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Efficient Funds in a Financial Market with Options: a New Irrelevance Proposition

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ABSTRACT

Under the same assumptions that Ross used to assert the existence of an efficient fund (on which a spanning set of options can be written) we prove that almost any portfolio is an efficient fund. From a constructive point of view, a randomly chosen vector of portfolio weights yields an efficient fund. When the Ross assumptions are relaxed, a limited notion of efficiency—maximal efficiency—is the best attainable. The maximally efficient funds are also everywhere dense in the portfolio space. Some implications are discussed and illustrative examples given.

THE THEORY OF CONSUMER and firm behavior in economies with a complete set of contingent claims is rather well-understood. These complete market structures[1] bring about competitive equilibria where consumers maximize utility, firms maximize profits, and prices adjust to match demand and supply. As shown in Debreu [7], all competitive equilibria are Pareto optimal and any Pareto optimum can be sustained as a competitive equilibrium.

It is commonly recognized, however, that complete markets have not come into existence (see, e.g., Hakansson [13, p. 166] or [14, p. 759]). Research into general equilibria in incomplete markets has taken two major courses. One group of papers have explored the existence (under restrictive assumptions) and limitations of equilibria in incomplete markets (see [8, 15, 16, 18, 19]). Another approach has been to explore the role of financial markets augmented by derived assets, such as options [1, 4, 8, 9, 10], supershares [11, 12, 13], portfolio insurance [6], etc., to bring about the allocational efficiencies of a complete market structure.

In an important paper by Ross [20], the following seminal result is proved: if investors distinguish only among states with differential payoffs of the primitive securities,[2] then a fully efficient market can be achieved by supplementing the primitive securities with options written on a single portfolio (of the primitive securities). Following Ross [20], such a portfolio (with distinct returns in each

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[1] A financial market is complete when investors can, in effect, trade in Arrow-Debreu [2, 7] certificates; this would be the case when the number of linearly independent securities is the same as the visualized states of the world and short sales are unrestricted.

[2] An equivalent condition is that there exist primitive securities such that for any two states of the world, there is at least one security yielding different payoffs.

state of the world) will be called an efficient fund. The existence of such an efficient fund has important welfare implications. Through ordinary puts and calls written on such an efficient fund, the allocational efficiency of a complete market structure is brought within the reach of financial markets (e.g., see Ross [20], Friesen [9, 10], Hakansson [14]).

The existence theorem of Ross [20] does not, unfortunately, provide any prescription for constructing an efficient fund. This gap is evident in the important research (following Ross [20]) which focuses on the power of options to complete financial markets. An apparent assumption implicit in most of this work is that the efficient fund (whose existence Ross proved) is some critical portfolio which would be difficult to construct. Some clever proxy for the efficient fund is relied upon to provide the spanning set of options.

Hakansson [14] uses the market portfolio as the underlying fund on which options or "supershares" are issued to improve the efficiency of the existing market structure.[3] He recognizes that the market portfolio may not be an efficient fund, but additional structure imposed on the beliefs of the investors[4] ensures that options or supershares written on the market portfolio does bring about a fully efficient market. Motivating his choice of the market portfolio, Hakansson [14] argues "the critical portfolio may of course be difficult to identify."[5]

Breeden and Litzenberger [4] proposes an interesting technique for the valuation of Arrow-Debreu securities (and therefore contingent claims, in general) based on computed values for European call options written on the market portfolio (aggregate wealth).[6] By defining the states of the world based on distinct returns of the market portfolio, Breeden and Litzenberger [4] ensure that the market portfolio can effectively play the role of an efficient fund (for that particular configuration of states). However, further restriction on investor beliefs (similar to Hakansson [13], see fn. 3) is required to guarantee the full efficiency of Arrow-Debreu complete markets. Banz and Miller [3] uses a similar approach to evaluate capital budgeting decisions, with the market portfolio in the role of the required efficient fund. in Friesen [9, 10], an equivalence is established between equilibria in financial markets with options and equilibria in "complete"

[3] Hakansson's pioneering work [12, 13, 14] explores in detail the operational and institutional issues in implementing the augmentation of allocational efficiency using options or supershares. His work, along with Ross [20], Breeden and Litzenberger [4], Banz and Miller [3], etc., has made a serious attempt to put the conceptual Arrow-Debreu framework in an operational context.

