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# Investment Policy Implications of the Capital Asset Pricing Model

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## ABSTRACT

The results of previous generalized Security Market Line (SML) tests of the Mean Variance (MV) and Linear Risk Tolerance (LRT) Capital Asset Pricing Models indicate that the models are empirically identical. A very widely accepted, but technically incorrect, explanation for the results is that with normal return distributions all expected utility maximizing risk-averse investors will pick MV portfolios. The paper shows that the generalized SML tests cannot distinguish between the MV model and a much wider variety of power utility LRT models than has previously been entertained. On the other hand, with approximately normal, or real world, return distributions the investment policies of the various models are shown to be different from each other, and from the MV policy in particular. To the extent the results of the portfolio selection calculations are robust, the results of, and implications drawn from, the tests of the macro pricing relations are not based on firm micro foundations.

EMPIRICAL TESTS OF POSITIVE theories of asset pricing have focused primarily on the linear risk return trade-off predicted by the mean variance capital asset pricing model (MV CAPM). The tests (Friend and Blume [9], Black, Jensen, and Scholes [2], Blume and Friend [4], Fama and MacBeth [7]) have provided the model with less than full-fledged support. Moreover, from a theoretical view, the model suffers from a number of well-known deficiencies. These two factors have led several authors (Roll [23], Rubinstein [25, 26], Hakansson [16], and Grauer [11]) to suggest that one of the power linear risk tolerance (LRT) utility functions may form the basis of a better positive theory of asset pricing, in particular, one consistent with risky assets being normal goods. But so far the results from more general empirical risk return tests have neither supported nor failed to support the suggestion. For example, Roll [23], based on weekly stock market data from the 1960s, attempted to distinguish between the growth optimal CAPM (which is consistent with logarithmic utility (only) and which in turn can be viewed as a special case of the LRT models) and the MV CAPM, by comparing the slopes and intercepts of the generalized security market lines (SMLs) predicted by the two models. The results indicated that at the macro level the two models were empirically identical. In a broader study of linear risk return trade-offs, that covered the 1934–71 period and employed monthly data, Grauer [11] did not find

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any statistical difference between six LRT CAPMs that included quadratic utility and powers ranging from .5 (square root utility) through zero (logarithmic utility) to $-5$.

Subsequently, both Roll [24] and Grauer [13] pointed out fundamental problems in testing the equilibrium implications of the CAPMs using the standard linear risk return methodology. Yet apparently most of the profession is of the opinion that the LRT models are, for all intents and purposes, empirically identical, and identical to the MV model in particular.[1] Some writers have attempted to explain the macroeconomic results in terms of more basic microeconomic (portfolio selection) considerations. For example, in comparing the growth optimal and MV models Fama and MacBeth ([8], p. 859) state:

> Given normally distributed one-period percentage portfolio returns, the Efficient Set Theorem applies. The growth-optimal portfolio is just the specific mean-variance portfolio that is optimal for the log utility function. Thus the two-parameter and growth-optimal models are mutually consistent.

Presumably, because security returns are approximately normally distributed and because the authors believe power utility functions can be closely approximated by quadratic utility functions, they would argue that the lack of difference found in the empirical tests was to be expected. But this may be a premature conclusion. First, the statements cited are technically incorrect. With exact normal distributions an expected utility maximizer, a growth optimizer with logarithmic utility, or any expected utility maximizer with a power utility function below, in (1) will invest exclusively in the risk-free asset. The reason is straightforward. If an investment is made in risky assets, end-of-period wealth will be normally distributed. But the power utility functions are not defined over the whole wealth axis. Therefore, the investor will commit all his funds to the risk-free asset. (See Hakansson [16]). If, on the other hand, returns are approximately normal the MV and power utility investment policies may be similar but that must be determined. This is the central point of this paper. Given that the generalized SML methodology used to generate the macro evidence is itself suspect, we may be premature in using either the normality argument or the apparent empirical identity of the models at the macro level to infer that the models are essentially identical from a portfolio construction viewpoint.

The paper proceeds as follows. Section I focuses on the macro implications of the LRT models. Two modest extensions of the evidence presented in the Roll and Grauer papers are noted. First, the results indicate that the generalized SML tests cannot distinguish between the MV CAPM and selected LRT CAPMs with powers ranging as low as $-30$. Second, the results appear insensitive to the choice of either an equally or value-weighted market proxy. Section II, which focuses on the micro implications of the LRT models, forms the central part of the paper.

[1] This paper is concerned exclusively with discrete-time models. It is well known that the assumption of a stochastic process of the Ito variety in continuous time leads to an MV CAPM. (See Merton [21]). It is also useful to recognize that, while some argue that the close empirical similarity of the models can be explained in terms of the LRT models reducing to the MV model in continuous time, the proposition has not been tested. With the exception of a test of Merton's continuous-time MV model by Jensen [18], the empirical literature has focused on testing discrete-time models.

