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Notes on Multiperiod Valuation and the Pricing of Options

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ABSTRACT

A mean-variance risk-return tradeoff relationship is derived for the diffusion process limiting case of a state-preference model, with aggregate consumption serving as a pivotal variable. The model is compared to other recent models along the dimensions of generality and tractable implementation. The incorporation of stochastic interest rates in general equilibrium and arbitrage-based valuation models is examined, and an extension to earlier methods is discussed, in connection with the implementation of "robust" general valuation procedures.

THIS PAPER DEVELOPS AN integrative discussion of some recent developments in the theory of multiperiod valuation. Our aim is *not* to survey the rapidly burgeoning literature in this area in an exhaustive way. Instead, it is our goal to highlight some results of our own and others that have a bearing on judgments regarding promising and robust directions of enquiry in this area of research. In doing so, areas that have been particularly emphasized are: (1) the incorporation of aggregate consumption as a pivotal variable in general and specialized multiperiod valuation formulae, and (2) the incorporation of endogenized dynamics for the riskless rate of interest in Black-Scholes [4] - type option pricing methodology. Throughout, our emphasis is on the internal consistency, generality, and simplicity of different models insofar as full-information valuation in a general equilibrium context is concerned, rather than on issues pertaining to empirical testing and econometric methodology.

The starting point for much of our analysis is Rubinstein's [28] important paper on multiperiod valuation, and Hakansson's [15] related contributions. In this paper, Rubinstein considers a multiperiod state-preference model of general exchange equilibrium—with no *explicit* modeling of production/investment—in order to develop pricing relationships linking the prices of elementary (Arrow-

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Debreu) securities, and more general payoff streams, to specific measures of expectations and "systematic risk," i.e., risk that is linked closely to the randomness of key economic aggregates. Most of the empirically identifiable and testable pricing relations that Rubinstein develops are derived under the assumption of "weak aggregation," discussed in Rubinstein [25]. These are restrictions on individual investor beliefs and preferences which imply that the equilibrium price system of the actual economy is also the equilibrium price system of an equivalent economy of homogeneous "composite" individuals whose endowment, preference, and beliefs data are simple averages of those of the original economy.[1] We are able to extend Rubinstein's model to show that, in the limiting case of continuous-time *diffusion-process* dynamics for asset returns, testable restrictions among identifiable expected return and risk measures of different assets are implied quite generally even in the absence of "weak aggregation." The resulting risk-return tradeoffs are linear relations linking instantaneous expected returns of assets to the instantaneous covariances of returns with aggregate consumption. Our result overlaps with the stochastic control of diffusions development in Breeden [5].

The strengths and weaknesses of this consumption-based relative returns relation—along the dimensions of generality and tractable application—are then compared to some recent research on equilibrium models of valuation, starting from the contributions of Rubinstein [28], and the pathbreaking earlier work of Merton [20] on diffusion process models. We focus closely on the nature and rationality/justifiability of assumptions or conclusions regarding the intertemporal dynamics of the riskless rate of interest and aggregate "price of risk" terms, both of which are important constituents of most of these pricing relations, as they were in the seminal two-period models of Sharpe [30] and Lintner [19].

Our conclusions regarding methods of valuation that are relatively robust and tractable then lead us naturally to some extensions of the Black-Scholes [4]-Merton [21] methodology for the pricing of contingent claims or derived securities, whose payoff patterns are functionally linked to those of other traded securities. The specific issue we emphasize is that of incorporating intertemporally stochastic instantaneously riskless rates of interest in these models, and the merits and generality of the resulting valuation methodology relative to other methods. The seminal contribution of Cox, Ingersoll, and Ross [12] to the theory of the term structure of interest rates in general equilibrium models contains concurrently developed results that are closely related to our discussion here, though the formal development of their model is done within the context of a homogeneous investor ("weak aggregation") economy. Our focus is on the generality and tractability of a small set of relatively robust insights across a spectrum of different models, rather than on the important alternative task of developing a set of specific pricing formulae under "restrictive" assumptions (which add to the intuitive understanding of specialized problems.)

[1] The existence of such "composite individuals," for arbitrary endowment distributions, *requires* identical linear coefficients of response of individuals' Arrow-Pratt risk-tolerance functions to consumption, as shown in Brennan and Kraus [8]. Different sufficient conditions are discussed in detail by Rubinstein [25].

In Section I, we sketch the Rubinstein [28] model, and then extend it to the relative asset pricing relation for diffusion processes. Section II contains comparisons with, and discussions of, the key features of other recent equilibrium valuation models. Section III deals with the incorporation of stochastic interest rates in arbitrage-based option pricing models, and its implications for the equilibrium valuation methodology suggested by Breeden and Litzenberger [6]. A set of summary remarks in Section IV concludes the paper.

I. The Model of Intertemporal Returns

As in Rubinstein [28], we consider a multiperiod state-preference general equilibrium model with a single commodity, for an economy populated by investors with temporally additive, state-independent, strictly concave utility functions of consumption who act to maximize the expected utility of their state-contingent consumption plans. We do *not* assume that the "weak aggregation" conditions of Rubinstein [25] hold. The market for securities is assumed to be competitive, frictionless, and sufficiently diverse so that Pareto-optimal allocations that would have resulted from a literally complete market, also arise in this economy. This requires that securities exist to span all the "meta states" of different aggregate consumption at each date, *provided* probability beliefs are "sufficiently" homogeneous across individuals—for reasons we note immediately below. We assume completely homogeneous probability beliefs—the reader is referred to Rubinstein [29] for a treatment of heterogeneous beliefs—because we make no attempt to model information-transfer processes that would surely arise with heterogeneity. The model we deal with is perfectly consistent with production/investment decisions (being undertaken by competitive value maximizing firms) serving to determine the endogenized dynamics of the key aggregate consumption variable.

