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Author(s): T. Agmon, A. R. Ofer, A. Tamir
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Variable Rate Debt Instruments and Corporate Debt Policy

T. AGMON,* A. R. OFER* and A. TAMIR**

ABSTRACT

Sustained high rate of inflation has led to the creation of debt instruments with variable interest rate. The availability of these debt instruments presents management with the problem of the choice of the optimal debt portfolio. This paper deals with this problem assuming a given, and optimal, debt to equity ratio. Given expected monetary value maximization, an efficient frontier is derived in terms of the expected net income and probability of bankruptcy, where net income is defined as operating income minus debt repayment. This efficient frontier is shown to be also mean-variance efficient. It is also shown that in most cases the optimal debt portfolio includes more than one debt instrument. In other words, the firm will avoid the policy of minimizing the expected cost of its debt repayments or the policy of minimizing the costs of bankruptcy. The optimal solution itself is affected by market variables like the relative expected cost of different debt instruments and by firm specific variables like the variability of its operating income stream, and the covariance between the operating income and the debt repayments.

I. Introduction

ONE OF THE MAIN concerns of a financial manager according to the three classic questions posed by Ezra Solomon is:

"How should the funds required (for investment) be financed?"

In the standard paradigm of a perfect market with one currency and no inflation, the two sources are equity and debt. In reality, however, the financial manager is faced by a number of different debt instruments. In particular the sustained high rate of inflation which characterizes some parts of today's world has led to the creation of debt instruments which have a variable interest rate. The variable rate may be linked to a centrally determined rate of interest such as the prime rate in the U.S. or the LIBOR (London InterBank Offer Rate) or to some price index like the CPI (Consumer Price Index). Another way of effecting a variable rate debt instrument is by borrowing in a foreign currency. For example, an American firm may issue bonds in the Eurobonds market denominated in Deutsche Marks or in Swiss Francs. The debt repayment is linked in this case to the exchange rate between the U.S. dollar and the foreign currency. In all these cases, the future debt repayments are variable and uncertain in nominal terms and can be described by a random variable.

The availability of variable rate debt instruments presents management with a new problem; namely, the choice of an optimal debt portfolio.

* Faculty of Management, Tel-Aviv University
** Department of Statistics, Tel-Aviv University

The question of the optimal debt portfolio is a specific part of the general problem of the optimal capital structure of the firm. Modigliani and Miller [11] have shown that given the ideal conditions of a perfect capital market the value of the firm is invariant to the capital structure. Recent works by Kraus and Litzenberger [8], Scott [14] and Kim [7] demonstrated that when taxes and bankruptcy costs are introduced the firm has an optimal financial structure. The deductability of interest charges from taxable income increases the after-tax operating income of the levered firm. Therefore, it acts as an inducement to increase financial leverage. However, if as the result of the financial leverage the firm cannot meet its debt obligations, it is forced into bankruptcy and incurs bankruptcy costs. Thus, the existence of bankruptcy costs acts as a counter incentive to increase the debt to equity ratio. The issue of bankruptcy costs and their nature is discussed in the literature.[1] It is sufficient here to point out that the existence of bankruptcy costs curtails the debt capacity of the firm, that is the supply of debt capital, and it reduces the willingness of the shareholders to borrow unlimited amounts. Thus the shareholders bear the expected costs of bankruptcy as it affects their optimal financial decisions. For example, Kim [7] has shown that the shareholders will not use what he terms "maximum debt capacity". In this paper, it is assumed that the firm has already decided on its optimal capital structure taking all the relevant considerations into account and now the financial manager should decide on the optimal composition of the debt.

The existence of several debt instruments implies that for a given debt to equity ratio the expected debt repayment is a decision variable. However, different combinations of debt instruments will produce a different probability of bankruptcy and hence different expected bankruptcy costs. Thus, given a certain level of bankruptcy costs, the optimal debt portfolio is a function of the expected debt repayment and the probability of bankruptcy.