Because the equilibrium pricing equations are derived from investors' more basic micro investment choices, we might reasonably expect that, if there is no difference in the models at the macro level, there should be (little or) no difference in the models at the portfolio selection level. In order to illustrate that this does not necessarily follow, the MV and power utility investment problems and two means of generating discrete state opportunity sets are formulated. With short sales constraints, the mix of risky assets for utility functions with powers greater than zero differs from the mix of risky assets for negative power utility functions and the MV model. Moreover, margin constraints are shown to be binding for the more restrictive, but widely studied, isoelastic utility functions with powers greater than zero. Naturally, the binding margin constraints accentuate the differences in the risky asset mix. Section III contains a summary.

## I. The Macro Focus: The Linear Risk Return Trade-offs

The standard assumptions that borrowing and lending can take place at the risk-free rate; all investments are infinitely divisible and can be sold short; taxes are nonexistent; and all investors share homogeneous beliefs and identical decision horizons, allow the derivation of the LRT CAPMs from the LRT utility functions for which the separation property holds, namely,

$$u_i(w) = 1/\gamma(w + a_i)^\gamma, \qquad \gamma < 1; \tag{1}$$

$$u_i(w) = -(a_i - w)^\gamma, \qquad \gamma > 1 \quad \text{and} \quad a_i \text{ large}; \tag{2}$$

$$u_i(w) = -\exp(a_i w), \qquad a_i < 0; \tag{3}$$

where the $a_i$ are allowed to vary among investors to reflect differing degrees of risk aversion, but, with the exception of negative exponential utility, a common $\gamma$ value must be shared by all investors.[2]

Let $z_j$ be the dollar amount invested in asset $j$ (asset 1 being riskless), $w_0$ and $w_1 = \sum_{j=2}^{J} z_j(r_j - r) + w_0 r$ be initial and end-of-period wealth, $r_j$, $r$, and $r_m$, be unity plus the rate of return on asset $j$, the riskless asset, and the market portfolio, respectively, $E(\cdot)$ be the expectation operator, and $\text{cov}(\cdot)$ be covariance.

In LRT economies, where the aggregation property holds,[3] Rubinstein [25] derived a generalized valuation equation that was given empirical content by Grauer [11]. For the power and quadratic families of the LRT utility functions the generalized valuation equation is given by:

$$E(r_j) = r + [E(r_m) - r]\text{Risk}_j, \tag{4}$$

where $\text{Risk}_j = \dfrac{\text{cov}(r_j, (d + r_m)^c)}{\text{cov}(r_m, (d + r_m)^c)}$, $c = \gamma - 1$, and $d = a/w_0$.

Different values of $c$ and $d$ yield a series of simple two-parameter valuation equations that linearly relate "reward" (expected return) to "risk," but only for

[2] Quadratic utility is a special case of (2), where $\gamma = 2$ and (1) reduces to $\log(w + a_i)$ when $\gamma = 0$.

[3] Rubinstein's aggregation theorem provides alternative sets of sufficient conditions under which equilibrium security rates of return are determined as if there existed only identical individuals, whose resources, beliefs, and tastes are composites of the (at least partially) heterogeneous resources, beliefs, and tastes of actual individuals in an economy.

the special case of quadratic utility ($c = 1$, $d = -1$) will the risk measure be the familiar beta coefficient.[4]

An empirical analogue of (4) is given by the cross-sectional regression equation

$$\bar{r}_j - \bar{r} = \gamma_0 + \gamma_1 \text{Risk}_j + e_j, \tag{5}$$

where $\bar{r}_j - \bar{r}$ is the average realized excess return on a security, and $\text{Risk}_j$ is a risk measure estimated from ex post data. If one of the LRT models describes security pricing, then from (4) and (5), the intercept should equal zero and the slope should equal the excess return on the market, $\bar{r}_m - \bar{r}$. The model that yields the best estimate of $\gamma_0$ and $\gamma_1$ should then provide the best estimate of the market's risk aversion characteristics.