In a general equilibrium model with complete markets, the classical first-order conditions for the individual's expected utility maximization problem imply the following basic relation between prices, probabilities, and consumption choices. For any given individual and any given time-period, the marginal utility of the consumption allocation chosen for a given state of nature is proportional to the ratio of: (1) the price of the elementary Arrow-Debreu security having an unit payoff in that state of nature, to (2) the subjective probability of that state of nature for the individual. If probability beliefs are homogeneous across individuals then—without any restriction on utility functions other than state-independence, time-additivity, and strict concavity—these first-order conditions imply that the ordering of price-probability ratios across states of nature at a given time-point is the same as the ordering of *each* individual's marginal utility of consumption and thus (by strict concavity) the inverse of the ordering of *each* individual's consumption. In order to be consistent with market clearing conditions, the ordering of price/probability ratios must be the inverse of the ordering of aggregate per capita consumption.[2] This ordering relationship is basic to the pricing results in Rubinstein [28], and in this paper.

[2] Each individual i has at time τ his expected utility $\sum_{t \geq \tau} (\rho^i)^{t-\tau} E_\tau U^i(C^i(t))$, where $C^i(t)$ is his random consumption at time t, and E_τ is the conditional expectations operator. Each individual

The same logic of market clearing also implies that if two states have the same aggregate consumption, then *each* individual has the same consumption in those two states. Thus Pareto-optimal allocations "only" require spanning of different aggregate consumption "meta-states" by available securities. Variants of these observations, along with some generalizations to less than fully homogenous beliefs, are contained in Hakansson [15], Rubinstein [27] and, in a general form, in Breeden and Litzenberger [6].[3] It should be noted that these authors all deal, strictly speaking, with finite horizon, finite state-space models with nonstochastic population. Among these restrictions, only the nonstochastic population one cannot be relaxed without impairing the relationship between price-probability ratios and aggregate consumption across states of nature.[4] (Indeed, given this, it is unimportant to distinguish aggregate and per capita consumption, since the relationship only applies to states at a given time-point.)

In a discrete-time model like that of Rubinstein [28], the above-mentioned monotonicity of the relationship between price-probability ratios and aggregate consumption is *not* sufficient to enable Rubinstein to eliminate the specific relationship of marginal utility to consumption from any of his pricing formulae. We now show that, in the limiting case of continuous-time diffusion processes, the monotonicity may be utilized to "factor out" the relationship between marginal utility and consumption, for the limited purpose of obtaining a *relative* pricing equation relating the intertemporal expected returns of securities to a risk measure that is an identifiable convariance of security returns with aggregate consumption.

Let $\{S(t)\}$ denote the states of nature at time t, $\{X[S(t)]\}$ denote time-state contingent payoffs, and $Z_{S(\tau)}[S(t)]$ be the price of the state-contingent Arrow-Debreu security to state $S(t)$ at time $\tau < t$—conditional on the state at τ, but that is often suppressed in the notation. For any asset with cash flows $\{X[S(t)]\}$, the price at $\tau = 0$ is given by

$$P_0 = \sum_{t=1}^{T} \sum_{S(t)} Z_0[S(t)]*X[S(t)] \tag{1}$$

chooses a state-contingent consumption stream $\{C^i[S(t)]\}$ for states $\{S(t)\}$ to maximize his expected utility, subject to the budget constraint $\sum_t \sum_{S(t)} Z_\tau[S(t)]C^i[S(t)] \leq W_\tau$, where W_τ is his endowment-contingent wealth at time τ, and $Z_\tau[S(t)]$ the prices at τ of elementary securities with state-contingent payoffs. The first-order conditions, in terms of marginal utilities, are that

$$\frac{U^{i\prime}(C[S(t)])}{U^{i\prime}(C(\tau))} = \frac{Z_\tau[S(t)]}{\pi_\tau[S(t)]} \forall i, \qquad t > \tau, S(t)$$

where $\{\pi_\tau\}$s refer to conditional probabilities of future states. The observations above all follow from this condition.

[3] Hakansson [15], and Breeden and Litzenberger [6] in a multiperiod model, also point out that when individual beliefs about probabilities of states of nature conditional on "meta-states" of equal aggregate (per capita) consumption are homogeneous, a set of linearly-independent-payoff securities spanning the meta-states is sufficient for "effective" completeness and Pareto-optimal allocations, since consumption allocations and prices for the existing assets would be the same in a literally complete market, in equilibrium.

[4] For a rigorous extension of the Hakansson and Breeden-Litzenberger results to models with a continuum of states, see Dothan and Williams [12].

where T represents the terminal date of the model. Although Rubinstein [28] deals with a finite-horizon model, all results discussed below, other than existence proofs, should carry over in a straightforward way to infinite horizon models, as is evident from the model of Brock [9]. Similarly, conditional on a particular state $S(1)$ at $t = 1$, the prices are given by

$$P_{S(1)} = \sum_{t=2}^{T} \sum_{S(t)} Z_{S(1)}[S(t)]X[S(t)]. \tag{2}$$

Equations (1) and (2) represent examples of what we term "levels relationships." On the other hand, the following pricing equation represents an example of a "period to period" or "intertemporal" pricing or returns equation, in our terminology,

$$P_0 = \sum_{S(1)} Z_0[S(1)]\{X[S(1)] + P_{S(1)}\}. \tag{3}$$

To show that (3) is indeed true, one may substitute from (2) for $P_{S(1)}$ in the right-hand side of (3) and utilize the (rational expectations) no-arbitrage condition

$$Z_0[S(t)] = \sum_{S(1)} Z_{S(1)}[S(t)]*Z_0[S(1)] \tag{4}$$

to obtain equation (1).[5] All of this is standard but worth clarifying at the outset.