Dating back to the work of Roy [13], there have been attempts in the financial literature to incorporate the default risk, or the probability of "ruin", into models of economic choice under uncertainty. Roy specifies the minimization of the probability of disaster as an investment criterion. Most of the researchers which have followed Roy present this element as a constraint.[2] For example, Telser [15] employs as a criterion the maximization of the expected rate of return subject to a constraint of a minimum return with a prespecified probability. Baumol [3] has set a similar constraint in the context of the mean-variance efficient frontier by employing what he called the "gain confidence" criterion.[3] The use of the probability of ruin as a constraint only, is somewhat arbitrary as the "accepted" level of this type of risk is set outside the optimization process itself. Furthermore, this approach implicitly assumes that the "acceptable" level of the risk of ruin is invariant to the costs associated with this occurrence.

The costs associated with "ruin", or bankruptcy costs, are explicitly accounted for in the model presented in this paper. This is done by incorporating these costs into the total cost of debt capital. The optimal debt portfolio is determined by

[1] See, for example, Baxter [4], Stanley and Girth [15], Warner [17] and Kim [7].

[2] See Pyle and Turnovsky [12] for a discussion of these models.

[3] For other examples of models which use the probability of ruin as a constraint, see Agnew et. al. [2] and Kahane [6].

the trade off between expected debt repayment and expected bankruptcy costs. Given a level of bankruptcy costs, the composition of the optimal debt portfolio could be determined. However, since the optimization problem should be solved for a whole range of bankruptcy costs, we have to explore a whole range of bankruptcy probabilities. This leads to the development of an efficient frontier in terms of expected debt repayments and the probability of bankruptcy. Thus, the recognition of a cost of bankruptcy results in treating the probability of bankruptcy as a parameter in the choice problem and not as a constraint the way it was previously done in the literature.

In the specification of this model it is assumed that the firm acts to maximize the expected monetary value of its shareholders' wealth. It should be pointed out that although the choice of the optimal debt portfolio involves a trade-off between risk and cost, this problem cannot be solved using the mean-variance approach in the classic fashion. A mean-variance optimization at the level of the firm is not a necessary condition for utility maximization. It is generally agreed that the maximization of the market value of the shareholders wealth in monetary term is a sufficient objective both from theoretical and managerial considerations. If the capital market functions well, this objective is consistent with the expected utility maximization of any given risk-averse individual investor. Thus there is no need for the firm to optimize along a mean-variance efficient frontier. In a perfect capital market the investor can perform the necessary diversification. Therefore for the firm to build an efficient portfolio is to offer a redundant service. Fama and Miller [5] in discussing optimal policies for the firm state that ".... in a perfect market consumers can combine the stocks of different firms so that in itself diversification on the part of an individual firm is of no particular value to them." In addition, the specific nature of bankruptcy risk cannot be captured completely by the classic mean-variance criterion and it does require a special model. This model has to reflect the all or nothing characteristic of the bankruptcy risk. Given these considerations we are presenting a model which reflects the specific nature of the bankruptcy risk. We then examine if there is correspondence between portfolios chosen according to our model and portfolios chosen according to the mean variance criterion.

In Section II, a model for the choice of an optimal debt portfolio is presented. The efficient frontier in terms of the probability of bankruptcy and the expected debt repayments is derived and analyzed, as well as the characteristics of the optimal composition of the debt given the capital structure of the firm. A special case where only two debt instruments exist is discussed in section III. This limited case, where only one debt instrument has variable repayment schedule, allows for a more detailed analysis of the determinants of optimal debt policy. The implications and the limitations of this approach are discussed in the fourth concluding section.

II. The Choice of the Optimal Debt Portfolio

In the derivation of the model for the optimal debt portfolio, the two period paradigm is employed. The firm buys all the factors of production in the first period; in the second period, it sells all its output and all its property, distributes its income among the different claimants (bondholders and shareholders) and disbands. As there exists uncertainty with regard to the second period, the operating income of the firm at this period is a random variable. This random variable is denoted Y. Part of the necessary investment at period one is financed by borrowing a certain amount which brings the debt-to equity- ratio to a prespecified target level.[4] To raise the amount of L dollars (an amount which is consistent with the target ratio), the firm can use and choose from N debt instruments, each of which has a different repayment schedule. The repayment of each debt instrument is linked to different economic indices and is therefore uncertain. Let x_j denote the amount of debt raised by using the j-th debt instrument where:

$$\sum_{j=1}^{N} x_j = L \tag{1}$$

Then the debt repayment (interest and principle) in period two is[5]:

$$\sum_{j=1}^{N} x_j R_j \tag{2}$$

where

R_j = A random variable describing the debt repayment per dollar debt, of the debt instrument j.

