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On The Direction of Preference for Moments of Higher Order Than The Variance

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EMPIRICAL AND THEORETICAL ATTACKS on the Markowitz mean-variance, portfolio theory have given impetus to the investigation of moments of higher order than the variance. Denoting w as an investor's wealth, and $\tilde{x}$ his incomes (a random variable), if the investors utility function, U, is solely dependent on the sum of his wealth and income we can define his utility as $U = U(\tilde{x} + w)$. The investor's rate of return on investment of w is defined as $\tilde{r} = \tilde{x}/w$ and his utility may be given by $U = U(rw + w)$. Letting $\mu = E(w + rw)$, representing the expected value on investment, expanding U in the familiar Taylor series, and taking the expected value of both sides gives:

$$E(U) = U(\mu) + \frac{U^2(\mu)}{2}\sigma^2 + \textstyle\sum_{i=3}^{\infty} \frac{\mu_i}{i!} U^n(\mu) \tag{1}$$

where U^n denotes the n-th derivative derivative of U and μ_i is the i-th central moment. In order to analytically study the expected utility of return on investment w, one must define the relevant moments of the distribution of return and the sign of the coefficient of each relevant moment. The mean and the variance can completely describe $E(U)$ generally for only certain distributions of return such as the Normal, the Uniform and the Binomial distributions. If (i) the distribution of returns to a portfolio is asymmetric, (ii) the investor's utility function is of higher order than the quadratic, and (iii) the mean and variance do not completely determine the distribution, then the third and perhaps higher moments and the sign of their coefficients must be considered.[1]

O For higher moments than the variance two questions may be asked: can the direction of preference (the sign of the coefficients in (1)) for each moment be determined on some a-priori grounds and if so, what is the preference direction for each moment? For the third moment the first question has received considerable attention (See: Arditti [1], Jean [4], Arditti and Levy [2], Kraus and Litzenberger [6], Levy and Sarnat [7], and Simkowitz and Beadles [10]). Arditti [1], and Kraus and Litzenberger [6] have implied a positive preference direction for the third moment ($U^3(w) > 0$) in answer to the second question as regards skewness (normalized third moment). Other conditions such as "clientele effects" have been posited as indications of investor preferences for the third moment. The fourth and higher moments have received relatively little attention apparently because of the difficulty of dealing with the fourth and higher moments. Kaplanski [5] has indicated that it is difficult to discern just what the fourth

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[1] If any of these three conditions does not hold, then the mean and the variance are sufficient.

statistical moment measures. Kraus and Litzenberger [6] argue that it is trivial to extend their model to the fourth and higher moments for separable utility functions. They do not make the extension attempting to dismiss those moments largely on the basis of Kaplanski's [5] results which provide necessary but not sufficient conditions for ignoring the fourth and higher moments.

It is the purpose of this note to aid the investigation of the use of higher moments in portfolio analysis by determining the higher moment preference direction of the usual investor who is consistent in direction of preference of moments. Specifically, we show for such an investor that the preference direction is positive (negative) for positive (negative) values of every odd central moment and negative for every even central moment. It is further shown that investors who are not strictly consistent in preference direction must exhibit on average negative preference for even central moments and positive preference for odd central moments.

The usual risk averse investor is assumed to have a utility function with the following first two derivatives:

$$U^1(w) > 0 \ \forall w, \quad \text{and} \tag{A1}$$

$$U^2(w) < 0 \ \forall w. \tag{A2}$$

An investor who is strictly consistent in preference direction for the n-th moment has a utility function for which:

$$\begin{aligned} &U^n(w) > 0 \ \forall w, \\ &U^n(w) = 0 \ \forall w, \quad \text{or} \\ &U^n(w) < 0 \ \forall w. \end{aligned} \tag{A3}$$

For an investor who is not strictly consistent in preference direction it is true that either:

$$\begin{aligned} &U^n(w) \geq 0 \ \forall w \quad \text{or} \\ &U^n(w) \leq 0 \ \forall w. \end{aligned} \tag{A3*}$$

Assumptions (A1) and (A2) are assumptions requiring positive marginal utility of wealth and consistent risk aversion respectively. Assumption (A3) requires that the investor exhibit strict consistency for moment preference, i.e., that the n-th moment will always be associated with indifference, $U^n(w) = 0$, or the same preference direction regardless of wealth level (the value of the coefficient of the n-th moment in (1) will always be positive, zero, or negative). Assumption (A3*) only requires that the preference direction of the n-th moment be a non-strict monotone function at each wealth level. This condition allows the possibility that over some ranges of wealth there may be indifference to the n-th moment that was preferred (positively signed) over another range of wealth, but does not allow for aversion (negative signed) to the n-th moment at a range of wealth that was positively signed (preferred) over another range of wealth.