Following Rubinstein [28], we define

$$Z_0'[S(1)] = Z_0[S(1)]/\pi_0[S(1)] \tag{5a}$$

$$G[S(1)] = X[S(1)] + P_{S(1)} \tag{5b}$$

where $\{\pi_0\}$'s are the conditional state-probability beliefs held at time zero. Using Equation (3) and the definition of covariance, we obtain that

$$\begin{aligned} P_0 &= E_0(Z_0'G) \\ &= E_0(G)E_0(Z_0') + \mathrm{Cov}_0(Z_0', G) \\ &= \frac{E_0(G)}{r_{f1}} + \frac{\mathrm{Cov}_0(Z_0', G)/E_0(Z_0')}{r_{f1}} \end{aligned} \tag{6}$$

where E_0 is the conditional expectations operator at time 0, and r_{f1} is the gross (one plus) the one-period riskless rate of interest between periods zero and one. The last step leading to (6) follows from the fact that $\frac{1}{r_{f1}} = \sum_{S(1)} Z_0[S(1)]$. Clearly, the levels relationship (1) can also be expressed in an analogous form. It is also obvious that an intertemporal pricing relationship like (6) holds, with appropriate relabeling, *between any two periods.*[6]

We can simplify (6) further by writing it in terms of gross returns $R[S(1)] = G[S(1)]/P_0$ to get

$$E_0(R) = r_{f1} + \mathrm{Cov}_0(-Z_0', R)/E_0(Z_0'). \tag{7}$$

[5] The basic characteristic of an intertemporal equation is that it relates "today's" prices to future prices as well as cash flows, over a horizon that is shorter than the full horizon of the model. Garman [13] has emphasized the nature of the restriction implied by Equation (4), and used it to develop pricing results for diffusion processes.

[6] Beja [2] exlicitly noted this form of the pricing relationship.

We define a one-period *aggregate consumption security* as one which provides a next period payoff that is perfectly correlated with the next period aggregate consumption C, and denote its payoff, price, and return by G_c, P_c and R_c. Since (7) holds for any portfolio, we have

$$E_0(R_c) = r_{f1} + \mathrm{Cov}_0(-Z'_0, R_c)/E_0(Z'_0). \tag{8}$$

Substituting (8) into (7), we obtain

$$E_0(R) - r_{f1} = \{E_0(R_c) - r_{f1}\} \frac{\mathrm{Cov}_0(-Z'_0, R)}{\mathrm{Cov}_0(-Z'_0, R_c)}. \tag{9}$$

It was discussed before that in Rubinstein's [28] model there is a strictly monotonic (decreasing) relationship between R_c and Z'_0, across states of nature at any future point in time. This follows from the fact that R_c is proportional to aggregate consumption, and Z'_0 is the price-probability ratio, having the same-ordering across states of nature as that of each individual's marginal utility of consumption. (Indeed, Z'_0 equals *each* individual's marginal rate of substitution between current and future consumption in the state.) We also needed the assumption that utility functions are time-additive and strictly concave. In what follows, we assume that each individual's utility function is thrice differentiable, so that when the states of nature map into a set of uncountably infinite aggregate consumption meta states that constitute some interval of the real line, the strictly decreasing function mapping R_c to Z'_0 is twice differentiable. The application of the above-mentioned characterization relating R_c and Z'_0 in a continuum of states model only requires a straightforward extension if we have atomless probability and price density measures across states of nature, given the restrictions on agents' utility functions assumed above, and a finite number of decision/consumption points.

Implicit in the methods used in Rubinstein's [28] model and here is the assumption that the number of time-points is finite. Brock [9] suggests that the extension of similar characterizations (of especially the relationship between R_c and Z'_0) is straightforward when the number of time-points is countable. As we proceed to the limit by decreasing the time-interval between successive decision/consumption points towards zero, then, provided all security prices (including the limiting aggregate consumption rate and the security with perfectly correlated returns) follow Ito diffusion processes, the twice differentiable strictly decreasing functional relationship between R_c and Z'_0 approaches that of perfect negative instantaneous correlation. As a result, the covariances in Equation (9) between inter-period returns and Z'_0 approach the following instantaneous covariances given by Ito's Lemma to be

$$\mathrm{Cov}(-Z', R) = \frac{-\partial Z'}{\partial R_c} \mathrm{Cov}(R, R_c) \tag{10a}$$

$$\mathrm{Cov}(-Z', R_c) = \frac{-\partial Z'}{\partial R_c} \mathrm{Var}(R_c). \tag{10b}$$

Equations (9) and (10 a-b) together imply the following instantaneous pricing relation, which does *not* involve the (local) "preference information" $\partial Z'/\partial R_c$:

$$E(R) - r_f = \frac{\{E(R_c) - r_f\}}{\text{Var}(R_c)} \text{Cov}(R, R_c)^{7,8} \tag{11}$$

An alternative way to state the *same* result is just to use the information:

$$\text{Cov}(-Z', R) = -\frac{\partial Z'}{\partial C} \text{Cov}(R, C)$$

along with Equation (7), to show that for any two assets i and j, the instantaneous expected returns are related by:

$$\frac{E(R_i) - r_f}{E(R_j) - r_f} = \frac{\text{Cov}(R_i, C)}{\text{Cov}(R_j, C)} \tag{12}$$

Equation (11) is, of course, very similar in form to the intertemporal asset pricing relationship derived by Merton [20] in his seminal paper, for the simple (assumed) case of intertemporally stochastically independent lognormal diffusion processes for returns on securities. The substitution of the instantaneous rate of return R_m of the market portfolio in place of R_c in (11) gives us Merton's equation, which is in the same spirit as the classic CAPM of Sharpe [30] and Lintner [19]. The difference here is the greater generality and the fact that the "pivotal" role is played by an aggregate consumption security rather than the market portfolio (necessarily). An alternative way to make the same point is to note that the complex multiple-factor pricing equation of the general diffusion-process model in Merton [20] has been replaced by a simple one-factor relationship, without

[7] It is worth noting that the simplification in (9) that results from Equations (10 a–b) is very similar in spirit to the essential pricing logic of the mean-variance capital asset pricing model of Sharpe [30] and Lintner [19], in a discrete-time two-period model with bivariate normality of returns. Bivariate normality of R and R_c would imply that (see Rubinstein [28]):

$$\text{Cov}(R, Z'(R_c)) = E\left(\frac{\partial Z'}{\partial R_c}\right) * \text{Cov}(R, R_c)$$

and

$$\text{Cov}(R_c, Z'(R_c)) = E\left(\frac{\partial Z'}{\partial R_c}\right) * \text{Var}(R_c).$$

Furthermore, in a *two-period* model, R_c is identical to R_m, the return on the market (aggregate wealth) portfolio.