[4] The necessary and sufficient conditions required by Hakansson [13, 14] to ensure a fully efficient market by supplementing the primitive securities with a full set of options written on the market portfolio are as follows:

(i) investor preferences depend only on wealth;

(ii) probability beliefs π_{is} conditional on aggregate wealth w are homogeneous; i.e., $\pi_{is/w} = \pi_{s/w}$ for all investors $i \in I$, where I is the index set of investors.

[5] See Willig [22] for a discussion of [14] where the difficulties anticipated in calculating an efficient fund are mentioned. He also questions the generality of Hakansson conditions, given in fn. 3.

[6] As pointed out by Breeden and Litzenberger [4], any reference portfolio can be chosen as the underlying portfolio. Again, by defining the states of the world based on distinct returns of the chosen reference portfolio, and efficient fund for that particular definition of the state space is given by the chosen reference portfolio. Important extensions to the multiperiod context use aggregate consumption as the relevant efficient fund.

Arrow-Debreu models. But Friesen assumes a "fineness" property for stock prices which guarantees that there exists some stock which would essentially play the role of an efficient fund. Other researchers [6, 17] have explored the role of portfolio insurance instead of options in a similar context.

The above literature clearly underscores the importance of an efficient fund in incomplete markets. As mentioned above, all the previous work has made use of some proxy for an efficient fund. However, restrictive assumptions about investor beliefs and/or the definition of the states of the world has been evoked to guarantee that the chosen proxy fund would be an efficient fund.

In this paper, the following question is examined: for a general specification of the states of the world and investor beliefs and preferences, how difficult is it to construct an efficient fund? Are these efficient funds very specialized (critical) portfolios, or are they common?

Under the assumptions made by Ross [20] to prove the existence of an efficient fund, it will be shown that almost any portfolio is an efficient fund; i.e., a portfolio picked at random will be an efficient fund, w.p. 1. Thus, the construction of an efficient fund is as simple as choosing a random vector of portfolio weights q and constructing a portfolio using those weights. In a sense (to be made precise in Section II), the choice of an efficient fund obeys an irrelevance proposition. At the same time, a prespecified portfolio (like the market portfolio or aggregate wealth) may not be an efficient fund! These results and their implications for related work are discussed in Section II.

In Section III, we drop the Ross assumption that investors distinguish only among events associated with differential payoffs on securities. In other words, there might be distinct states (say war and peace) in which all securities have the same payoffs (see fn. 2). Of course, an efficient fund then no longer exists. A limited notion of efficiency is defined for any portfolio, based on the partition of the state space induced by the returns of that portfolio.[7] It is shown that almost any portfolio is a "maximally efficient" fund; i.e., given the existing market structure, a portfolio picked at random will have maximal efficiency possible w.p. 1. Again, the choice of a maximally efficient fund satisfies an irrelevance proposition. Nevertheless, it is easy to construct examples where the market portfolio does not have maximal efficiency. Illustrative examples are provided.

Essential notation and definitions are given in Section I. Concluding remarks and some directions for further work are given in Section IV.

I. Definitions and Notation

In this section, we list the definitions and explain the notation that will be used.

In the state-space approach, the random events that might occur are subsets of elementary points or "states" in a (probability) state space S. In this paper, we let S be of finite cardinality, say $|S| = m$. A typical asset $\tilde{a}$ then is viewed as a function,

$$\tilde{a}: S \rightarrow R$$

[7] Our definition is motivated by Hakansson [13], where the fineness of the partition is related to a weak dominance relation among market structures.

i.e., for each $s \in S$, $\tilde{a}(s) \in R$ (is the return in dollars per unit investment). Therefore, all securities belong to R^m.