As noted earlier, Roll and Grauer found no difference between the MV, square root, log, generalized log, $-1$, and $-5$ power CAPMs. But it is possible that if more risk averse (say $-10$, $-20$, and $-30$) powers or a value- (rather than an equally) weighted market proxy were considered, differences in the models might be detected. To check out these possibilities the usual portfolio grouping procedures were followed to form twenty portfolios. The twenty portfolios consist of New York Stock Exchange common stocks, contained on a merged Compustat and Center for Research in Security Pricing data base, formed into portfolios on the basis of historical beta estimates calculated against an equally weighted market index in formation periods.[5] (The data comprise a subset of the data used by Grauer [11]). The returns on the portfolios were then calculated from the underlying securities in the portfolios in subsequent test periods. The eight sets of four equal subperiods and one full period risk measures (composed of five of the six risk measures previously considered by Grauer plus the $-10$, $-20$, and $-30$ powers), used in (5), were calculated over the corresponding four equal subperiods and the full period from the time series of portfolio returns and the equally and value-weighted market portfolio proxies. The proxy for the risk-free rate was the one-month rate on U.S. Treasury Bills subsequent to 1941 and the monthly rate on Banker's Acceptances prior to 1941.[6]

[4] For quadratic utility saying that $d \equiv a/w_0 = -1$ is a notational convenience to emphasize the similarity between the models. For the power utility models, the constant $d$ is a well-defined economic quantity that guarantees borrowing and lending cancel in aggregate. In the case of quadratic utility, $d$ is not defined, but as it is a constant, it does not affect the covariance.

[5] See Jensen [18] for a summary of the research on the MV CAPM. Blume [3], Friend and Blume [9], Miller and Scholes [22], Black, Jensen, and Scholes [2], Blume and Friend [4], Roll [23], and Fama and MacBeth [7] have originated and developed the econometric procedures employed here.

[6] The careful reader will note slight differences between the test procedure described here and in Grauer [11]. The tests in that article correspond to six replications of Fama and MacBeth. The individual securities were grouped into portfolios six times, according to each of the six risk measures considered there. Then, instead of running (5), as in this paper, in average return form, once for the full period and once in each of the four subperiods, an analogue of (5) was run in return form in each of the 456 months of the sample. The tests were then performed on the time series of the slope and intercept terms generated from this process. In this paper, we used the return data for twenty portfolios grouped by beta alone (according to the Fama-MacBeth technique as described in Grauer) and employed a testing procedure similar to Friend and Blume, Black, Jensen, and Scholes, and Roll. The reason for the change in the macro test methodology (and the need for the replication) is to be able to produce macro results that are directly comparable to the micro results, which in turn must be calculated on one (rather than six) portfolio grouping. The replication has two added benefits. It allows us to investigate whether the close empirical similarity of the models holds over a wider range of much more risk averse powers and whether the choice of an equally weighted proxy for the market portfolio may have contributed to our inability to distinguish between the models.

To conserve Journal space and because the results are similar to those reported in Roll and Grauer, we simply summarize them.[7] The results of the cross-sectional regressions (5) showed that all the models receive mixed support with the strongest support occurring in the first and fourth subperiods. This occurred whether an equally or a value-weighted index was used as a proxy for the market portfolio. Moreover, the similarity of the results across all the regression statistics for all the models was striking. Perhaps more surprising, the −20 and −30 power models appeared to provide the closest estimates of the theoretically correct slope and intercept terms.

The close similarity of the regression results across all the models can be appreciated when it is noted that over any time period for a given market proxy, the eight sets of LRT measures were nearly identical to each other, i.e., not just highly correlated with, but virtually identical to each other.[8] The reason for this close similarity of the risk measures is not immediately apparent. For example, it may have been caused by the 'compactness' of the returns over the sample period. And the use of portfolios, whose returns tend to be more symmetric than those of individual securities, may have contributed to our inability to distinguish between the models. In any event, the results indicate that the models receive mixed support and are indistinguishable whether one employs an equally weighted, or a more desirable, value-weighted proxy for the market portfolio.

However, one must be careful in interpreting this evidence. The MV (and implicitly the generalized) SML test methodology has not been without its critics. Fama [6] suggested that no unambiguous test of the MV CAPM has appeared in the literature because of the poor choice of the proxy for the market portfolio. Presumably, then, results generated from a value-weighted rather than an equally weighted market proxy might be judged to be more meaningful. Yet Roll [24] has raised a more fundamental issue suggesting that the SML tests have little or no meaning unless the "true" market portfolio is observed. For this reason, it is suggested that the macro results can only be judged to appear to support the contention that the MV and LRT CAPMs are virtually identical.

This is the central issue addressed in the paper. Can we take the macro evidence at face value? The majority of the profession appears to believe we can. Recognizing that the macro pricing equations are derived from more basic investment behavior, it is commonly argued that with (approximately) normal return distributions all risk-averse investors will pick (approximately) MV portfolios. But if it can be demonstrated that, with approximately normal or real world return distributions, the investment policies differ with the degree of investor risk aversion, then we have reason to question the conventional wisdom. In other words, if the investment policies differ, the macro results are without micro foundations. Perhaps more important, we may be premature in using the apparent empirical identity of the models at the macro level to infer that the models are essentially identical from a portfolio construction viewpoint.