Bankruptcy will occur when the firm cannot meet its debt obligations in period two, that is when:

$$Y - \sum_{j=1}^{N} x_j R_j < 0. \tag{3}$$

Bankruptcy is costly. The costs arise from having to liquidate the firm in "distress" prices and from fees that have to be paid to a third party.[6] The cost of bankruptcy to the shareholders can be thought of as the equivalent of an insurance premium paid by the shareholders to assure the suppliers of debt capital against incurring bankruptcy cost.[7] Given such an arrangement, if bankruptcy does occur, the insurance company bears the bankruptcy cost. The premium for such an insurance policy is a function of the costs associated with bankruptcy, BC, and the probability of bankruptcy, α. The premium is paid in period one, when the debt capital is raised. For simplicity of exposition, it is assumed that the insurance premium equals the present value of the expected bankruptcy costs. In essence, the insurance premium is part of the cost associated with debt financing.[8]

[4] We assume here that L is an exogeneous variable that has been determined beforehand. We do not address the issue of the optimal debt to equity ratio, although the optimal amount of debt could be affected by the choice of the debt instruments.

[5] Since taxes have a proportional effect on all the debt instruments, we assumed them away. R_j can be defined as debt repayment after taxes.

[6] In the multiperiod case, the shareholders can also incur bankruptcy cost that will arise from "indirect" costs of reorganization. These costs will be incurred even if the firm is not forced into bankruptcy. See Baxter [4].

[7] For a discussion of this hypothetical device, see Levy and Sarnat [10], pp. 234-240.

[8] Note, the insurance covers only the bankruptcy costs. The debt is still risky. The income of the

The income of the shareholders in period two equals the firm operating income minus the debt repayments minus the expected bankruptcy costs.[9] The objective of the firm is to maximize the expected monetary value of the shareholders. This objective can be stated as follows:

$$\text{Max}\,[E - \alpha \cdot BC] \tag{4}$$

where

$$E = E[Y - \textstyle\sum_{j=1}^{N} x_j R_j]$$

$$\alpha = \text{the probability of bankruptcy}$$

$$\alpha \cdot BC = \text{the expected bankruptcy costs}$$

We will refer to E as expected net income, and $E - \alpha \cdot BC$, the expected net income minus the expected bankruptcy costs, is referred to as the expected monetary value (EMV).

The choice to raise an amount x_j by the j-th debt instrument will affect the amount of expected debt repayment and hence the expected net income, E. It will also affect the probability of bankruptcy α and thus the expected bankruptcy costs. Due to the nature of the trade off between the expected net income and the probability of bankruptcy in the objective function, it is clear that the optimal debt portfolio should have the highest expected net income (or the lowest expected debt repayment) for a given α, the probability of bankruptcy. That is, an optimizing pair (E_o, α_o) should satisfy the property that E_o is the maximum expected net income that can be achieved for a given α_o. Similarly, α_o should be the minimum probability of bankruptcy that can be obtained for a given E_o.

Different firms may have different bankruptcy costs, so that the whole set of optimizing pairs of E and α should be identified. In order to do so, an efficient frontier is developed that treats the probability of bankruptcy as a parameter. This is done by first restricting the probability of bankruptcy to a given level and then parameterizing it. In general, the optimal debt portfolio will be determined by the shape of the (E, α) efficient frontier, and by the level of the bankruptcy costs. We turn next to the problem of deriving the (E, α) efficient frontier.

We start by considering the problem of finding a debt portfolio $x = (x_1, \cdots, x_N)$ which will maximize the firm's expected net income E, subject to the constraint that the probability of bankruptcy will not exceed a safety level of α. This model can be stated as follows:

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debt holders, D, can be specified as follows:

$$D = \begin{cases} \sum_{j=1}^{N} x_j R_j & y \geqslant \sum_{j=1}^{N} x_j R_j \\ y & y < \sum_{j=1}^{N} x_j R_j \end{cases}$$

That is, if bankruptcy occurs, the debt holders will not get all that is owed to them, but they will not have to pay the bankruptcy costs. These will be paid by the insurance company.