$S = \{s_1, s_2, \cdots, s_m\}$ is specified in sufficient detail to have as subsets all random events of interest in the economy. The state space specifications based on the returns or payoffs on special assets (e.g., market portfolio, as in [3] or [4]) or assets in general would be special cases of the above definition.

There are only a finite number of marketed capital assets, shares of stock, bonds, or as we call them (following Ross [20]), primitive assets. The set of primitive assets, say A, of cardinality "p", is assumed to be invariant and cannot be altered; i.e., production decisions are precluded. Arranging the p primitive assets columnwise, we can form an $m \times p$ matrix (which will also be denoted by A). For any $s \in S$, a row of A, say $A(s)$ lists the returns from the p primitive assets in state s. Given the state space S and the set of primitive securities A, we say that the Ross assumption is satisfied if $s \neq s'$ implies $A(s) \neq A(s')$; i.e., the rows of A are distinct. In other words, the Ross assumption is satisfied if for any pair of distinct states s, $s' \in S$, there is at least one primitive asset $\tilde{a}$ such that $\tilde{a}(s) \neq \tilde{a}(s')$.

A portfolio $\tilde{e}$ is an efficient fund if $\tilde{e}(s) \neq \tilde{e}(s')$ for all s, $s' \in S$, $s \neq s'$. Thus, an efficient fund has distinct returns in all states of the world. An illustrative example can clarify the above ideas.

Example 1

$$S = \{s_1, s_2, s_3, s_4\}, p = 2, \tilde{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \tilde{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$A = \{\tilde{a}_1, \tilde{a}_2\} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \\ 0 & 2 \end{bmatrix}$$

The Ross assumption is satisfied, since

$$A(s_1) = [1 \quad 1] \neq A(s_2) = [1 \quad 2] \neq A(s_3) \neq A(s_4).$$

The portfolio

$$\tilde{e}_1 = 2\tilde{a}_1 + 1\tilde{a}_2 = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}$$

is efficient since

$$\tilde{e}_1(s_1) \neq \tilde{e}_1(s_2) \neq \tilde{e}_1(s_3) \neq \tilde{e}_1(s_4), \quad \text{i.e., all components of } \tilde{e}_1$$

are distinct.

In Section III, we drop the Ross assumption. If A, the set of primitive securities, does not satisfy the Ross assumption, then no portfolio is an efficient fund. For this case, we define a limited notion of efficiency as follows.

Given the set of primitive securities A, the portfolio $\tilde{e}$ is maximally efficient if for all $s, s' \in S$, $s \neq s'$, and $A(s) \neq A(s')$ implies $\tilde{e}(s) \neq \tilde{e}(s')$. The motivation for the above definition is illustrated by an example.

Example 2

$$S = \{s_1, s_2, s_3, s_4\}, p = 2, \tilde{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \tilde{a}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

Clearly, the Ross assumption is not satisfied since $A(s_1) = A(s_4)$. No efficient fund exists because any portfolio $\tilde{e}$ of these primitive securities would have $\tilde{e}(s_1) = \tilde{e}(s_4)$. In fact, the best any fund can do is to distinguish between states where the rows of A are distinct. This motivates the notion of a maximally efficient fund.

A portfolio

$$\tilde{e}_1 = 1\tilde{a}_1 + 1\tilde{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 3 \end{bmatrix}$$

is a maximally efficient fund; the components of $\tilde{e}_1$ are distinct in all states which have distinct rows for A. Another portfolio

$$\tilde{e}_2 = 2\tilde{a}_1 - 1\tilde{a}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

is not a maximally efficient fund since $\tilde{e}_2(s_2) = \tilde{e}_2(s_3)$ even though $A(s_2) \neq A(s_3)$.

II. An Irrelevance Proposition

In this section, we ask the following questions: given that the primitive securities satisfy the Ross assumption, how do we construct an efficient fund? How clever do we have to be to select a portfolio which is an efficient fund?