[7] However, the full set of regression results is available from the author upon request.

[8] However, it is interesting to note that the risk measures calculated against the value-weighted market proxy were uniformly larger than the same risk measures calculated against the equally weighted market proxy. Yet, because the value-weighted market proxy has both a smaller average return and standard deviation than its equally weighted counterpart, the results from (5) generated from the two proxies were essentially identical.

## II. The Micro Focus: The Investment Policies

If there is no empirical difference between the MV and LRT models at the macro level, and if one derives the models on the basis of the expected utility maximizing behavior of investors at the micro level, then we might reasonably expect there should be (little or) no difference in the models' investment policies. While it may seem to be a simple proposition to compare investment policies, information (the generation of a complete joint probability distribution) and computation problems associated with the expected utility model have seriously impeded attempts to compare the investment policies derived from the MV and LRT models.[9] In an attempt to overcome these problems a sophisticated numerical analysis routine (Best [1]) is combined with two ways of estimating joint return distributions. First, the joint return distributions are estimated using a market model that employs a discrete approximation to normally distributed market returns. Second, the return distributions are simply estimated as historical joint frequency distributions. The central idea in varying the method of estimating the return distributions and the constraints imposed on the investment problems is to show that, while the specific investment policies may be sensitive to changes in the joint return distributions, for any particular joint return distribution the investment policies are different and these differences do not depend on any one specific set of assumptions.

### A. *The Formulation of the MV and Power Utility Investment Problems*

*The MV model.* Lintner's [19] algorithm is employed to solve the MV portfolio selection problem. Let $x_j$ be the proportion of the amount invested in asset $j$ to the total amount invested in risky assets. Maximizing the ratio of excess return to standard deviation on the portfolio leads to the optimal unlevered MV portfolio. Formally the MV portfolio selection problem with a risk-free asset is:

$$\max \theta = \frac{\sum_{j=2}^{J} x_j E(r_j) - r}{[\sum_{j=2}^{J} \sum_{k=2}^{J} x_j x_k \sigma_{jk}]^{1/2}}, \tag{6}$$

subject to $\sum_{j=2}^{J} x_j = 1$, and, with no short sales, $x_j \geq 0$ for all $j$.

*The power utility models.* The expected utility investment problem is characteristically formulated as a one-step procedure with the degree of leverage determined at the same time as the investment in risky assets. Let $v_j$ be the fraction of wealth invested in asset $j$, $j = 1, \cdots, J$, $\pi_s$ the probability of occurrence of state $s$, and $m_j$ the margin requirement on asset $j$. Assuming equal borrowing and lending rates the unconstrained power utility investment problem can be formulated as:[10]

[9] Dexter, Yu, and Ziemba [5], Maier, Peterson, and VanderWeide [20], and Grauer [10, 12] provide exceptions. The formulation of the discrete market model, with approximately normal returns considered here, follows Grauer [10].

[10] An unconstrained policy refers to a lack of constraints on the fractions of wealth invested in the risky assets. The formulation of the expected utility problems essentially follows Hakansson [14]. Hakansson and Miller [17] have proven that (8) is never binding at the optimum. Technically, with no short sales or margin purchases, and $r_{js} \geq 0$ for all $j,s$ (8) and $v_j \geq 0$ for all $j$, together are redundant. Nonetheless, it proves convenient to program (8) explicitly into all the expected utility problems, even if it is redundant in some cases, so that the numerical analysis routine does not consider a negative wealth level as it searches for the optimum policy.

$$\max_{\{v_j\}} \sum_s \pi_s 1/\gamma(\textstyle\sum_{j=2}^J v_j(r_{js} - r) + r)^\gamma, \qquad \gamma < 1 \tag{7}$$

subject to

$$\textstyle\sum_{j=2}^J v_j(r_{js} - r) + r \geq 0 \qquad \text{for all} \quad s. \tag{8}$$

With no short sales we have the added constraints that $v_j \geq 0$ for all $j$. With margin purchases and no short sales the power utility problem is given by (7) subject to (8) and

$$v_j \geq 0 \qquad \text{and} \quad \textstyle\sum_{j=2}^J m_j v_j \leq 1. \tag{9}$$

The fraction of wealth borrowed or lent[11] is given by $v_1 = 1 - \sum_{j=2}^J v_j$.