[8] It is, of course, the instantaneous change in the flow rate of consumption that is directly related to the return on the aggregate consumption security over that instant. Given continuous diffusion process dynamics for the flow rate of consumption C and price-probability ratios Z' for different states—these ratios now being ratios of price and probability *densities* over possible future states—instantaneous perfect correlation between Z' and R_c results. Note that Z' has the local drift rate $-r_f$, and it is the Z' "originating" at unity for the current date-state that enters the *intertemporal* return equation. See Kreps [18] for a discussion of the isomorphism between equilibrium results obtained with stochastic control of diffusions, and equilibrium with one-shot expected utility maximization by agents subject to a budget constraint dependent on a linear price functional, when net trades need only satisfy subspace restrictions. The results here should be extendable to incomplete markets, where the extension of the pricing functional on the full state-space is non-unique. This is suggested by the development in Breeden [5].

any exogenous assumptions about the intertemporal process of security returns other than diffusion process dynamics. It should be re-emphasized that we have nowhere used any assumption of exogenously given dynamics for aggregate consumption. The equilibrium dynamics may well have arisen in a model in which Pareto-optimal competitive value-maximizing investment decisions endogenously determined aggregate consumption. In the sections below we apply these criteria of generality and simple tractable applicability to other recent models on multiperiod valuation.

The very appealing generality of the instantaneous relative returns Equation (11) does not, however, imply that it can be used in a straightforward way to provide "levels" pricing relationships of the type of Equation (1) in terms of a "vector" of observables that is of a lower "dimension" than the full set of (consumption-state contingent) elementary security prices. In the general case with diffusion but no utility specialization, there is a conflict between expression in simple form versus usefulness for valuation purposes of the intertemporal "instantaneous" returns equation approach, and other approaches that directly attack the problem of evaluating the "levels" relationship of Equation (1), discussed in Sections II and III. The major problem with the "instantaneous" returns relation embodied in Equation (11) is that we have no compelling reasons for assuming that either the riskless rate r_f or the "price of risk" term $\frac{E(R_c) - r_f}{\operatorname{Var}(R_c)}$ is non-stochastic —except in the very special case of aggregate power utility and a geometric lognormal random walk in aggregate consumption, discussed in Rubinstein [28]. This argues against the usefulness of (11) as a recursion relationship that can be "folded back" in a dynamic programming fashion for purposes of valuing cash flow streams. However, (11) is still potentially useful for testing the theory behind it.

II. Other Equilibrium Valuation Models

General equilibrium intertemporal pricing relationships of the "type" of (11) include the following. For the case of aggregate power utility, Rubinstein [28] obtains (using discrete-time methods) an intertemporal returns relationship similar to (11), with risk premia proportional to appropriate rate of return covariances with the market (i.e., aggregate wealth) portfolio, and not just the aggregate consumption portfolio. He assumes, in addition to aggregate power utility, that cash flows are jointly lognormally distributed with aggregate consumption, which is assumed to follow a geometric lognormal random walk. The two assumptions jointly have the effect in his model of producing a proportional mapping between aggregate consumption and the value of the market portfolio, which permits insertion of R_m in place of R_c in Equation (11). In this special case in Rubinstein's paper, the interperiod riskless rate of interest is nonstochastic over time. His assumptions also have the result of making the "market price of risk" $\left|\frac{E(R_m) - r_f}{\operatorname{Var}(R_m)}\right|$ non-stochastic.

In a related unpublished paper, Rubinstein [26] showed that random walks in aggregate consumption arose in a model with endogenized production, if real technologies had intertemporally identically distributed, constant returns to scale payoffs, and aggregate utility was of the power type. A contribution of Brock's [9] work has been to demonstrate this same result in the context of an infinite-horizon model. The reason this occurs is that, given the homotheticity of power utility and the intertemporal constancy of technology, a constant proportion of aggregate endowment is reinvested in real technologies in equilibrium. Prescott and Mehra [23] have also obtained such a result in an infinite-horizon model. The equilibrium model developed by Cox, Ingersoll, and Ross [11], in a seminal study of the term structure of interest rates, can also be used to provide this result. Cox, Ingersoll, and Ross also consider more general technologies that are briefly discussed in Section III.

A similar intertemporal "mean variance" pricing relationship in terms of the market portfolio, without (necessarily) the simplification of a non-stochastic (over time) riskless rate or market price of risk, also holds in the case of logarithmic (aggregate) utility. Both Rubinstein [29] and Kraus and Litzenberger [17], discuss such results. Merton [20] noted the result for diffusion processes and logarithmic utility, which makes all the "hedging" terms drop out in his multifactor general model. Rubinstein [29] provides an extensive discussion of both intertemporal and levels pricing relations obtainable for the logarithmic utility case.[9]

Two other recent models have considered multiperiod economies in which a market portfolio oriented mean-variance relative returns equation, (11) with R_m substituted for R_c, hold *without* completely exogenous assumptions about the intertemporal process of rates of return. The first, by Stapleton and Subrahmanyam [31], considers a discrete-time economy in which all asset cash flows are distributed multivariate normal, and investors all have negative exponential utility functions, which is a particular case of preferences that satisfy the "weak aggregation" property of Rubinstein [25]. The authors *further* assume that the interperiod riskless rate of interest is nonstochastic over time. They then show that a market-based mean variance relative returns equation holds in equilibrium with a "market price of risk," $\frac{E(R_m) - r_f}{\mathrm{Var}(R_m)}$, that is non-stochastic over time. This case, along with Rubinstein's [28] power (or logarithmic) utility with geometric lognormal random walks in consumption, constitute two situations in which the market-based risk/return tradeoff equation has stable "parameters," the riskless rate and the market price of risk, and this is favorable for the use of such a pricing relation for obtaining levels pricing results, along the lines of Myers and Turnbull [22], for example.