To motivate our results in relation to other important work focusing on options and efficiency, let us consider the following example.

Example 3

Let $S = \{s_1, s_2, s_3, s_4\}$ index a 4-state world, where there are only two primitive securities in the market, 200 shares of $\tilde{a}_1$ and 100 shares of $\tilde{a}_2$,

$$\tilde{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \tilde{a}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

By definition, the market portfolio

$$M = 200\,\tilde{a}_1 + 100\,\tilde{a}_2 = \begin{bmatrix} 600 \\ 200 \\ 600 \\ 600 \end{bmatrix}$$

The Ross assumption is satisfied for $A = \{\tilde{a}_1, \tilde{a}_2\}$. The market portfolio M is not an efficient fund; and, by puts or calls on the market portfolio, it is clear that one cannot obtain four linearly independent options to span R^4. Assuming arbitrary preferences and beliefs for the investors, the options or supershares written on the market portfolio M will not bring about the allocational efficiency of complete markets. Since the market portfolio is not an efficient fund, prices of options written on it will not specify the prices of Arrow-Debreu securities (as in [4]).

Since the Ross assumption is satisfied, we know that an efficient fund exists. What portfolio weights $q = (q_1, q_2)$ would yield an efficient fund?

The answer is provided (for the general case, with a set A of p primitive securities satisfying the Ross assumption) by the following theorem.

THEOREM 1. *Given a set A of primitive assets satisfying the Ross assumption, almost any*[8] *portfolio is an efficient fund.* (Arditti and John [1]).

Proof: The formal proof is given in Appendix A. A simple geometric motivation can be provided based on Example 3. Let $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ denote the vector of portfolio weights used to form the generic portfolio $q_1\,\tilde{a}_1 + q_2\tilde{a}_2$. The strategy is to show that the set $NE = \{q: q_1\tilde{a}_1 + q_2\tilde{a}_2$ is not an efficient fund$\}$ is a "negligible" set in $q_1 - q_2$ space (R^2). It can be worked out that NE is the union of the 1-dimensional subspaces $q_1 = 0$, $q_2 = 0$, $q_1 + 2q_2 = 0$, and $q_1 - 2q_2 = 0$ (shown in Figure 1 below). The set NE has Lebesque measure[9] zero in R^2.

The theorem clearly indicates that an efficient fund is not hard to find. It is not a "critical" portfolio to be constructed cleverly. An arbitrarily-picked vector q of portfolio weights would yield an efficient fund with probability one. Thus, the rule for constructing an efficient fund is as simple as follows: choose a vector of portfolio weights at random. The resulting portfolio will be an efficient fund w.p. 1. (In Example 3 above, throwing a dart at Figure 1 will produce a suitable $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ of portfolio weights).

[8] We use "almost any" portfolio in the customary measure-theoretic sense. When we cannot assert that a certain property is satisfied by all portfolios, the next best assertion which can be made seems to be that almost every portfolio satisfies the property. If we assign some probability measure on the set of all portfolios, then the set of "exceptional" portfolios is shown to have a probability zero. An arbitrary portfolio thus satisfies the property with probability 1. An important case is where the probability measure is given by a density with respect to ordinary p-dimensional Lebesgue measure on the set R^p of all portfolios; one can cover all such probabilities simultaneously by proving that the set of exceptional portfolios has a Lebesgue measure zero.

[9] The Lebesgue measure can be loosely thought of as an extension of the ordinary concept of length in R^1, or area and volume in R^2 and R^3, respectively to general subsets of R^p.

A p-rectangle $C \subseteq R^p$ is a nonempty set of the form $C = \{x: a_i \leq x_i \leq b_i, i = 1, \cdots, p\}$. The measure of C is $\prod_{i=1}^{p} (b_i - a_i)$. A subset $S \subset R^p$ has measure zero if for every $\in > 0$ there exists a countable family of p-rectangles covering S, the sum of whose measures is less than $\in$.