[9] Recall that the shareholder paid in period one the present value of the expected bankruptcy costs.

$$\text{Max } E = E\{Y - \textstyle\sum_{j=1}^{N} x_j R_j\}$$
$$\text{s.t. } Pr(Y - \textstyle\sum_{j=1}^{N} x_j R_j < 0) \leqslant \alpha$$
$$\textstyle\sum_{j=1}^{N} x_j = L, \qquad x_j \geqslant 0, \qquad j = 1, \cdots, N. \quad (5)$$

Let us assume that one debt instrument has a fixed nominal repayment scheme and N-1 debt instruments have variable payments schemes which may be linked to different indices. Let the N-th debt instrument be the one with a fixed nominal repayment. Using R_N, the nominally fixed debt repayment as a benchmark, and comparing all other debt payments to R_N, we define

$$R'_j = R_j - R_N, \qquad j = 1, 2, \cdots, N-1$$

Following the same procedure, let us define a benchmark income after debt services so that

$$Y' = Y - LR_N$$

then the maximization problem becomes:

$$\text{Max } E = E(Y' - \textstyle\sum_{j=1}^{N-1} x_j R'_j)$$
$$\text{s.t. } Pr(Y' - \textstyle\sum_{j=1}^{N=1} x_j R'_j < 0) \leqslant \alpha$$
$$\textstyle\sum_{j=1}^{N-1} x_j \leqslant L, \qquad x_j \geqslant 0, \qquad j = 1, \cdots, N-1. \quad (6)$$

The solution of (6) and the model to be presented below are based on the following assumption: the variables $(Y', R'_1, \cdots, R'_{N-1})$ have a joint multivariate normal distribution.

Let $\Omega = (\sigma_{jk})$ denote the covariance matrix of the variables $(Y', R'_1, \cdots, R'_{N-1})$. Also let $e_0 = E(Y')$ and $e_j = E(R'_j), j = 1, 2, \cdots, N-1$.

Using the above assumption and definitions the maximization problem becomes:

$$\text{Max } E = x_0 e_0 - \textstyle\sum_{j=1}^{N-1} x_j e_j$$
$$\text{s.t. } x_0 e_0 - \textstyle\sum_{j=1}^{N-1} x_j e_j + Z(\alpha)\sqrt{x'\Omega x'} \geqslant 0$$
$$\textstyle\sum_{j=1}^{N-1} x_j \leqslant L, \qquad x_j \geqslant 0, \qquad j = 1, 2, \cdots, N-1 \qquad x_0 = 1 \quad (7)$$

where $Z(\alpha)$ is defined by $\phi(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} e^{-t^2/2}\, dt = \alpha$ and $x = (x_0, -x_1, \cdots, -x_{N-1})$.

It is assumed that the firm restricts itself to $\alpha \leqslant 1/2$. In other words, the firm will never choose a debt portfolio for which the probability of bankruptcy exceeds one half. If $\alpha < 1/2$ then $Z(\alpha) < 0$.

The solution of the maximization problem described by (7) yields an optimal debt portfolio for a given level of α. That is, the portfolio derived by solving (7) has the highest expected net income for a given level of α.

To generate the (E, α) efficient frontier, we solve problem (7) parametrically for all $\alpha \leqslant ½$.

Using the above notations, the (E, σ) efficiency set corresponding to our model is obtained by solving the following problem parametrically for all $\sigma \geqslant 0$

$$\text{Max } E = x_o e_o - \sum_{j=1}^{N-1} x_j e_j$$

$$\text{s.t.} \quad \sqrt{x'\Omega x} \leq \sigma$$

$$\sum_{j=1}^{N-1} x_j \leq L \qquad x_j \geq 0, \qquad j = 1, 2, \cdots, N-1 \qquad x_o = 1 \tag{7a}$$

Let $\sigma = \sigma(E)$ be the (E, σ) frontier curve. Then, $\sigma(E)$ is a monotone, increasing, convex function in its interval of definition, which we denote by $[E_{\min}, E_{\max}]$. $\sigma(E)$ is also differentiable for all, but finitely many values of E.

Using $E = E(\alpha)$ to denote the (E, α) frontier curve, it is easily observed that $E(\alpha) \geq 0$, and it is also increasing in α. An investigation of the (E, σ) and (E, α) frontiers appears in Agmon, Ofer and Tamir [1], which is a detailed version of this paper. Here we only state the most significant theorems proved in Agmon, Ofer and Tamir [1].