[9] A major difference between the logarithmic and power (aggregate) utility cases is the following. When aggregate consumption does *not* follow a geometric random walk, power utility pricing formulae of the "levels" type can be "simply" expressed only in terms of aggregate consumption, but not necessarily the market (aggregate wealth) portfolio. This is because the simplicity of the mapping between aggregate consumption and wealth breaks down. The same is *not* true of logarithmic utility—there is always a simple proportionate relation between wealth and consumption, even when the consumption dynamics are consistent with rates of return processes for risky securities, the interperiod riskless rate, and the "price of risk" that are *not* intertemporally stationary.

However, on closer examination, the simplification in pricing achieved by Stapleton and Subrahmanyam [31] appears to be neither of great incremental importance, nor as consistent with a general equilibrium set-up as Rubinstein's [28] special case. The lack of incremental importance arises from the fact that, with multivariate normality and negative exponential utility, Rubinstein's [28] results can be used to provide a very simple levels pricing relationship in terms of (a) covariance of cash flows with aggregate consumption, and (b) prices of discount bonds, *without* making any exogenous assumptions about the dynamics of the interperiod riskless rate of interest.[10] Second, the assumption of an interperiod riskless rate that is non-stochastic over time is consistent with negative exponential utility if (Gaussian) aggregate consumption follows an arithmetic or additive normal random walk over time, if we restrict ourselves to closed economies. However, in a model in which endogenized Pareto-optimal real investment decisions determine the dynamics of aggregate consumption, it is difficult to think of any simple set of assumptions about technologies that is consistent, given exponential utility, with an additive random walk in consumption.[11] If we do *not* restrict ourselves to closed economies, it is not clear why the riskless rate should be a preferred candidate for exogeneity relative to prices of risky assets.

[10] From the first-order conditions to the individual investor's problem, it follows that, for any ith investor with utility function (for each period's consumption) U^i and consumption C^i, and for each t, $\tau < t$,

$$Z'_{S(\tau)}[S(t)] = (\rho^i)^{t-\tau} U^{i'}(C^i(S(t))/U^{i'}(C^i(S(\tau))$$

where U' denotes marginal utility and $\rho^i < 1$ is the i-th investor's "impatience factor." Using this and properties of the bivariate normal distribution, Rubinstein [28] obtains the following relationships.

$$P_0 = \sum_{t=1}^{T} \frac{E(X_t) - \theta_t^i \mathrm{Cov}(X_t, C_t^i)}{R_{Ft}} = \sum_{t=1}^{T} \frac{E(X_t) - \theta_t \mathrm{Cov}(X_t, C_t)}{R_{Ft}}$$

where $\theta_t^i = E(-U^{i''}(C_t^i))/E((U^{i'}(C_t^i))$, $\theta_t = 1/\sum_i 1/\theta_t^i$ and $1/R_{Ft}$ is the price of the t-period discount bond. The valuation formula can be considerably simplified if all investors have exponential utility functions, with possibly different coefficients, i.e., if $\theta^i = \lambda_i$, then we can write:

$$P_0 = \sum_{t=1}^{T} \frac{E(X_t) - \theta\, \mathrm{Cov}(X_t, C_t)}{R_{Ft}}$$

where $\theta = \dfrac{1}{\sum_i 1/\lambda_i}$ is independent of the probabilities associated with aggregate consumption (and its distribution) at t. Of course, in view of nonnegativity constraints that are sensible for consumption, the bivariate normality case should be considered only as an approximation.

[11] Between any two adjacent (for simplicity) periods, denoted 0 and 1, without loss of generality, the one-period short rate of interest r_f satisfies the relationship:

$$\frac{1}{r_f} = E_0(Z') = \frac{E_0\{U^{i'}(C^i(S(1)))\}}{U^{i'}(C^i(0))} \cdot \rho^i \ \forall_i$$

where the i superscript stands for the ith investor and ρ^i is his/her "impatience factor" in the temporally additive utility function. If $U^i = -e^{-\lambda_i C^i}$, r_f is independent of $C^i(0)$ only if $E_0\{e^{-\lambda_i(C^i(1)-C^i(0))}\}$ is independent of $C^i(0)$, which can come about if aggregate consumption follows an additive or arithmetic normal random walk with a nonstochastic variance. With stochastic constant returns to scale stationary technologies and exponential utility, the assumption of nonstochastic riskless rates implies nonstochastic investment levels and *expected* future consumption, which is inconsistent with an additive random walk in consumption.

A second model, due to Constantinides [10], generalizes the results of Rubinstein [28], and offers what appears to be the most general set of conditions under which an interperiod or intertemporal market-based mean-variance rate of return relation holds. The conditions on preferences, technology and (assumed) homogeneous probability beliefs are as follows. Preferences must satisfy Rubinstein's [25] "weak aggregation" criteria. Risky real investment technologies must have end of period productivity distributions that are functions of *only* the level of investment and intertemporally independently distributed random variables, and no other "state variables." Further, technological uncertainty may either be of a diffusion process kind or the probability distributions of productivity may be from the family of "separating distributions" (Ross [22])—which result in two-fund portfolio separation with concave utility in a two-period problem-*provided* investment technologies are constant returns to scale. Normality of technological payoffs and negative exponential utility satisfy these conditions. But while they do give rise to a market-based mean variance returns relation, neither the equilibrium riskless rate, nor the market price of risk are nonstochastic over time, in equilibrium.

