expectations (no tildes), this implies

$$r_p^\circ = r_p - \pi - [c_p + \pi.r_p] + [\sigma^2(\tilde{\pi}) + \pi^2] \text{ where } c_p = \text{cov}(\tilde{r}_p, \tilde{\pi})$$

Now, for shorter and shorter intervals, the means and (co)variances in this expression all decrease roughly in proportion to the time span; but they stay at the same relative order of magnitude. That, then, means that the terms $\pi.r_p$ and π^2 go to zero by the *square* of the speed of the other terms. Consequently, for very short intervals:

$$r_p = r_p - \pi - c_p + \sigma^2(\tilde{\pi}) \tag{1.7a}$$

Similarly, neglecting covariances among cross-terms as being of second order of smalls, one obtains for the real variance

$$\sigma_p^{\circ 2} = \sigma_p^2 - 2c_p + \sigma^2(\tilde{\pi}) \tag{1.7b}$$

exactly as in the linear approach.

Clearly, price uncertainty and covariance with inflation now affect both the real variance and the expected real return, so that neither can be a priori classified as a good or a bad.

With (1.7) the attitude of the risk-avert investor w.r.t. price uncertainty and hedging against inflation would be summarized as follows:

$$\frac{\partial V}{\partial \sigma^2(\tilde{\pi})} = \frac{\partial V}{\partial \sigma_p^{\circ 2}} + \frac{\partial V}{\partial r_p^\circ} = \frac{1}{2}\frac{\partial V}{\partial r_p^\circ}(2 - \eta) < 0 \quad \text{if } \eta > 2 \tag{1.8}$$

[4] More rigorously (since real and nominal returns cannot be *both* normally distributed): if portfolios can be ranked either on the basis of $(r_p^\circ, \sigma_p^{\circ 2})$, or (r_p, σ_p^2, c_p), then the log-utility investor's portfolio will be real as well as nominal efficient. Ito price dynamics are a way to justify the first two moments (whether nominal or real) as sufficient statistics.

where, it will be recalled, η is relative risk-aversion; and

$$\frac{\partial V}{\partial c_p} = \frac{\partial V}{\partial \sigma_p^{\circ 2}}(-2) + \frac{\partial V}{\partial r_p^{\circ}}(-1) = \frac{\partial V}{\partial r_p^{\circ}}(\eta - 1) > 0 \quad \text{if } \eta > 1 \tag{1.9}$$

Unlike (1.4) and (1.5) (from the linear approach), the attitudes of investors w.r.t. price uncertainty and covariance with inflation implied by the Ito approximation are in perfect agreement with the theory. Obviously, the difference of both approximations w.r.t. price uncertainty is not so important, as $\sigma^2(\tilde{\pi})$, being the same for all assets, cannot affect portfolio choice. However, the additional covariance with inflation term which will turn up in the required expected real return constraint in the Min $\sigma_p^{\circ 2}$ problem *will* have an effect, showing up in the fact that the real and nominal efficient set *always* share *at least one* portfolio.

II. Real and Nominal Efficient Portfolios with the Same Expected Nominal Return

According to the approximation (1.7), the real efficient set is derived by solving, for all values of the required expected real return r_z°

$$\text{Min } Z'[V - lC' - Cl' + ll'\sigma^2(\tilde{\pi})]Z$$
$$\text{s.t.} \begin{cases} Z'[R - C - l\pi + \sigma^2(\tilde{\pi})] = r_z^{\circ} \\ Z'l = 1 \end{cases}$$

Using the last constraint, the objective function and the first constraint can be simplified to

$$\text{Min } Z'VZ - 2Z'C + \sigma^2(\tilde{\pi})$$
$$\text{s.t.} \begin{cases} Z'(R - C) = (r - c)_z \\ Z'l = 1 \end{cases} \tag{2.1}$$

The RHS of the first constraint is expressly written as $(r - c)_z$, not $r_z - c_z$, to show that the investor cares not about the expected nominal return and covariance with inflation in themselves, but only about their difference. The first order conditions w.r.t. Z immediately yield

$$Z = (1 - \lambda/2)V^{-1}C + \lambda/2V^{-1}R + \kappa/2V^{-1}l \tag{2.2}$$

It is obvious that with $\lambda/2 = 1$, unit relative risk-aversion, this collapses to a nominal efficiency condition. Moreover, if (R, l, C) is not of full rank, that is, if real numbers m and n exist such that

$$C = ml + nR$$

then (2.2) becomes

$$Z = [(1 - \lambda/2)n + \lambda/2]V^{-1}R + [(1 - \lambda/2)m + \kappa/2]V^{-1}l \tag{2.3}$$

This is formally indistinguishable from a nominal efficiency condition, so that linear dependence of C and R trivially implies coincidence of nominal and real efficient sets. It is henceforth assumed that (R, l, C) *is* of full rank.