## B. *The Opportunity Set Described in Terms of Discrete State Probability Distributions*

### 1. Opportunities Estimated from a Discrete Market Model, with Approximately Normal Market Returns

There are two steps involved in this method of estimating returns: identifying a set of states and generating the returns in each state. First consider a discrete normal approximation to a market index as a means of generating a state set. Let $r_{js}$ be unity plus the rate of return on security $j$ in state $s$, $r_{ms}$ be unity plus the rate of return, and $\sigma_{r_m}$ be the standard deviation on the market. As the construction of a forecasting model of market returns is beyond the scope of the paper, the parameters used in estimating the mean and standard deviation of market returns are taken to be the historic time series average and standard deviation. Finally only for concreteness let $S$ equal twenty-five. The states[12] are centered at

$$r_{ms} = \mu_{r_m} + k_s\sigma_{r_m} \qquad k_s = -3.0, -2.75, -2.5, \cdots, 2.75, 3.0. \tag{10}$$

The second step is to determine the returns in state $s$. To do so, consider the historical market model given by

$$r_{jt} = \alpha_j + \beta_j r_{mt} + e_{jt} \qquad j = 2, \cdots, J \tag{11}$$

where $\alpha_j$, $\beta_j$, and $e_j$ are parameters specific to asset $j$. It is assumed that the

[11] The discussion of the investment policies in the text is structured in terms of investors with power utility functions exhibiting constant relative risk aversion. However, in the absence of binding margin constraints for a given power in (1), the $v_j$ are invariant across individuals, i.e,, whether $a_i = 0$ or not. This is one statement of the separation property of the LRT utility functions. The dollar amount invested by individual $i$ in risky asset $j$ is then determined as:

$$z_j^i = v_j\left(w_0^i + a_i/r\right),$$

while the amount borrowed or loaned is determined from the budget constraint

$$z_1^i = w_0^i - \textstyle\sum_{j=2}^J z_j^i.$$

On the other hand, if the margin constraints are binding, the solutions, the $v_j$, are only valid for the specific constant proportional risk aversion cases examined in the text.

[12] The probability of the twenty-three interior states occurring is calculated from the normal probability tables to be $Pr\{\mu_{r_m} + (k_s - .125)\sigma_{r_m} < r_{ms} < \mu_{r_m} + (k_s + .125)\sigma_{r_m}\}$, $k_s = -2.75, -2.5, \cdots, 2.75$. The remaining probability is then swept into the two outlying states.

random disturbances $e_j$ have the properties that

$$E(e_j) = 0 \tag{12a}$$

$$\text{cov}(e_j, e_k) = 0 \qquad j \neq k \tag{12b}$$

$$= \sigma^2_{e_j} \quad j = k \tag{12c}$$

$$\text{cov}(e_j, r_m) = 0 \qquad j = 2, \cdots, J. \tag{12d}$$

It is also assumed that $e_j \sim N(0, \sigma_{e_j})$. The return on security $j$ in state $s$ is then generated from a temporal or ex ante version of the market model

$$\begin{aligned} r_{js} &= \hat{\alpha}_j + \hat{\beta}_j r_{ms} + e_{js} \qquad && r_{js} \geqq 0 \\ &= 0 && \text{otherwise} \end{aligned} \tag{13}$$

where $\hat{\alpha}_j$ and $\hat{\beta}_j$ are parameters estimated from the time series regression of $r_{jt}$ on $r_{mt}$, i.e., from (11), and $e_{js}$ is calculated as a normal zero one random deviate times the standard error of estimate from the time series regression. Equation (13) explicitly recognizes that, even though we are modeling (approximate) normality, limited liability constrains unity plus the rate of return on common stocks to be greater than or equal to zero.

This way of estimating the joint return distribution represents a simple twist on Sharpe's [27] market model. Its appeal is that it uses two widely accepted means of describing security return distributions, the market model and approximate normality, in a way that is computationally tractable.

2. Opportunities Estimated from Historical Joint Frequency Distributions

An even simpler way to estimate a joint return distribution is to use a historical joint frequency distribution. One reason for making this type of estimate is that it involves no closed-form distributional assumptions. A second reason is to show that, while the investment policies may be sensitive to the way in which the return distributions are estimated, for any particular formulation of the joint return distributions, the investment policies differ.

### C. *The Results*

The investment policies are presented in Tables I–III.[13] The following elements are varied in the tables: (1) the way of estimating the return distribution, i.e., the market model method, using both equally weighted and value-weighted proxies for the market, and the historical joint frequency distribution method; (2) the constraints imposed on the investment problems, i.e., policies with and without margin constraints are noted; and (3) the time period over which the return distributions were estimated, i.e., the full period and fourth subperiod results are reported. In each period, the risk-free rate is taken to be the average risk-free rate in effect during that period. The title of the table indicates the period over which the return distributions were estimated and whether or not margin con-

[13] The results presented in Tables I–III are a representative subset of the investment policies actually calculated.