Given α, we suppose that $x(\alpha)$ solves (7), and use the notation $\sigma(\alpha) = (x'(\alpha)\Omega x(\alpha))^{1/2}$. The next theorem shows that the optimal debt portfolio derived by solving (7) also has the lowest variance among all portfolios with the same expected net income E. That is, the optimal debt portfolio, in terms of E and α, is also mean variance efficient.

THEOREM 1

Given α for which (7) is feasible, then $E(\alpha) = E_{\max}$, or else, $(E(\alpha), \sigma(\alpha))$ is on the (E, σ) – efficiency set defined by (7a). In particular, if $E(\alpha) + Z(\alpha)\sigma(\alpha) > 0$, then $E(\alpha) = E_{\max}$.

Other properties of the $E(\alpha)$ frontier are given in the following theorem, proved in Agmon, Ofer and Tamir [1].

THEOREM 2

Let $[E_{\min}, E_{\max}]$ be the interval, where the (E, σ) frontier is defined, and let $E_o = \text{Max}(E_{\min}, 0)$

(i) If $\sigma(E_o) > 0$, then the $E(\alpha)$ frontier is defined on an interval, $[\alpha_o, \alpha_1]$, where it is continuous and increasing. $\alpha_o > 0$, is defined by $\alpha_o = \phi(Z_o)$, where Z_o is such that the line $E + Z_o\sigma = 0$ is tangential to the frontier $\sigma(E)$. Furthermore

$$\alpha_1 = \phi(Z_1), \quad \text{where} \quad Z_1 = -E_{\max}/\sigma(E_{\max}), \text{ i.e. } E(\alpha_1) = E_{\max}.$$

(ii) If $\sigma(E_o) = 0$ and either $E_o > 0$, or $\sigma'(E_o+) = 0$, then $E(\alpha)$ is defined in the interval $[0, \alpha_1]$ where it is continuous and increasing. α_1 is defined as in (i).

(iii) If $\sigma(E_o) = E_o = 0$ and $\sigma'(0+) > 0$, then the (E, α) frontier consists of the isolated point $(E, \alpha) = (0, 0)$, and the continuous curve $E(\alpha)$, defined on an interval $[\alpha_o, \alpha_1]$. α_o and α_1 are defined as in (i).

As mentioned above, portfolios on the (E, α) frontier are also mean variance efficient. Recalling that the mean variance frontier is convex and piecewise quadratic, we show in Agmon, Ofer and Tamir [1] the following (local) concavity property of the (E, α) frontier near α_o, the left bound of its interval of definition.

THEOREM 3

Let $\sigma^2 = AE^2 + BE + C$ be a quadratic piece of $\sigma^2 = \sigma^2(E)$, containing $E(\alpha_o+)$. If $E(\alpha_o) > 0$, $\sigma^2(E(\alpha_o)) > 0$ and $\sigma^2(E)$ is differentiable at $E(\alpha_o)$ (e.g. $E(\alpha_o)$ is an interior point of the range of the above quadratic), then

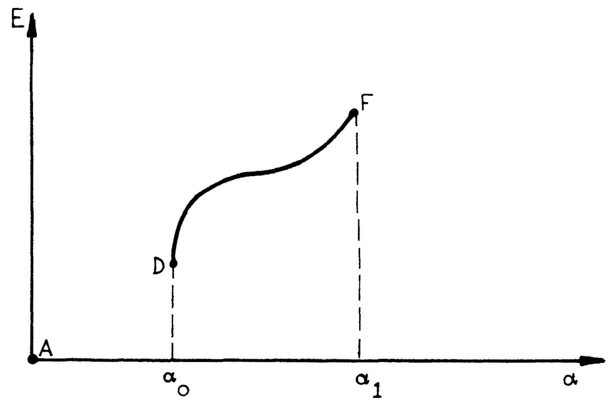
$$\text{(i) } BE(\alpha_o) + 2C = 0, \qquad \text{(ii) } E'(\alpha_o) = \infty, \qquad \text{and}$$
$$\text{(iii) } E(\alpha) \text{ is concave in some interval } [\alpha_o, \bar{\alpha}],\ \bar{\alpha} > \alpha_o.$$

These properties of the (E, α) frontier are described in Figure 1.