To express Z as a function of portfolio z's expected *nominal* return, use

$$(R, l)'Z = \begin{pmatrix} r_z \\ , \\ 1 \end{pmatrix}$$

$$= (1 - \lambda/2)(R, l)'V^{-1}C + (R, l)'V^{-1}(R, l) \begin{pmatrix} \lambda/2 \\ \kappa/2 \end{pmatrix}$$

$$= (1 - \lambda/2)(R, l)'V^{-1}C + A \begin{pmatrix} \lambda/2 \\ \kappa/2 \end{pmatrix}$$

Hence

$$\begin{pmatrix} \lambda/2 \\ \kappa/2 \end{pmatrix} = A^{-1} \left\{ \begin{pmatrix} r_z \\ , \\ 1 \end{pmatrix} - (1 - \lambda/2)(R, l)'V^{-1}C \right\} \tag{2.4}$$

and Z in (2.2) becomes

$$Z = V^{-1}(R, l)A^{-1} \begin{pmatrix} r_z \\ , \\ 1 \end{pmatrix} + (1 - \lambda/2)(V^{-1}C - V^{-1}(R, l)A^{-1}(R, l)'V^{-1}C)$$

$$= X_i + (1 - \lambda/2)H \tag{2.5}$$

where we used X_i and H as defined in (0.2). So H is the same hedging fund as the one identified by Manaster. The difference with his results is twofold:

(1) H makes the difference between a nominal and a real efficient portfolio with the same nominal return, rather than with the same real expected return (as the linear approximation seems to imply); and

(2) the required amount of investments in H depends on risk aversion instead of being the same (unity) for all portfolios.

Since by construction z and x_i have the same expected nominal return, it suffices to compare their nominal variances. Using Manaster's result that $\mathrm{cov}(\tilde{r}_{x_i}, \tilde{r}_h) = 0$, x_i nominally efficient, one immediately obtains

$$\sigma_z^2 = \sigma_{x_i}^2 + (1 - \lambda/2)^2 H'VH \tag{2.6}$$

(2.5) and (2.6) show that there *always* is one real efficient portfolio which is also nominally efficient, viz. the log-utility solution ($\lambda/2 = 1$). A sufficient (and necessary—see Manaster's proof) condition for both sets to coincide *entirely* is that $H = \mathbf{0}$.

To obtain a graphical representation of the case $H \neq \mathbf{0}$, $\lambda/2$ can be expressed in return terms. The first line in (2.4) yields

$$\lambda/2 = \frac{(c, -b)}{|A|} \begin{pmatrix} r_z \\ , \\ 1 \end{pmatrix} - (1 - \lambda/2) \frac{(c, -b)}{|A|} \begin{pmatrix} C'V^{-1}R \\ C'V^{-1}l \end{pmatrix}$$

In this expression one recognizes Black [2]'s basic portfolios p and q, with composition vectors

$$P = V^{-1}R/b$$

$$Q = V^{-1}l/c$$

Their expected returns are covariances-with-inflation are

$$r_p = R'P = a/b$$
$$r_q = R'Q = b/c$$
$$c_p = C'P = C'V^{-1}R/b$$
$$c_q = C'Q = C'V^{-1}l/c$$

Therefore

$$(1 - \lambda/2) = \frac{1 - \dfrac{1}{b}\dfrac{r_z - r_q}{r_p - r_q}}{1 - \dfrac{c_p - c_q}{r_p - r_q}} \tag{2.8}$$

The denominator is the same for all portfolios (independent of r_Z). Hence, $(1 - \lambda/2)^2 H'VH$ is a parabola in expected return with one root, which corresponds to the log-solution; or the horizontal distance between σ_z^2 and σ_x^2 is a quadratic function of r_z that is zero in the log-solution.[5]

III. Nominal and Real Efficient Portfolios with the Same Expected Real Return

The first order conditions for real efficiency can be written as

$$Z = V^{-1}C + V^{-1}(R - C, l)\begin{pmatrix}\lambda/2 \\ \kappa/2\end{pmatrix} \tag{3.1}$$