It is instructive to compare the diffusion process case of Constantinides' [10] result to our consumption-based pricing relation (11), which does *not* require similar specialization of utility. Without "weak aggregation," it is *not* possible to go from (11) to the corresponding market-based pricing relation even with Constantinides' assumptions about the intertemporal stationarity of the "real opportunity set" of investment technologies. The basic driving force behind Constantinides' result is that the equilibrium riskless rate of interest becomes a function only of aggregate wealth in his set-up. Without weak aggregation, it may also depend, given aggregate wealth, on the distribution of wealth across investors. Thus, in the terminology of Merton [20], state variables not uniquely related to aggregate wealth come to influence the full market investment opportunity set perceived by investors. Another way to make the same point is that, because of the dependence of the riskless rate on the wealth distribution, there is not (necessarily) a one-to-one mapping between aggregate wealth and aggregate consumption, in the *absence* of "weak aggregation," *despite* the assumed stationarity of the probability distributions of returns on risky assets.[12] If such a one-to-one mapping were to exist, then one could readily substitute R_m in place of R_c in the diffusion-process pricing relation (11). Cox, Ingersoll, and Ross [11] have made some headway in investigating pricing results in a model in which the "weak aggregation" property holds, but the productivity of real technologies depends on many state variables.

From the perspective of seeking robust valuation relations that do not require

[12] An apparently appealing approach to tractable relative pricing is to build models with constant returns to scale investment technologies, including a riskless one, and have endogenized competitive value-maximizing investments. That "ties down" the capital market returns to the technological (average and marginal) productivities and, if portfolio separation is ensured through, for example, normal or, in continuous time, lognormal distributions, the market price of risk is also stationary. However, for this to happen, we have to be sure that no "corner solutions" occur with respect to the technologies, particularly the riskless one, which seems implausible in general without *some* restrictions on preferences.

a great deal of utility specialization, the above observations on conditions required to have a simple market-based analog of (11) prevail come as disappointing news. Even in situations in which such an analog intertemporal returns equation holds, the nonstationarity of key "parameters" like the riskless rate and the market price of risk suggest reliance on directly derived levels pricing relations, often in terms of aggregate consumption rather than the market, as in the negative exponential utility case. Tractable reliance on market portfolio based relative pricing formulae appears justified only for logarithmic utility and for power utility with geometric random walks in aggregate consumption.[13]

However, as we noted in Section I, use of the instantaneous intertemporal consumption based relative pricing relation (11), as a recursion relation to derive levels formulae, faces analogous problems of nonstationarity. The outlook for valuation without utility aggregation is *not* completely hopeless, nevertheless. In the next section, we go on to examine the option-theoretic levels pricing methodology of Breeden and Litzenberger [6], focusing particularly on the specification and handling of intertemporally stochastic instantaneously riskless rates of interest in the option pricing methods employed. We show there that fairly robust and tractable valuation techniques centered around aggregate consumption are available, especially when technologies exhibit the type of stationarity demanded by the model of Constantinides [10], *without* recourse to a great deal of utility specialization.

III. Option-Theoretic Valuation and Interest Rates

One approach to the problem of tractable levels pricing relationships which do *not* depend on specialization of preferences, is the option-theoretic treatment of state-contingent security prices developed by Breeden and Litzenberger [6], with closely related work by Banz and Miller [1] and Garman [14] on the pricing of the "supershares" of Hakansson's [15] model. Two key observations form the basis for this approach. First, as discussed in Section I, in Pareto-optimal capital markets, across states of *equal* aggregate consumption at the same point in time, (a) each individual's consumption, and (b) the ratio of the "current" price of the state-contingent elementary security to the "current" (conditional) probability of the state, are equalized. As a result, the levels pricing relation, Equation (1) can be expressed as

$$P_0 = \sum_{t=1}^{T} \sum_{c(t)} Z_0[c(t)] * E_{c(t)}\{X[S(t)]\} \tag{13}$$

[13] It should be noted that "weak aggregation" models that allow for the possibility of marginal utility of consumption being finite at zero consumption, are of dubious generality beyond the case of homogeneous investors, because current proofs ignore the possibility of "corner solutions" with Kuhn-Tucker inequalities holding at zero consumption. The exponential utility case is a clear "culprit," and it is discussed only as an approximate model; with normality, nonnegativity is not satisfied even in the homogeneous investor case. A similar qualification exists for Rubinstein [29], except when *all* investors have strictly logarithmic utility. Outside the "weak aggregation" framework, this type of reservation does *not* apply to, for example, a model where all investors have power utilities with different powers.

where $c(t)$ is a meta (group of) state(s) of equal aggregate consumption at t, $E_{c(t)}$ are expectations conditional on $c(t)$, and $Z_0[c(t)]$ is the price at time zero of a "meta state" security that pays off one unit of the commodity conditional on (a particular level of) $c(t)$.

The second key observation constituting this approach relies on the modern theory of pricing of options or contingent claims, initiated by the seminal work of Black-Scholes [4], to provide estimates of the prices $Z_0[c(t)]$. Consider an aggregate consumption security for time t, which pays off (same arbitrary multiple of) $c(t)$ conditional on aggregate consumption being $c(t)$ at t. Its price at time 0, $V_0[c(t)]$ is

$$V_0[c(t)] = \sum_{c(t)} c(t) * Z_0[c(t)]$$

In a continuum of $c(t)$ model, this can instead be written as

$$V_0[c(t)] = \int_0^{\infty} c(t) Z_0[c(t)]\, dc(t) \tag{14}$$

where Z_0 is now a "price density." If a call option with exercise price K on the aggregate consumption security for t is written, its price $H_0[V_0, K]$ would be

$$H_0[V_0, K] = \int_K^{\infty} [c(t) - K] Z_0[c(t)]\, dc(t). \tag{15}$$

It is easy to see that

$$\left. \frac{\partial^2 H_0}{\partial K^2} \right|_{K=c^*(t)} = Z_0[c^*(t)] \tag{16}$$

where $c^*(t)$ is any particular value of $c(t)$. Hence if methods to evaluate $H_0[V_0, K]$ exist which are *not* dependent on utility specialization, *then* we have a robust pricing methodology for evaluating $Z_0[c(t)]$ and, hence, any other payoff pattern. Notice, though, that such prices are required for separate aggregate consumption securities paying off at every date in future.