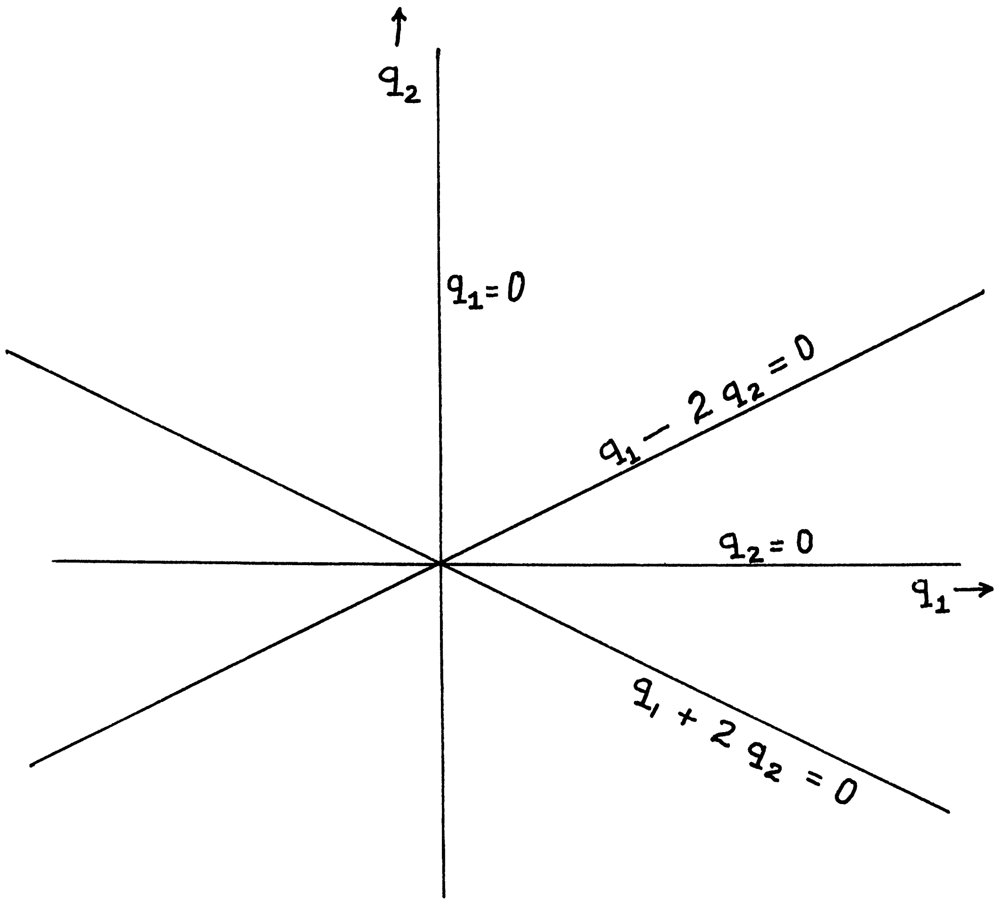

Figure 1.

At this point, a skeptic might ask: "Since there exist portfolios which are not efficient funds, what if the random choice happens to be a 'bad' (not efficient) portfolio?" In response, we can assert a result about efficient funds which is stronger, in a constructive sense, than Theorem 1. If a particular choice of portfolio weights, say $\bar{q}$, does not produce an efficient fund, then in an immediate neighborhood of $\bar{q}$ there are infinitely many that will produce an efficient fund. This can be formalized as follows.

COROLLARY 1. *The set of portfolio weights yielding an efficient fund is open and dense everywhere.*

Proof: Follows from Theorem 1, noting *NE* is closed and has an emtpy interior in R^p.

This corollary is important. The set of "good" portfolio weights (i.e., yielding efficient funds) is open and dense everywhere. This implies that a "bad" portfolio can always be perturbed to yield a good one (while a good portfolio perturbed yields another good one).