[5] From Roll [10], $(r_z - r_q)/r_p - r_q)$ in the denominator of (2.8) is the fraction α_ι of a nominally efficient portfolio that is invested in basic fund q:

$$X_\iota = V^{-1}(R, l)A^{-1}\begin{pmatrix} r_z \\ , \\ 1\end{pmatrix} = \alpha_\iota P + (1 - \alpha_i)Q,$$

where $\alpha_\iota = (r_\iota - r_q)/(r_p - r_q)$, and P, Q, r_p, r_q as just defined in the text. So for $\lambda/2 = 1$, $\alpha_\iota = b$, and

$$X_{\log} = bP + (1 - b)Q.$$

Also from Roll [10]: $\sigma_p^2 = a/b^2$, $\sigma_q^2 = 1/c$, $\mathrm{cov}(\tilde{r}_p, \tilde{r}_q) = 1/c$.
So

$$r_{\log} = a + b/c - b^2/c$$
$$\sigma_{\log}^2 = a + 1/c - b^2/c$$

and hence

$$r_{\log} - r_q = a - b^2/c = \sigma_{\log}^2 - \sigma_q^2$$

So the log portfolio is on a 45 degree straight line through q (which itself is the vertex portfolio). This property also holds in the *real* mean-variance plane, where the vertex portfolio is $lV^{\circ-1}/lV^{\circ-1}l$. The unit slope reflects the unit relative risk aversion.

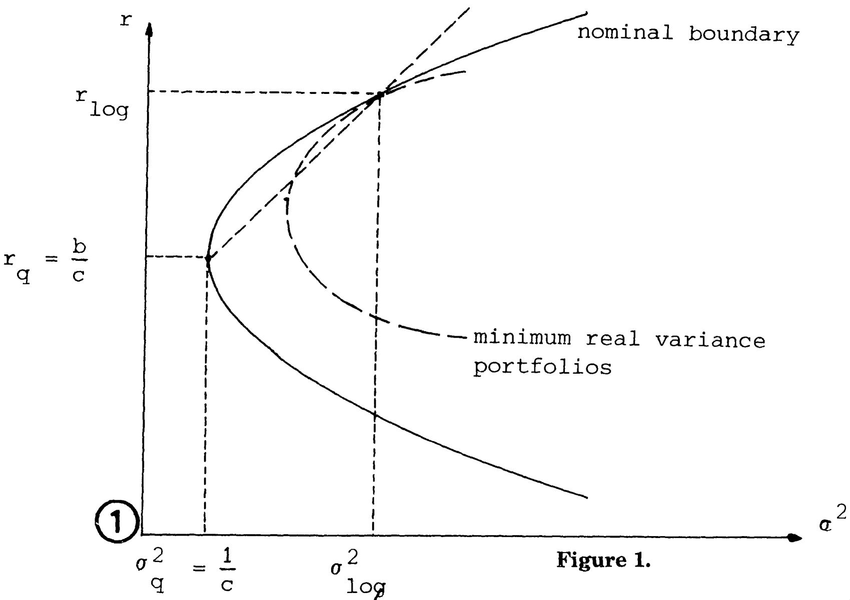
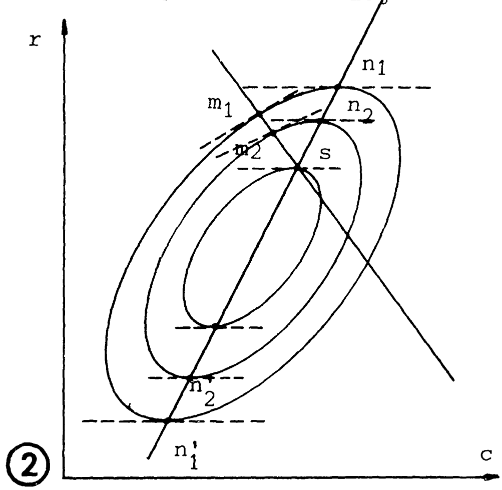


Figure 1.