In figure 1, the segment DF presents the case where the mean variance frontier does not pass through the origin (i.e., the pair $(E, \sigma^2) = (0, 0)$ is not a mean variance efficient solution), then the (E, α) frontier is defined, continuous, and increasing for all α in some interval $[\alpha_0, \alpha_1]$ where $\alpha_0 > 0$ and $\alpha_1 \leqslant 1/2$. α_0 and α_1, the boundaries of the (E, α) frontier are determined by the means, the variances and the covariances of the operating income Y and the debt payments $R_1, \cdots, R_N$.

When the point $(E, \sigma^2) = (0, 0)$ is a mean-variance efficient portfolio, then the (E, α) efficient set consists of the segment DF described above and the isolated point A, where $(E, \alpha) = (0, 0)$.

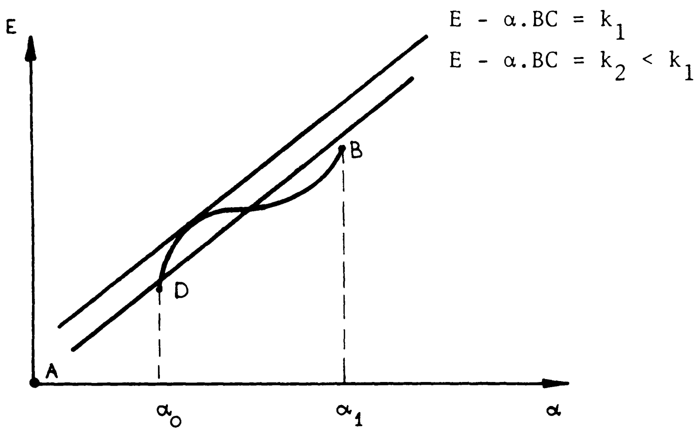
Also note that the frontier is concave at the neighborhood of α_0. Furthermore, at $\alpha = \alpha_0$, the frontier has a 90° slope.[10] In general, the frontier is not concave throughout its entire interval of definition.



The objective function specified in (1), namely the maximization of $K = E - \alpha \cdot BC$ can be described in the (E, α) plane for different levels of K. The slope of the objective function line equals BC and higher lines represent higher expected

[10] This holds unless some degeneracy is present in the covariance matrix. This degeneracy may lead to a situation where the assumptions of Theorem 3 are not met. This point is further elaborated in Agmon, Ofer and Tamir [1]. To simplify the discussion, we assume that this degeneracy does not occur.

monetary values to the shareholders. Thus it follows that the firm will try to achieve the highest K for which the line $E - \alpha \cdot BC = K$ contains possible (E, α) values. Therefore, the firm will choose the debt portfolios where the line $E - \alpha \cdot BC = K$ is tangent to the (E,α) frontier. This is demonstrated in Figure 2 below.



The shape of the (E, α) frontier and slope of the objective function, which equals the bankruptcy costs, will determine the tangency point and hence the optimal debt portfolio. The composition and characteristics of the optimal debt portfolio depend on the section of the (E, α) frontier at which the tangency occurs. The optimal debt portfolio can only be on the following sections of the (E, α) frontier:

1. The concave portions of the (E, α) frontier.
2. At the point $(E(\alpha_1), \alpha_1)$
3. At the point $(E, \alpha) = (0, 0)$, if $(E, \sigma^2) = (0, 0)$ is mean variance efficient.

The point D, where $(E, \alpha) = (E(\alpha_0), \alpha_0)$, cannot be an optimal debt portfolio. The implications of these possible tangency points with regard to the characteristic of the optimal debt portfolio are further discussed below, with references to Figure 2.