In a constructive sense, perturbing a bad portfolio $\bar{q}$ can be implemented in various ways. A straightforward strategy is the method of perturbation used in linear programming for resolving degeneracy. For $t > 0$, define the p-dimensional perturbation (column) vector $\bar{t} = [t\ t^2\ t^3\ \cdots\ t^p]$. It can be shown that $\bar{q} + \bar{t}$ will be a good portfolio for all t, $0 < t < t_0$ (t_0 is some positive number).

To illustrate, let us go back to Example 2. $\bar{q} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a bad portfolio $\left(\text{yielding a nonefficient fund, } 2\tilde{a}_1 + 1\tilde{a}_2 = \begin{bmatrix} 6 \\ 2 \\ 6 \\ 6 \end{bmatrix} \right)$.

Perturb $\bar{q}$ to

$$\bar{q} + \bar{t} = \begin{bmatrix} 2 + t \\ 1 + t^2 \end{bmatrix}.$$

$\bar{q} + \bar{t}$ is a good portfolio for any value of t, $0 < t < \frac{1}{2}$.

The corollary has implications for incorporating attractive institutional features into an efficient fund construction. For example, if an efficient fund were required with long holdings of all primitive securities, it poses no problems: select an arbitrary q with all positive components (from the interior of the positive orthant of R^p). Since the good portfolios are everywhere dense, such a portfolio is easily found. Hakansson [14] considers conditions under which a superfund can be constructed from the market portfolio such that the need for individual investors to go short in any security is eliminated or the trading necessary is minimized. Clearly, with almost all portfolios to choose from, such attractive institutional features can be incorporated into our efficient fund formation.

III. Maximal Efficiency

In this section, the Ross assumption is dropped. The existing set of primitive securities A may not have distinct return vectors in distinct states of the world. As argued in Section I, if the Ross assumption is not satisfied for A, then an efficient fund will not exist. The best that any portfolio can do is to have distinct returns in states where the rows of A are distinct. Thus we look for maximally efficient[10] funds as defined in Section I.

Example 4

Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ index a 5-state world, where there are only two primitive securities in the market, 200 shares of $\tilde{a}_1$ and 100 shares of $\tilde{a}_2$,

$$\tilde{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \quad \tilde{a}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \quad \begin{array}{l} \text{Market portfolio} \\ M = 200\tilde{a}_1 + 100\tilde{a}_2 = \end{array} \begin{bmatrix} 600 \\ 200 \\ 600 \\ 600 \\ 600 \end{bmatrix}$$

[10] Note that for a set of securities A, if the Ross assumption is satisfied fortuitously, then a maximally efficient fund is an efficient fund.

Since $A(s_1) = A(s_4)$, the Ross assumption is not satisfied. Any portfolio $\tilde{e}$ would have $\tilde{e}(s_1) = \tilde{e}(s_4)$, thus an efficient fund does not exist. However, consider

$$\tilde{e}' = 1a_1 + 1a_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 5 \\ 3 \end{bmatrix}.$$

$\tilde{e}'$ is a maximally efficient fund since $\tilde{e}'(s) \neq \tilde{e}'(s')$ whenever $A(s) \neq A(s')$, $s \neq s'$, for all $s, s' \in S$. Note that the market portfolio, in this example, is not a maximally efficient fund.

Now the general strategy for constructing maximally efficient funds is the same as that of the earlier section: choose a portfolio at random.

THEOREM 2. *Almost any portfolio is a maximally efficient fund.*

Proof: The formal proof is given in Appendix A. Example 4 can be used to motivate the proof. Since all primitive securities have identical returns in states s_1 and s_4, redefine the state space as $\hat{S} = \{\{s_1, s_4\}, s_2, s_3, s_5\}$ where the original states with identical returns for all primitives are lumped together as one state in $\hat{S}$. The Ross assumption is satisfied (with respect to $\hat{S}$) and almost any portfolio $\tilde{e}$ is an efficient fund (from Theorem 1). But $\tilde{e}$ is maximally efficient with respect to S, the original state space.