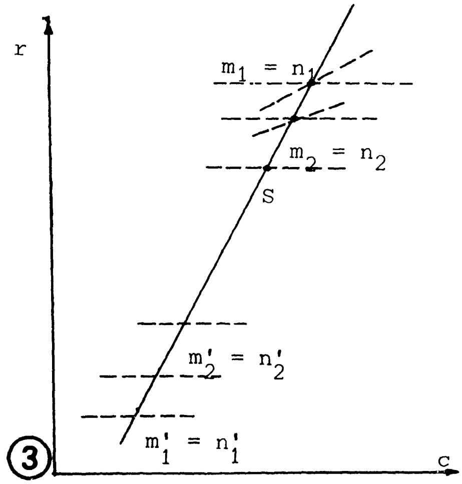


Figure 2. Nominal and Real Efficient Sets in the $r - c$ Plane: The General Case
Figure 3. The Case of Linear Dependence in (C, R, l)

Using the constraints allows one to remove the multiplier terms:

$$(R - C, l)'Z = \begin{pmatrix} (r-c)_z \\ , \\ 1 \end{pmatrix}$$

$$= (R - C, l)'V^{-1}C + D\begin{pmatrix} \lambda/2 \\ \kappa/2 \end{pmatrix} \tag{3.2}$$

where $D = (R - C, l)'V^{-1}(R - C, l)$.

So

$$\begin{pmatrix} \lambda/2 \\ , \\ \kappa/2 \end{pmatrix} = D^{-1}\left\{\begin{pmatrix} (r-c)_z \\ , \\ 1 \end{pmatrix} - (R-C,l)'V^{-1}C\right\} \tag{3.3}$$

and

$$Z = V^{-1}(R-C,l)D^{-1}\begin{pmatrix} (r-c)_z \\ , \\ 1 \end{pmatrix} + G$$

$$\text{with } G = V^{-1}C - V^{-1}(R-C,l)D^{-1}(R-C,l)'V^{-1}C.$$

To bring out X_j, the composition of an nominally efficient portfolio with the same expected real return, choose $(r-c)_z$ (or X_j, alternatively), such that

$$\begin{aligned}\begin{pmatrix} (r-c)_z \\ , \\ 1 \end{pmatrix} &= (R-C,l)'X_j = (R-C,l)'V^{-1}(R,l)A^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} \\ &= (R-C,l)'V^{-1}\left\{(R-C,l)A^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} + C\frac{(c,-b)}{|A|}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix}\right\} \\ &= DA^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} + ((R-C,l)'V^{-1}C)\,\frac{1}{b}\frac{r_j-r_q}{r_p-r_q}\end{aligned}$$

This results, after some rearrangement, in

$$Z = V^{-1}(R,l)A^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} + \left(1 - \frac{1}{b}\cdot\frac{r_j-r_q}{r_p-r_q}\right)\cdot G = X_j + \left(1 - \frac{1}{b}\frac{r_j-r_q}{r_p-r_q}\right)\cdot G \tag{3.5}$$

The conclusion is that the hedging fund (required to transform a nominally efficient portfolio into a real efficient portfolio with the same expected real return) is different, in general, from the hedging fund h in the previous section. However, the amount of g to be added to x_j is also a linear function of the expected return. Equation (3.5) writes his required amount in terms of nominal returns, but real returns are easily brought in by noting that

$$\frac{r_j-r_q}{r_p-r_q} = \alpha_j, \text{ the proportion of } x_j \text{ invested in basic portfolio } p$$

Then, since

$$r_j^\circ = r_q^\circ + \alpha_j(r_p^\circ - r_q^\circ)$$

one can write

$$(1 - \alpha_j/b) = \left(1 - \frac{1}{b}\cdot\frac{r_j^\circ - r_q^\circ}{r_p^\circ - r_q^\circ}\right) \tag{3.6}$$

Since z and x_j have the same expected real return, one only has to compare their real variances. This requires knowledge of the properties of g:

1. $(R - C, l)'G = \begin{pmatrix} 0 \\ , \\ 0 \end{pmatrix}$ (zero investment and zero expected real return) (3.7)