At first sight, the arbitrage-based option pricing formula of Black-Scholes [4] seems to offer just such an alternative method, because it requires no direct knowledge of $Z_0[c(t)]$ to evaluate $H_0[V_0, K]$, provided we know $V_\tau[c(t)]$ to be following a diffusion process whose variance function can be estimated. However, the Black-Scholes options pricing formula assumes that the instantaneously riskless rate of interest is nonstochastic over time which, as we saw in the last section, is not a consistent general equilibrium implication for a wide class of models. Indeed, when Breeden and Litzenberger [6] fully work out their valuation procedure along the lines indicated (using the Black-Scholes formula), they find an asset pricing formula for lognormal diffusion in terms of mean cash flows, covariances with aggregate consumptions, and interest rates, that is essentially the same as that obtained by Rubinstein [28], in his special case of power utility and geometric (exponential in continuous time) lognormal random walks in aggregate consumption.

The result above is not surprising, because it is precisely this special case of Rubinstein's that produces an interperiod riskless rate that is nonstochastic over time. Indeed, by pursuing this discrete-time valuation methodology for his special case, Rubinstein [28] was able to obtain a formula for pricing options that is identical to the Black-Scholes pricing formula, namely that option price

$$H[S, K] = SN(Z^* + \sigma) - KR_F^{-1}N(Z^*). \tag{17}$$

Here, S is the current price of the underlying security, the logarithm of whose terminal price relative at the maturity of the option is distributed normally with variance σ^2, R_F^{-1} is the current price of a discount bond maturing at the same time as the option, K is the exercise price, $N(\cdot)$ is the cumulative normal distribution function, and

$$Z^* = \frac{\log_e(S/K) + \log_e(R_F)}{\sigma} - \frac{1}{2}\sigma$$

Breeden and Litzenberger [6] and Brennan [7] were able to strengthen the link between the Black-Scholes and Rubinstein models by showing that, for $H_0[V_0, K]$ to satisfy (17), it was *necessary*, given lognormality of $c(t)$ and a "weak aggregation" setting, that the aggregate utility function be of a power type.

Merton [21] extended the Black-Scholes option formula to a restrictive case of stochastic (over time) interest rates, in a case where the prices of discount bonds and underlying security prices followed "lognormal" diffusion processes with variances of rates of return being independent of price. The appropriate no-arbitrage hedging argument was obtained by considering a zero net investment "three asset hedge" of the underlying security, the option, and a discount bond of the same maturity. The restrictiveness of stochastic processes employed—quite serious because discount bond prices could rise arbitrarily above unity—arose from the constraint that the three hedge weights had to satisfy *four* constraints of (i) adding to zero, (ii and iii) "neutralizing" both sources of uncertainty, and (iv) providing zero net return. The resulting pricing formula was almost as simple as that of Black-Scholes [4], of a similar form as (17), except that now the variances of both rates of return (on the security and the bond) and their covariance were involved.[14]

Recent results in Bhattacharya and Fehr [3] and, as part of a much larger effort by Cox, Ingersoll, and Ross [11], show that arbitrage-based option pricing methodology under diffusion processes can be extended to more general stochastic interest rate processes. For example, Bhattacharya and Fehr [3] showed that, if the term-structure of discount bond prices were Markov with one (stochastic) state variable, conveniently taken to be the instantaneously riskless rate r, then

[14] It is important to point out that the Merton [21] case is quite distinct from the Rubinstein [28] case, which results in a formula identical to Black-Scholes. The distinction is blurred by the fact that Rubinstein only mentions joint lognormality of aggregate consumption and the security price, without explicit mention of geometric random walks in the former, in the segment on option pricing. However, without a geometric random walk—nonstochastic proportionate expected increments which produce nonstochastic interest rates with power utility—lognormality of aggregate consumption is not possible for *every* future date. It is as yet unclear as to precisely what general equilibrium diffusion model special case gives rise to the Merton [21] special case of interest rate dynamics.

the following pricing result could be obtained. For options or any general contingent claim on an underlying security S, Merton's [21] three-asset hedge can be extended to a four-asset hedge, made up of S, the option, a discount bond of same maturity, and the instantaneously riskless asset, to provide a Black-Scholes [4] type differential equation for pricing that can be solved subject to appropriate boundary conditions.[15] The differential equation requires data on the current values of all the assets in the hedge, the variances and covariances of returns on the security and the discount bond, *and* the levels relationship between the instantaneously riskless rate and the price of the discount bond. No explicit knowledge of the expected returns on the security and the bond are required, as with the earlier dynamic hedging arguments. The solutions to the differential equation are not as *analytically* tractable as those for the Black-Scholes-Merton special cases. The methodology is robust, in that if the term-structure of interest rates follows an n-state-variable diffusion process, and there is an n-dimensional basis of traded asset prices whose movements span these state variables, then an $(n + 3)$ asset hedge can be constructed.

It appears to us that a "marriage" of the "four-asset hedge" type of option valuation procedure with the option-theoretic procedure of Breeden and Litzenberger [6], is capable of providing a practical and robust method for generating consistent $Z_0[c(t)]$ prices. This is true provided aggregate consumption securities, more precisely the prices $V_0[c(t)]$ and the variances of their dynamics, can be estimated/approximated reasonably well. The procedure is practical for three reasons: (1) the analysis of financial market data can be centralized as a table of $Z[c(t)]$ with practitioners "only" needing to estimate $E_{c(t)}\{X[S(t)]\}$, the expected cash flows conditional on aggregate consumption, in (13); (2) the four asset hedge differential equations in three variables can be numerically solved fairly generally; and (3) the data requirements do *not* include estimation of expected returns. In effect, the procedure does not directly require full knowledge of the joint distribution of all the relevant variables, and thus it achieves the kind of "informational economy" for the analyst that Black-Scholes [4] first surprised us with. (For an alternative approach requiring more structure see Dothan and Williams [12].)