straints were imposed. Each panel of a table reports the characteristics of the investment policies based on a particular return distribution. The body of the tables records, first, the mix of risky assets. (To conserve space, only the nonzero entries are reported). For the MV model, the mix is calculated directly from (6). For the power utility models, the mix is calculated by forming the set of ratios, the $\{v_k / \sum_{j=2}^{J} v_j\}$, where the $v_j$'s are the solution to (7). Next, the sum of the $v_j$'s is recorded. In the case of constant proportional risk aversion ($a_\iota = 0$ in (1)), this gives the fraction of wealth invested in risky assets. Alternatively, one minus the sum of the $v_j$'s gives the amount borrowed or loaned. Although we will be concerned almost exclusively with the mix of risky assets, this information implicitly allows one to judge the degree of leverage employed by different constant proportional risk-averse investors. Finally, the expected monthly rate of return and standard deviation of return for the unlevered portfolios (i.e., the mix of risky assets) are reported.

Panel A of Table I shows the mix of risky assets for the MV and various power utility models when the joint return distribution is generated from a discrete market model and an equally weighted market proxy. Note that the mix of risky assets is very similar for the negative power utility models. But the mix for the positive powers differs from the mix for the MV and other power utility models. More specifically, the .6 power utility function's mix of (.461, .539) invested in portfolios 9 and 14 differs from the mix (.243, .757) exhibited by the −30 power function and the mix (.280, .720) of the MV model. Differences also show up in the expected monthly returns and standard deviations of the optimal unlevered portfolios, but the differences in these two dimensions are not as pronounced.

Panel B shows the mix of risky assets when a value-weighted market proxy is used to generate the joint return distribution. As one would expect, given that the forecasted joint return distributions differ from panel A to panel B, so does the mix of risky assets. But the primary observation remains: for a given approximately normal return distribution, the mix of risky assets differs across the power utility functions and the MV model.

Comparisons without margin constraints appear to be most in the spirit of models of security pricing that ignore such factors as differential tax rates, transaction costs, etc. Nonetheless, one cannot help but notice that the positive power utility models, with constant proportional risk aversion $a_\iota = 0$, (and certainly those with $a_i > 0$), would be bumping into real world margin constraints. Note, for example, from either panel A or B, someone with logarithmic utility would want to invest more than six times his wealth in risky assets. With real world margin constraints, this would not be possible. Let us therefore consider the case where there is a 50 percent margin constraint. It is important to recognize, however, that with the added realism of binding margin constraints, we will be comparing the mix of risky assets for a more restricted set of models. With binding margin constraints, the separation property no longer holds, and the mix of risky assets is only valid for the constant proportional risk aversion utility functions from which it was calculated.[14]

Panels A and B of Table II show the mix of risky assets for the same two joint-

[14] See fn. 11.

**Table I**

Investment Policies not Subject to Margin Constraints, Based on Returns Generated from the 1934–71 Period[a]

| | Power | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | .6 | .5 | .25 | 0 | −5 | −10 | −20 | −30 |
| **Panel A: Characteristics of Policies Based on Returns Generated from a Discrete Market Model and an Equally Weighted Market Proxy** | | | | | | | | | |
| Portfolio Number 9 | .280 | .461 | .395 | .280 | .241 | .239 | .241 | .242 | .243 |
| Portfolio Number 14 | .720 | .539 | .605 | .720 | .759 | .761 | .759 | .758 | .757 |
| Sum of $v_j$[b] | | 7.508 | 7.406 | 6.988 | 6.240 | 1.253 | .684 | .358 | .243 |
| $E(r_p) - 1$[c] | .02142 | .02151 | .02148 | .02142 | .02140 | .02140 | .02140 | .02140 | .02140 |
| $\sigma_{r_p}$[d] | .05225 | .05330 | .05291 | .05225 | .05252 | .05252 | .05252 | .05252 | .05252 |
| **Panel B: Characteristics of Policies Based on Returns Generated from a Discrete Market Model and a Value-Weighted Market Proxy** | | | | | | | | | |
| Portfolio Number 9 | .423 | .400 | .370 | .372 | .350 | .360 | .363 | .365 | .365 |
| Portfolio Number 14 | .577 | .600 | .630 | .628 | .650 | .640 | .637 | .635 | .635 |
| Sum of $v_j$[b] | | 7.539 | 7.336 | 7.040 | 6.104 | 1.188 | .649 | .340 | .230 |
| $E(r_p) - \bar{1}$[c] | .02098 | .02092 | .02085 | .02085 | .02080 | .02082 | .02083 | .02083 | .02084 |
| $\sigma_{r_p}$[d] | .05298 | .05284 | .05266 | .05267 | .05256 | .05261 | .05262 | .05263 | .05263 |

[a] The table records the mix of risky assets, i.e., the $\{v_k/\sum_{j=2}^{21} v_j\}$. To conserve space, only the nonzero entries are recorded.