Note that zero investment ($l'G = 0$) implies that $(R - C)'G$ indeed is the real expected return on g. For further use, note also that $(R - C)'G = 0$ implies

$$(R, l)'G = (C, l)'G = \begin{pmatrix} C'G \\ , \\ 0 \end{pmatrix} \tag{3.8}$$

2. From zero investment, it follows that

$$G'[V - lC' - Cl' + ll'\sigma^2(\tilde{\pi})]G = G'VG \geqslant 0 \ (= 0 \quad \text{if } G = \mathbf{0}) \tag{3.9a}$$

and further, from the definition of G,

$$= C'G \geqslant 0 \tag{3.9b}$$

3. Similarly, from zero investment one obtains that for any X

$$G'[V - lC' - Cl' + ll'\sigma^2(\tilde{\pi})]X = G'VX - G'C$$

When X_j is nominally efficient, this becomes (see Equation (0.2)):

$$= G'V\left\{ V^{-1}(R, l)A^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} \right\} - G'C$$

$$= G'(C, l)A^{-1}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} - G'C \quad \text{(from (3.8))}$$

$$= G'C\frac{(c, -b)}{|A|}\begin{pmatrix} r_j \\ , \\ 1 \end{pmatrix} - 0 - G'C \quad \text{(from zero investment)}$$

$$= -\left(1 - \frac{1}{b}\alpha\right)G'C = -\left(1 - \frac{1}{b}\cdot\alpha\right)G'VG \quad \text{(from (3.9b))} \tag{3.10}$$

With expression (3.10) the real variance on z is found to be

$$\sigma_z^{\circ 2} = \sigma_{x_j}^{\circ 2} - \left(1 - \frac{1}{b}\cdot\frac{r_z^\circ - r_p^\circ}{r_p^\circ - r_q^\circ}\right)^2 \cdot G'VG \tag{3.11}$$

This gives us a similar picture as before: when $G = \mathbf{0}$, both sets coincide entirely. However, when $G \neq \mathbf{0}$, the distance between both parabolae, in the real mean-variance space, is a quadratic function of expected real return (with one root) rather than a constant. This one root, which corresponds to the log-utility solution, shows that both sets always have at least one portfolio in common.

IV. The Case H = 0

For nominally efficient portfolios j, there obviously exists a linear relationship between expected nominal return and covariance with inflation, because any such portfolio is itself a linear combination of the two basic funds p and q and because expected return as well as covariances are additive:

$$\left.\begin{aligned} r_j &= r_q + \alpha_j(r_p - r_q) \\ c_j &= c_q + \alpha_j(c_p - c_q) \end{aligned}\right\} \Rightarrow r_j = (c_j - c_q)\frac{r_p - r_q}{c_p - c_q} \tag{4.1}$$

Now, if $H = \mathbf{0}$, then, as noted by Manaster:

$$C = (R, l)A^{-1}(R, l)'V^{-1}C \tag{4.2}$$

However, there are an infinite number of portfolio vectors U satisfying $U'(R, l) = (r_j, 1)$, as soon as the number of assets exceeds two. For each of these inefficient portfolios one finds that, under (4.2),

$$c_u = U'C = U'(R, l)A^{-1}(R, l)'V^{-1}C = X_j'C = c_j \tag{4.3}$$

That is, all portfolios with the same expected return must have the same covariance with inflation. So, for any inefficient portfolio or single asset, (4.1) also holds, and C and R must be linearly dependent.

In that case, also $(R - C - l\pi + l\sigma^2(\tilde{\pi}))$ is of course linearly dependent on C. Since in the CAPM this real expected returns vector is linearly dependent on the vector of real betas, by implication the latter vector must be linearly dependent on C. This last implication was also pointed out by Manaster. Congruence of both sets if $H = \mathbf{0}$ is now obvious: if everything which can be said about covariances with inflation is summarized by R and l, then C is mathematically redundant, and nominal and real efficient sets must coincide, in both the linear and the Ito approximation.

V. Geometric Interpretation

In this final section, a more general minimum-variance problem is discussed which explains the similarities of some of Manaster's and my results, and provides a geometric interpretation of the relation between nominal and real efficient sets.

From (1.1) it is easily shown that, whether one uses a linear or an Ito approximation, the rational investor always dislikes nominal variance—ceteris paribus, that is, for given expected nominal return and covariance with inflation. So his portfolio must be a member of the more general class of portfolios y satisfying

$$\begin{aligned} &\text{Min } Y'VY \\ &\text{s.t. } Y'C = c_y \\ &\quad\;\; Y'R = r_y \\ &\quad\;\; Y'l = 1 \end{aligned}$$

The first-order condition yields

$$Y = \nu/2V^{-1}C + \lambda/2V^{-1}R + \kappa/2V^{-1}l = V^{-1}(C, R, l)\begin{pmatrix} \nu/2 \\ \lambda/2 \\ \kappa/2 \end{pmatrix} \tag{6.1}$$

The multiplier ν is related to the investor's degree of money illusion. Perfect money illusion obtains when $\nu = 0$ (leading to nominal efficiency); with Ito mathematics, the absence of money illusion requires $\nu/2 = 1 - \lambda/2$; finally, Manaster studied the case $\nu/2 = 1$.