All the assertions of Section II can be made about the construction of maximally efficient funds. A randomly chosen vector q of portfolio weights yields a maximally efficient fund w.p.1. The set of portfolio weights yielding a maximally efficient fund is open and dense everywhere (follows from Corollary 1). Thus, the perturbation strategy can be used to obtain a good portfolio from a particular bad choice. Further, desired institutional features can be incorporated into a particular choice of a maximally efficient fund.

A comment about the market portfolio not being maximally efficient in Example 4 is in order. In a given market structure, the market portfolio is a specific aggregate portfolio. So, it is possible that the market portfolio is not maximally efficient even though almost any portfolio is. However, the market portfolio can be perturbed to yield a maximally efficient fund.

IV. Conclusion

In incomplete markets, the welfare opportunities opened up by derived assets written on efficient funds of existing primitive securities has attracted attention. Ross [20] shows the existence of an efficient fund (under the assumption that the existing primitives can collectively distinguish among the states of the world). It is proved that almost any portfolio is an efficient fund. In other words, to construct an efficient fund, all one has to do is to form a portfolio with a randomly chosen vector of portfolio weights. Further it is shown that an arbitrarily small perturbation of any portfolio yields an efficient fund. It is important to note that the market portfolio need not necessarily be an efficient fund, for a given market structure.

When the Ross assumption is not satisfied, then efficient funds will not exist.

However, a maximally efficient fund can be constructed, as before, by choosing a random vector of portfolio weights.

The observation that good portfolios are everywhere dense is important from an operational point of view. This enables the incorporation of attractive institutional features (as in Hakansson [14]) into the construction of efficient or maximally efficient funds. This will have interesting implications for the role and characteristics of financial intermediaries.

It is clear that the partition of the state space induced by the maximal efficiency attainable through the existing primitive assets will be an important factor in the efficiency of an incomplete market structure. It will be interesting to explore the welfare properties resulting from the coupling of information structures of the agents in the economy and the maximal efficiency attainable.

Appendix A

Proof of Theorem 1: (See Arditti and John [1]).

Given that the Ross assumption is satisfied for the set $A = \{a_1, a_2, \cdots, a_p\}$ of primitive assets, it will be shown that the set of *n*on-*e*fficient portfolios,

$$NE = \{q \in R^p : Aq \text{ is not an efficient fund}\}$$
$$= \{q \in R^p : Aq \text{ does not have distinct components}\}$$

has a Lebesgue measure zero in R^p.

For any $s, s' \in S$, $s \neq s'$ define $Q_{ss'} = \{q : A(s)q = A(s')q\} = \{q : [A(s) - A(s')]q = 0\}$. Since by the Ross assumption, $[A(s) - A(s')] \neq 0$, it is easy to see that the subspace $Q_{ss'} \neq R^p$, is a proper subspace of R^p, and hence of dimension $\leq p - 1$. Thus, the Lebesgue measure of $Q_{ss'}$ in R^p is zero. Since $NE = \cup_{s,s'} Q_{ss'}$ is a finite union of sets $Q_{ss'}$ in R^p (each of which has zero Lebesgue measure), then NE has a Lebesgue measure zero in R^p as required (see [21], p. 304). Q.E.D.

Proof of Theorem 2:

Given a set of primitive assets, $A = \{a_1, a_2, \cdots, a_p\}$, and the state space $S = \{s_1, s_2, \cdots, s_m\}$, form the partition $\hat{S}$ of S lumping together the states with identical returns for all securities, i.e., $\hat{S} = \{w_1, w_2, \cdots, w_k\}$ where $s, s' \in w_i$ for some $i = 1, 2, \cdots, k$ implies $A(s) = A(s')$. Now with respect to the new state space $\hat{S}$, A satisfies the Ross assumption.

Using Theorem 1, almost any portfolio $\tilde{e}$ is an efficient fund with respect to $\hat{S}$. By definition $\tilde{e}$ is a maximally efficient fund with respect to S, the original state space. Thus, almost any portfolio is maximally efficient.

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