[b] Sum of the $v_j$ indicates the total fraction of initial wealth invested in risky assets by investors with power utility functions exhibiting constant relative risk aversion ($a_i = 0$).

[c] Expected monthly rate of return on unlevered portfolios.

[d] Standard deviation of the monthly rates of return on unlevered portfolios.

**Table II**

Investment Policies Subject to Fifty Percent Margin Constraint, Based on Returns Generated from the 1934–71 Period[a]

| | Power | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | .6 | .5 | .25 | 0 | −5 | −10 | −20 | −30 |
| **Panel A: Characteristics of Policies Based on Returns Generated from a Discrete Market Model and an Equally Weighted Market Proxy** | | | | | | | | | |
| P'folio #7 | | .200 | .003 | | | | | | |
| P'folio #9 | .280 | .450 | .513 | .415 | .364 | .239 | .241 | .242 | .243 |
| P'folio #14 | .720 | .349 | .484 | .585 | .636 | .761 | .759 | .758 | .757 |
| Sum of $v_j$[b] | | 2. | 2. | 2. | 2. | 1.253 | .684 | .358 | .243 |
| $E(r_p) - 1$[c] | .02142 | .02172 | .02154 | .02149 | .02146 | .02140 | .02140 | .02140 | .02140 |
| $\sigma_{r_p}$[d] | .05255 | .05740 | .05376 | .05301 | .05277 | .05252 | .05252 | .05252 | .05252 |
| **Panel B: Characteristics of Policies Based on Returns Generated from a Discrete Market Model and a Value-Weighted Market Proxy** | | | | | | | | | |
| P'folio #5 | | .104 | .086 | .061 | .094 | | | | |
| P'folio #7 | | .575 | .420 | .215 | .115 | | | | |
| P'folio #9 | .423 | .321 | .494 | .724 | .838 | .360 | .363 | .365 | .365 |
| P'folio #14 | .577 | | | | | .640 | .637 | .635 | .635 |
| Sum of $v_j$[b] | | 2. | 2. | 2. | 2. | 1.188 | .649 | .340 | .230 |
| $E(r_p) - 1$[c] | .02098 | .02098 | .02284 | .02265 | .02255 | .02082 | .02083 | .02083 | .02083 |
| $\sigma_{r_p}$[d] | .05298 | .06788 | .06540 | .06280 | .06183 | .05261 | .05262 | .05263 | .05263 |

[a] The table records of the mix of risky assets, i.e., the $\{v_k/\sum_{j=2}^{21} v_j\}$. To conserve space, only the nonzero entries are recorded.

[b] Sum of the $v_j$ indicates the total fraction of initial wealth invested in risky assets by investors with power utility functions exhibiting constant relative risk aversion ($a_i = 0$).

[c] Expected monthly rate of return on unlevered portfolios.

[d] Standard deviation of monthly rates of return on unlevered portfolios.

**Table III**

**Investment Policies Subject to a Fifty Percent Margin Constraint, Based on Returns Generated from the July 1962–December 1971 Subperiod[a]**

| | Power | | | | | | |
|---|---|---|---|---|---|---|---|
| | 2 | .5 | 0 | −5 | −10 | −20 | −30 |
| **Panel A: Characteristics of Policies Based on Returns Generated from a Discrete Market Model and an Equally Weighted Market Proxy** | | | | | | | |
| P'folio #9 | .162 | | | .126 | .129 | .130 | .131 |
| P'folio #14 | .838 | 1. | 1. | .874 | .871 | .870 | .869 |
| Sum of $v_j$[b] | | 2. | 2. | 1.089 | .595 | .311 | .211 |
| $E(r_p) - 1$[c] | .01927 | .01949 | .01949 | .01932 | .01931 | .01931 | .01931 |
| $\sigma_{r_p}$[d] | .04948 | .05035 | .05035 | .04964 | .04963 | .04963 | .04962 |
| **Panel B: Characteristics of Policies Based on Returns Generated from a Discrete Market Model and a Value-Weighted Market Proxy** | | | | | | | |
| P'folio #7 | | .117 | | | | | |
| P'folio #9 | .503 | .246 | .400 | .436 | .439 | .441 | .442 |
| P'folio #14 | .497 | .637 | .600 | .564 | .561 | .559 | .558 |
| Sum of $v_j$[b] | | 2. | 2. | 1.280 | .700 | .367 | .248 |
| $E(r_p) - 1$[c] | .02084 | .02094 | .02087 | .02086 | .02086 | .02086 | .02086 |
| $\sigma_{r_p}$[d] | .04775 | .04933 | .04796 | .04785 | .04785 | .04784 | .04784 |
| **Panel C: Characteristics of Policies Based on a Joint Frequency Distribution of Historical Returns** | | | | | | | |
| P'folio #3 | .269 | 1. | .905 | .242 | .244 | .246 | .246 |
| P'folio #6 | .383 | | .095 | .398 | .395 | .394 | .394 |
| P'folio #14 | .348 | | | .331 | .332 | .332 | .333 |
| P'folio #16 | | | | .029 | .028 | .028 | .027 |
| Sum of $v_j$[b] | | 2. | 2. | .674 | .369 | .193 | .131 |
| $E(r_p) - 1$[c] | .01521 | .01660 | .01650 | .01513 | .01513 | .01513 | .01513 |
| $\sigma_{r_p}$[d] | .05291 | .06116 | .06027 | .05256 | .05257 | .05258 | .05259 |