Proceeding as in Roll [10] one obtains:

$$\begin{pmatrix} \nu/2 \\ \lambda/2 \\ \kappa/2 \end{pmatrix} = B^{-1} \begin{pmatrix} c_y \\ r_y \\ 1 \end{pmatrix} \tag{6.2}$$

where

$$B = (C, R, l)'V^{-1}(C, R, l) = \left(\begin{array}{c|c} C'V^{-1}C & C'V^{-1}(R, l) \\ \hline (R, l)'V^{-1}C & A \end{array}\right)$$

(6.2) assumes that V and (C, R, l) are of full rank, otherwise B cannot be inverted and the problem has no general solution. With (6.2), the portfolio vector Y can be expressed as a function of the constraints, viz.

$$Y = V^{-1}(C, R, l)B^{-1} \begin{pmatrix} c_y \\ r_y \\ 1 \end{pmatrix} \tag{6.3}$$

This general minimum-variance problem illustrates why Manaster's hedge portfolio h emerges regardless of the value of ν. Inverting the partioned matrix B yields

$$B^{-1} = \begin{pmatrix} e & f & g \\ f & h & i \\ g & i & j \end{pmatrix}$$

where

$$\begin{aligned} e &= (C'V^{-1}C - C'V^{-1}(R, l)A^{-1}(R, l)'V^{-1}C)^{-1} \\ &= (C'H)^{-1} \end{aligned} \tag{6.4a}$$

[6]

$$(f, g) = -eC'V^{-1}(R, l)A^{-1} \tag{6.4b}$$

$$\begin{pmatrix} h & i \\ i & j \end{pmatrix} = A^{-1} + \begin{pmatrix} f \\ , \\ g \end{pmatrix} e^{-1}(f, g) \tag{6.4c}$$

Using (6.4) in (6.3), and rearranging, one obtains

$$Y = X_\iota + (ec_y + fr_y + g)H \tag{6.5}$$

with X_ι and H as defined in Manaster's and in this paper. From (6.2), the weight associated with fund H indeed is the multiplier $\nu/2$.[7] *So fund h always emerges*

[6] Since, by assumption, B is positive definite, e must be strictly positive. So when V and (C, R, l) are of full rank, the case $H = \mathbf{0}$ is indeed ruled out.

[7] So a portfolio y may be found in two steps: first, ignore the c_y constraint, and find X_ι satisfying $(r_y, 1)$; then add a portfolio h with zero investment and zero expected nominal return (see Manaster's paper) in such amounts that also the c_y constraint is met. We see that $ec_y + fr_y + g = (c_y - C'X_\iota)/C'H$, so that the weight of h depends on the difference between the target c_y and the inflation covariance of the nominal efficient portfolio x_ι.

as soon as the investor does not entirely disregard covariance with inflation. This explains the similarities of Manaster's and my results.

A geometric representation of this three-dimensional minimum-variance boundary provides additional insights. The nominal variance of a portfolio y is, from (6.3),

$$\begin{aligned}\sigma_y^2 &= (c_y, r_y, 1)B^{-1}(c_y, r_y, 1)' \\ &= ec_y^2 + 2fc_yr_y + 2gc_y + hr_y^2 + 2ir_y + j\end{aligned} \tag{6.6}$$

Equation (6.6) is a paraboloid in the (σ_y^2, c_y, r_y) space opening in the direction of the variance axis. For fixed c_y, as well as for fixed r_y, the minimum variance boundary is a parabola; for fixed nominal variance, as well as for fixed real variance, skewed ellipses emerge.[8]