First we observe that the (E, α) frontier is increasing in α. Therefore, for all bankruptcy costs sufficiently small, including zero costs, the optimal debt portfolio is the one with lowest expected debt repayment; namely, the right boundary of the frontier, point B. The debt portfolio that will result in the lowest expected net income and the lowest probability of bankruptcy is represented on the frontier by point D, or point A provided it exists. At point D, the (E, α) frontier has a 90° slope. Therefore, for finite bankruptcy costs, the portfolio represented by point D will never be chosen as the optimal debt portfolio. At point A the expected net income and the probability of bankruptcy are both equal to zero. The existence of point A assures that the firm will not have to choose a debt portfolio that will result in a negative expected monetary value for the shareholders. In all other cases, the optimal debt portfolio will be on the concave parts of the (E, α) frontier. At these parts of the (E, α) frontier, the optimal debt portfolio is a mixed portfolio which consists of more than one debt instrument. It is interesting to note that there exists discontinuity in the optimal level of the bankruptcy probability, α^*, with regard to changes in the cost of bankruptcy, BC. This is a result of the fact that the (E, α) frontier is not concave throughout its entire interval of definition. Thus, even very small changes in BC might cause the tangency point to move from one concave part of the frontier to another concave part and bring about a discontinuous change in the optimal level of the bankruptcy probability. Finally, it should be pointed out that although all debt portfolios on the (E, α) frontier are mean-variance efficient, not all the portfolios which are mean variance efficient are optimal debt portfolios. First, it is shown in Agmon, Ofer and Tamir [1] that the debt portfolios which are below a certain point on the (E, σ) frontier are not (E, α) efficient. Furthermore, even if a portfolio is on the (E, σ) frontier and also is (E, α) efficient, it will not be an optimal debt portfolio if it is positioned on a convex part of the (E, α) frontier. We conclude, therefore, that being (E, σ) efficient is only necessary but not sufficient for being an optimal debt portfolio.

Thus, in the general case, the firm will not choose the debt portfolio with the lowest probability of bankruptcy and it will raise its debt using more than one debt instrument. The optimal debt portfolio will maximize the shareholders' expected monetary value and will be mean variance efficient. To give the properties of the general case a more specific interpretation, as well as to explore the effects of the basic relationships in the covariance matrix Ω and the expected debt repayment on the debt financing decisions of the firm, a special case where the firm is faced by only two debt instruments is presented and analyzed in the following section.

III. Optimal Composition of Debt—A Balance Between Market and Firm Specific Variables

The choice model developed and analyzed in the preceding section is applied now to a situation where the firm can choose between two debt instruments, one with nominally fixed interest rate, and one with a variable rate. This rather simplified case allows for an exposition of the relationship between market variables and the firm specific variables which affect the choice of the optimal debt portfolio.

Retaining the one period paradigm of the last section the two debt instruments can be characterized by their repayment schedule. If employed, the "nominal" debt instrument has a nominally fixed charge against the income of the firm, whereas the repayment of the variable rate debt instrument can be described by a random variable. The maximization problem can then be restated as follows:

$$\begin{aligned} &\text{Max } E = e_0 - x_1 e_1 \\ &s.t.\quad e_0 - x_1 e_1 + Z(\alpha)[(1, -x_1)\Omega\left(\begin{smallmatrix}1\\ -x_1\end{smallmatrix}\right)]^{1/2} \geqslant 0 \\ &0 \leqslant x_1 \leqslant L \end{aligned} \tag{8}$$

where:

$$e_0 = E(Y - LR_N)$$

$$e_1 = E(R_1 - R_N)$$

$$\Omega = \begin{bmatrix} a^2 & c \\ c & b^2 \end{bmatrix}$$

R_1, R_N = the debt repayment on the variable interest rate and the nominally fixed rate debt respectively

a^2 = the variance of the firm's operating income.

b^2 = the variance of the variable debt repayment.

c = the covariance between the operating income and the variable debt repayment.

The solution of the above problem, as well as the shape of the (E, α) frontier depends on market variable like the value of e_1, the expected value of the difference between the repayment of the two debt instruments, and b^2 the variance of the nominal value of the repayment of the variable rate debt. It also depends on firm specific variables like the variance of the firm's operating income, a^2, and the covariance between the firm operating income and the variable debt repayment.

The question of the value of e_1 is discussed in the economic literature.[11] The issue is whether e_1 is zero, positive, or negative; in other words, whether one of the debt instruments is regarded as more risky by suppliers of debt and therefore carries a risk premium. This issue has not been fully resolved. Under different assumptions, e_1 can be negative, zero, or positive. Consequently, all these possibilities are examined. The mathematical derivation is done in Agmon, Ofer and Tamir [1]; here we present the results only.