Two main questions arise regarding the issue of generality, which we interpret mainly in the sense of not requiring utility specialization. The first point concerns the fact that, even assuming a one-state variable process for the term structure

[15] Bhattacharya and Fehr [3] show that the contingent claim/option price H on a stock of price S satisfies the differential equation (and appropriate boundary conditions)

$$\tfrac{1}{2}\,\sigma^2 S^2 H_{SS} + \rho\sigma VSPH_{SP} + \tfrac{1}{2}\,V^2P^2H_{PP} - H_\tau + r(SH_S + PH_P - H) = 0$$

where subscripts denote partial derivatives, τ is maturity remaining, σ^2, V^2 are the variances of instantaneous rates of return on the stock S and the discount bond of price P having the same maturity as the option, ρ is the instantaneous correlation coefficient between the dynamics of S and P, and r is the instantaneously riskless rate of interest which is *functionally related* to P because the term structure is *assumed* to follow a one-state-variable Markov process. σ, V and ρ are allowed to be functions of S and P. If *none* of them are, i.e., each is a possibly time-dependent parameter—an unrealistic assumption for the dynamics of the discount bond since then its price can rise arbitrarily above unity—then the bracket multiplying r drops out because H becomes homogeneous of degree one in S and P, as shown by Merton [21]. The Black-Scholes [4] solution arises when P has nonstochastic dynamics. Note that all instantaneous expected rates of return drop out of the equation.

of discount based prices, the particulars of the levels relationship between discount bond prices and the instantaneous riskless rate do depend on the utility functions of investors, as Cox, Ingersoll, and Ross [11] have noted. Nevertheless, sound empirical methods may proceed on the basis of estimated relationships, rather than delving deeply into the theoretical levels relationships, the latter task being tractable only for some special cases of "weak aggregation," mainly logarithmic utility in conjunction with assumptions about technology. The second question that arises about generality relates to the assumption of a one-state-variable process for the term structure of discount bond prices. It is easy to show that, if the productivity of real investment technologies is constant returns to scale with probability distributions that are stationary, then aggregate consumption follows a one-state-variable Markov process.[16] As a result, the prices of discount bonds of all maturities follow one-state-variable Markov processes with respect to aggregate consumption. This is because the price of a discount bond maturing at t as τ is just the sum/integral of $Z_\tau[c(t)]$ over $c(t)$, conditional on $c(\tau)$, and the ratio of $Z_\tau[c(t)]$ to the conditional probability (density) of $c(t)$ is proportional to the ratio of marginal utility of each investor's consumption in the $c(t)$ state to that in the $c(\tau)$ state. See footnotes 10 and 11.

There has been only one comprehensive effort to implement option-theoretic "state-price" calculations empirically to date. Banz and Miller [1] have recently implemented calculations of "state-prices," with two important differences with the methodology elaborated here: (1) they have considered the market portfolio, empirically a proxy for the aggregate wealth portfolio with value $\sum_{t>\tau} V_\tau[c(t)]$ at time τ, as the underlying "security"; and (2) their treatment of stochastic interest rates is somewhat heuristic, in that they calculate state-prices for "small" intervals using the Black-Scholes formula for a deterministic interest rate, but allow the level of the interest rate to be a state variable that conditions the parameters for their calculations. The first difference is of consequence because we do not generally expect a one-to-one mapping between aggregate consumption and aggregate wealth, without "weak aggregation" of preferences, even if real technologies have stationary productivity distributions, because the riskless rate of interest can depend on the wealth distribution. As to the extent of error this creates, especially relative to methods which would have to be based on imperfect empirical proxies for aggregate consumption securities, it is very difficult to provide an answer. However, the second difference, *vis-à-vis* accurate implementation of the information on interest rate dynamics in the pricing model, should be eliminated to improve the price estimates.[17]

[16] Using the methods of Brock [9], it is easy to show (analogous to his Equation 3.14) that for any ith investor, and any active jth risky technology, marginal utilities of current and future consumption at the next date at a Pareto-optimal allocation (produced by competitive value-maximizing investment decisions) are linked by

$$U^{i\prime}(C_t^i) = \rho^i E_t\{U^{i\prime}(C_{t+1}^i)R_{j,t+1}\} \;\forall\; i, j.$$

Here ρ^i is the ith investor's impatience factor in his/her temporally additive utility function, and $R_{j,t+1}$ is the random (marginal) productivity of real investment in the jth technology. Consider two states A, B at time t of the same aggregate C_t. For reasons disccussed before, each state has the same C_t^i for every i. As a result, the investment allocation of State A that satisfies the system of equations above

IV. Concluding Remarks

We have attempted to provide a wide-ranging, integrative discussion of some closely related topics in multiperiod valuation theory. As stated in the beginning, our goal has been to highlight contributions that pertain to the design of fairly robust valuation methodology. The incorporation and discussion of various contributions are to be understood in the context of this goal, which has been adhered to the possibly detrimental exclusion of elaborating many interesting specific pricing results in a comprehensive and rigorous fashion. The reader is especially referred to Cox, Ingersoll, and Ross [11] for an extremely rich discussion along such lines. Extensions that consider costly market-making, information costs, etc., are also desirable—for a beginning, see Hakansson [16].

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is also a solution to the corresponding system of equations in State B, provided $R_{j,t+1}$ has the same probability distribution at A and B. As a result, the probability distribution of C^{i}_{t+1}, C_{t+1} is the same conditional on B as it is conditional on A.

[17] As Michael Brennan has pointed out to me, although Banz and Miller [1] define their future states in terms of the market portfolio *value*, their multiperiod implementation, at least in their examples, appears to assume that states are defined in terms of the previous period rate of return. Thus, in multi-period implementation, two paths that appear to lead to the same state really do not, necessarily. Furthermore, definition of states in terms of the market portfolio value is essentially equivalent to using the market-based mean-variance CAPM—Equation (11) with R_m substituted for R_c—as a recursion relationship.

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