These ellipses provide a convenient way of visualizing nominal and real efficient sets on this paraboloid. Nominal efficiency requires maximal expected nominal return for given nominal variance. If F_1 is an ellipse in the r_y, c_y plane containing all portfolios with the same $\sigma^2 = k_1$,

$$F_1 = ec_y^2 + 2fc_yr_y + 2gc_y + hr_y^2 + 2ir_y + j - k_1 = 0,$$

then along this ellipse maximal expected return is obtained when

$$\left.\frac{dr_y}{dc_y}\right|_{F_1} = -\frac{2(ec_y + fr_y + g)}{2(fc_y + hr_y + i)} = 0$$

or simply

$$ec_y + fr_y + g = 0 \tag{6.7}$$

which is of course independent of σ_y^2. So nominal efficient portfolios are found where a plane (6.7), parallel to the variance axis, cuts the paraboloid. From (6.2), we see that on this plane, $\nu/2$ indeed is zero, and $Y = X_i$. Similarly, real efficient portfolios are found where expected real return is maximized among all portfolios with the same real variance; that is, where

$$F_2 = r_y - c_y$$

has the same slope as the fixed-real-variance ellipse

$$F_3 = ec_y^2 + 2fc_yr_y + 2(g-1)c_y + hr_y^2 + 2ir_y + j - k_3$$

where

$$k_3 = \sigma_y^2 - 2c_y \text{ (fixed real variance)}.$$

This condition is

$$\left.\frac{dr_y}{dc_y}\right|_{F_2} = 1 = \left.\frac{dr_y}{dc_y}\right|_{F_3} = -\frac{ec_y + fr_y + g - 1}{fc_y + hr_y + i} \tag{6.8}$$

[8] When $f = 0$, the ellipses in a fixed-variance plane would be symmetric to the r_y, c_y axes; and all parabolae in a fixed$-r_y$ or fixed c_y plane would be "parallel."

From (6.8) we can conclude that real efficient portfolios are found where the plane

$$(f+e)c_y + (h+f)r_y + (i+g) - 1 = 0 \tag{6.9}$$

cuts the paraboloid. Looking back at (6.2), this indeed represents all solutions where r_y and c_y are set such that $\nu/2 = 1 - \lambda/2$. The plane (6.9) is different from the locus of nominal efficient portfolios (6.7) because with B nonsingular (V and (C, R, l) of full rank), (f, h, i) cannot be $(0, 0, 0)$. So once again we conclude that when $C \neq ml + nR$, nominal and real efficient sets are different: they are located where *distinct* planes cut the paraboloid.

With a slight rearrangement of the real efficiency condition (6.8), we can represent the plane (6.9) on the fixed nominal variance ellipse F_1. Rewrite (6.8) as

$$-\frac{ec_y + fr_y + g}{fc_y + hr_y + i} = \frac{fc_y + hr_y + i - 1}{fc_y + hr_y + i}$$

The LHS is the general expression of the slope of the ellipse F_1; the right hand side equals, from (6.2), $(\lambda/2 - 1)/(\lambda/2)$. So for *real efficient* portfolios:

$$\left.\frac{dr_y}{dc_y}\right|_{F_1} = \frac{fc_y + hr_y + i - 1}{fc_y + hr_y + i} = \frac{\lambda/2 - 1}{\lambda/2} \tag{6.10}$$

while for *nominal efficient* portfolios, it will be recalled, the condition was

$$\left.\frac{dr_y}{dc_y}\right|_{F_1} = 0$$

So, on any F_1 ellipse, nominal efficient portfolios are found where the slope of the ellipse is zero, while real efficient portfolios are found where that slope is given by (6.10). There is one portfolio where this slope is also zero, viz. the case $\lambda/2 = 1$ (log-utility). This is illustrated in Figure 2, where points n, n' represent maximal and minimal nominal expected return for given nominal variance—the points where a portfolio y is also on the nominal minimum-variance boundary—, where m, m' are points which belong to the real minimum variance boundary, and where s represents the log-utility portfolio $(\lambda/2 = 1)$.[9]

Finally, consider the case where (C, R, l) is *not* of full rank. Then the ellipses of Figure 2 collapse to a (bounded) straight line, and the points m and n must obviously coincide (Figure 3). In such a case, nominal and real efficient sets will clearly coincide.

[9] In the general problem, $\lambda/2$ would be a kind of nominal risk aversion—the rate of substitution of nominal variance for nominal expected return. Without money illusion, however, nominal relative risk aversion equals real risk-aversion, as is easily checked:

$$-(W/I)\frac{\partial^2 U/\partial (W/I)^2}{\partial U/\partial (W/I)} = -W\frac{\partial^2 U/\partial W^2}{\partial U/\partial W}$$

This merely reflects the fact that the partial derivatives of U w.r.t. W hold constant the price level I.