If the expected value of the repayment of the variable debt instrument equals the nominally fixed debt repayment (i.e., $E(R_1) = R_N$), the (E, α) efficient frontier consists of one point. This point represents the optimal composition of debt. The actual debt composition depends on c, the covariance between the operating income and the variable debt repayment, and on the relationship between c and b^2, the variance of the debt payment of the variable rate debt. If $c < 0$, then the optimal debt portfolio includes only one debt instrument, the one with the nominally fixed interest rate. A mixed solution combining the two debt instruments applies in the case where $c > 0$ and $\frac{c}{b^2} < L$ and in this case the firm will employ $\frac{c}{b^2}$ of the variable debt instrument and $L - \frac{c}{b^2}$ of the nominally fixed debt instrument. If $c > 0$ and $\frac{c}{b^2} > L$, the firm will employ the variable rate debt instrument exclusively. If $E(R_1) > R_N$ and $c < 0$, the firm will employ only debt with nominally fixed interest rate. In the cases where $E(R_1) > R_N$ and $c > 0$ or when $E(R_1) < R_N$ and $\frac{c}{b^2} < L$, there exists a nontrivial (E, α) frontier similar to the one described in the preceding section. The optimal debt portfolio is determined in these cases by the tangency solution between the $k = E - \alpha \cdot BC$ line and the (E, α) frontier. Finally, when $E(R_1) < R_N$ and $\frac{c}{b^2} \geqslant L$, the debt portfolio will consist of only variable rate debt. These results are summarized in the table below.

[11] For example, see Levhari and Leviatan [9].

Table I

The Composition of Optimal Debt Portfolio under Different Assumptions with Regard to $E(R_1)$, R_N, and c

	$E(R_1) = R_N$	$E(R_1) > R_N$	$E(R_1) < R_N$
$c \leqslant 0$	Nominally fixed debt	Nominally fixed debt	Tangency Solution
$c > 0, \frac{c}{b^2} \geqslant L$	$x_1 = L$	Tangency Solution	$x_1 = L$
$c > 0, \frac{c}{b^2} > L$	$x_1 = \frac{c}{b^2}$	Tangency Solution	Tangency Solution

where x_1 = the proportion of the variable debt in the total debt

Table I demonstrates that the optimal composition of the debt varies with both the values of $[E(R_1) - R_N]$ and c. Consequently, the firm should have information with regard to some market variables like the value of $e_1 = E(R_1) - R_N$ as well as with regard to firm specific variables such as the bankruptcy costs and the covariance between the operating income of the firm and the debt repayment.

When the covariance, c, is positive (the last two rows in Table One), the variable rate debt instrument resembles a contingent claim. In this case, and given sufficiently large bankruptcy costs, the optimal debt portfolio will include the variable rate debt instrument, regardless of its relative price.

IV. Summary and Conclusions

The creation of variable rate debt instruments brings about a choice problem which is the optimal composition of the corporation debt portfolio. The solution for this choice problem, which was presented in the preceding sections, is based on a trade-off analysis between expected debt repayment and the expected bankruptcy costs. The special nature of bankruptcy risk makes the application of the mean-variance approach to this problem inadequate. Therefore, an efficient frontier was developed which balances the expected operating income minus debt repayment against the probability of bankruptcy. We showed that the feasible solutions which lie along this frontier are also mean-variance efficient. That is, the debt portfolios which are on the (E, α) frontier have the lowest variance in addition to having the lowest probability of bankruptcy among all portfolios with the same expected net income. However, it should be emphasized that the reverse does not hold true. We have shown that not all mean-variance efficient debt portfolios are (E, α) efficient.

The optimal composition of the debt for a given firm is a function of both market conditions and the nature of the income stream of the firm. Macroeconomic considerations determine the relative price of different debt instruments and thus the expected cost of debt from different sources to the firm. The relative riskiness of different debt instruments to the firm is determined by the combination of the effect of a given debt repayment schedule on the probability of bankruptcy, (α). The relative riskiness depends to a large extent on the covariance between the operating income of the firm and the debt repayments. Both the bankruptcy costs and the covariance factor are microeconomic, firm specific variables. Consequently, the optimal solution for the debt portfolio is determined by the interplay between market and firm variable and it may differ from one firm to another. In the special case when the firm is faced by only two debt instruments and only one with variable debt repayments, the optimal debt portfolio will include the variable rate debt instrument, provided that $c > 0$ and that the bankruptcy costs are sufficiently large. Operationally, the results of the analysis presented in this paper imply that firms should search for and issue variable rate debt instruments with a payment schedule that has a positive covariance with the firm operating income.

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