[a] The table records the mix of risky assets, i.e., $\{v_k/\sum_{j=2}^{21} v_j\}$. To conserve space, only the nonzero entries are recorded.

[b] Sum of the $v_j$ indicates the total fraction of initial wealth invested in risky assets by investors with power utility functions exhibiting constant relative risk aversion ($a_i = 0$).

[c] Expected monthly rate of return on unlevered portfolios.

[d] Standard deviation of monthly rates of return on unlevered portfolios.

return distributions considered in Table I, only this time a 50 percent margin requirement is in effect. Naturally, this real world constraint changes the mix of risky assets for those powers for which it is binding.[15] The most dramatic effect shows up in Panel B for the .6, .5, .25, and 0 powers that now concentrate their holdings in portfolios 5, 7, and 9, while the other power models for which the constraint is not binding continue to hold portfolios 9 and 14.

Table III also considers investment policies subject to a 50 percent margin requirement. However, two new elements are considered: the fourth subperiod, July 1962 to December 1971, and a second method of constructing a joint-return distribution. The distributions underlying the policies in panels A and B were generated from a discrete market model, where the market proxies—both equally

[15] Although this result is hardly surprising, it does not appear to be widely appreciated. However, it is explicitly mentioned in Grauer [10] and is implicit in Hakansson [15].

and value weighted—follow a discrete approximation to a normal distribution. Panel C considers a joint distribution that is estimated exclusively from historical data—specifically, as a joint-frequency distribution consisting of 114 equally likely monthly observations on the vector of 20 risky asset returns.[16] The results in Table III are similar to the results found in Tables I and II. The mix of risky assets differs with different forecasts. But for any given, approximately normal, or real world, return distribution, the mix of risky assets differs across the power utility and MV models. Again, we note that the mix for the negative powers, particularly the −10, −20, and −30 powers, are very similar in each case. It is also worth noting that the differences between the .5 and 0 and other powers are more pronounced than they would be with no margin constraints, precisely because the constraints are binding for the less risk-averse powers.

## III. Summary

At the macro level, the primary results are: (1) judged by the generalized SML tests, the MV and a very wide variety of power utility LRT models are indistinguishable; (2) in a pragmatic but somewhat limited sense, in light of Roll's critique, the results are not affected by the choice of either an equally or value-weighted proxy for the market portfolio. On the other hand, at the micro level, the key result is that, with approximately normal or real world return distributions, the models exhibit different investment policies. With short sales constraints, the mix of risky assets for utility functions with powers greater than zero differs from the mix of risk assets for negative power utility functions and the MV model. Moreover, margin constraints are binding for the more restrictive, but widely studied, isoelastic or constant proportional risk-aversion functions with powers greater than zero. Naturally, the binding margin constraints accentuate any differences in the risky asset mix.

It is widely accepted that in discrete time, if return distributions are normal, all risk averse expected utility decision makers will pick MV efficient portfolios. But technically this is incorrect. And while most people probably accepted the statement as a (perhaps very good) approximation, the results of this paper indicate that, with approximate normality, the mix of risky assesets differs. Now, in moving from the micro foundations to the macro implications of a CAPM, we must allow the theorist and the tester considerable license, as many factors, taxes, transactions costs, etc., have been ignored. At the same time, we should be cautious in accepting the results of the micro tests: first, because the generalized SML methodology used to generate the macro results is itself suspect, and second, because the portfolio selection results indicate that the macro theory is not based on firm micro foundations. Perhaps more important, the demonstration of the

[16] It is important to recognize that the return distribution generated from the market model that employs a discrete approximation to normally distributed market returns may differ quite significantly from the joint frequency distribution that is generated directly from historical data. The market model method explicitly generates a discrete joint return distribution that is consistent with normality of the market proxy. To the extent it uses historical data, it is only to provide realistic estimates of the parameters: the market proxy's mean and standard deviation and the portfolios' alphas, betas, and residual risks.

differences in the risky asset mix should caution us that we may be premature in using the apparent empirical identity of the models at the macro level to infer that the models are essentially identical at the portfolio selection level.

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