VI. Some Final Remarks

The traditional linear approximation for real returns was shown to have implications which are inconsistent with the mean-variance theory of utility of nominal and real wealth. However, an Ito approximation is "good enough" for mean-variance purposes. It follows that even if (C, R, l) is of full rank and therefore $H \neq \mathbf{0}$, still real and nominal efficient sets have one point in common—the log-utility investor's portfolio—rather than being strictly disjoint. Therefore the market portfolio in a world where investors hold real mean-variance efficient portfolios can still be nominally efficient even if $H \neq \mathbf{0}$; this case would obtain if all investors have logarithmic utility of end-of-period real wealth functions, or if the wealth-weighted mean relative risk-aversion in the market happens to be exactly unity.

Friend, Landskroner, and Losq [5] quote empirical evidence suggesting that for the U.S., the mean relative risk-aversion or market price of risk is greater than one and probably even in excess of two. However, they did not consider the whole market portfolio of *all* invested wealth, so that the validity of their estimates could be questionable. Arrow [1], on the other hand, says that relative risk-aversion should hover around one. In view of the problem of unobservability of the true market return (Roll [10]), it is unlikely that the issue will ever be settled empirically.

Finally, the result that log-utility investors hold nominally efficient portfolios is not at all new. The comments of Friend, Landskroner, and Losq [5] show that they are aware of this property. Stehle [11] uses log utility to duck the inflation problem and the related Purchasing Power Parity issue in his tests of segmented versus internationally integrated asset pricing models; he refers to an earlier article by Hakansson [6]. In most of the literature on inflation and mean-variance asset pricing, the inconsistency of the linear approximation with the properties of log-utility functions was, however, entirely overlooked, as evidenced by the list of references in Solnik's article [12].

It should be obvious that the differences of the Manaster–Solnik results with the conclusions derived here merely reflect the different starting points. As far as I am aware, they did not make any mathematical or logical errors within the traditional approach, and their efforts have clearly been of much help. The main issue, however, is that the linear approximation, however popular, turns out not to be "good enough" for mean–variance purposes. Since mean-variance theory itself can be viewed as a quadratic approximation, one ought to stick to quadratic approximations throughout.

The inadequacy of the linear approximation also shows up in other respects that are beyond the scope of the present paper. Two log-utility investors which face different inflation rates (for instance, because of differences in tastes, or deviations from Purchasing Power Parity) but identical investment opportunity sets, should obviously hold the same portfolio. Or two investors with the same investment, consumption opportunities, and equal tastes should hold the same portfolio, whatever the currency they happen to keep their books in. With linear approximations, none of these results obtain. With Ito mathematics, however, the portfolio compositions meet these requirements. It is possible that someone will turn up with a test which the Ito approximation fails. But until this is proven, the quadratic approximations are decidedly superior to the linear ones.

REFERENCES

1. K. Arrow. *Essays in the Theory of Risk-Bearing*. Chicago: Markham, 1971.
2. F. Black. "Capital Markets with Restricted Borrowing." *The Journal of Business* 45 (July 1972), 444–54.
3. E. F. Fama. Ordinal and Measurable Utility. In *Studies of the Theory of Capital Markets*, ed. M. Jensen. New York: Praeger, 1972.
4. S. Fischer. "The Demand for Index Bonds." *The Journal of Political Economy* No. 3 (1975), 509–34.
5. I. Friend, Y. Landskroner, and E. Losq. "The Demand for Risky Assets under Uncertain Inflation." *The Journal of Finance* (December 1967), 1287–97.
6. N. H. Hakansson. "On the Relevance of Price-Level Accounting." *Journal of Accounting Research* 7 (1969).
7. J. Long. Stock Prices, Inflation, and the Term Structure of Interest Rates." *Journal of Financial Economics* 1 (1974), 131–70.
8. R. C. Merton. "An Intertemporal Asset Pricing Model." *Econometrica* 41 (1973), 867–87.
9. S. Manaster. "Real and Nominal Efficient Sets." *The Journal of Finance* (March 1979), 93–102.
10. R. Roll. "A Critique of the Asset Pricing Theory's Tests, Part I." *The Journal of Financial Economics* 4 (1977).
11. R. Stehle. "An Empirical Test of the Alternative Hypothesis of National and International Pricing of Risky Assets." *The Journal of Finance* 32 (1977), 492–502.
12. B. Solnik. "Inflation and Optimal Portfolio Choice." *The Journal of Financial and Quantitative Analysis* (December 1979).