Theorem 1: Investors exhibiting positive marginal utility of wealth for all wealth levels, consistent risk aversion at all wealth levels, and strict consistency of moment preference will have positive preference for positive skewness (negative preference for negative skewness) i.e.:

(A1), (A2), and (A3) imply $U^3(w) > 0$

Proof:

(1) Assume $U^3(w) < 0 \; \forall w$, or $U^3(w) = 0 \; \forall w$

(2) The Mean Value Theorem states that for $w_2 > w_1 \; \exists \; \bar{w} \epsilon (w_1, w_2)$ for which $U^1(w_2) - U^1(w_1) = U^2(\bar{w})(w_2 - w_1)$.

(3) Reorganizing (2) yields: $U^1(w_2) = U^1(w_1) + U^2(\bar{w})(w_2 - w_1)$.

(4) By (1) we have: $U^2(w_1) \geq U^2(\bar{w})$ and $U^1(w_2) \leq U^1(w_1) + U^2(w_1)(w_2 - w_1)$.

(5) For $w_2 \geq w^* = w_1 + \dfrac{U^1(w_1)}{-U^2(w_1)}$ we have $U^1(w_2) \leq 0$ and $U^1(w_2) < 0$ for $w_2 > w^*$.

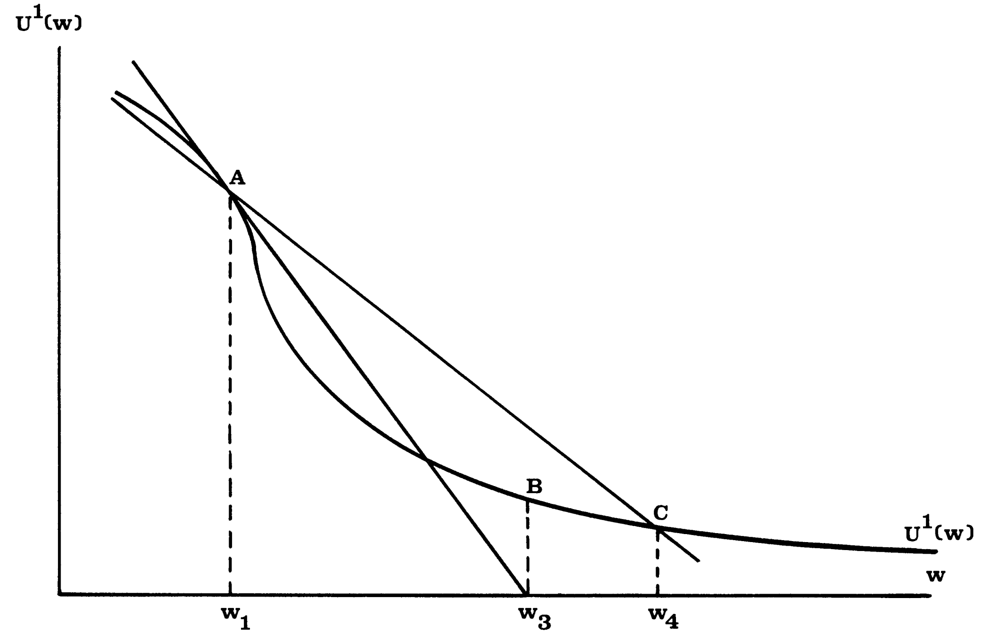
(6) This contradicts (A1), thus by (A3); $U^3(w) > 0$.

Q.E.D.

These results are consistent with the conclusions arrived at by Arditti [1], Kraus and Litzenberger [6] and Simkowitz and Beadles [10]. If an investor is assumed to have a negative preference for the third moment, μ_3, it is easy to see that this would imply negative marginal utility meaning of course that the investor would prefer less wealth to more wealth. The results of Theorem 1 indicate that the investor will be willing to accept a lower expected value from his investment of w in portfolio A, than in portfolio B if both portfolios have the same variance, $\mu_2^A = \mu_2^B$, if portfolio A has greater positive skewness, $\mu_3^A > \mu_3^B$, and all higher moments are the same. In a-priori economic terms it seems consistent that the investor would prefer an asset or portfolio with high probability of a return greater than the expected value compared to an investment with a high probability that his return will be less than the expected value. Risk averting investors prefer positive skew over no skew and over negative skew in the distribution of returns or wealth.

The Figure 1 illustrates the $U^3(w)$ greater than zero result. Marginal utility is illustrated as the curve labelled $U^1(w)$ which passes through points A, B, and C. The line segment through A and w_3 is tangent to the marginal utility curve at the point A and therefore has slope equal to the second derivative of utility at w_1. Although in general it is possible for the slope to be decreasing (thus the Aw_3 segment), for w greater than or equal to w_3 the slope of the segment from A to a point of the marginal utility curve (such as C) must increase or marginal utility would become zero or less by w_3. This indicates that $U^3(w)$ must be positive at some time and must be positive "on average". If the individual is consistent in sign of all derivatives (meaning consistent in moments preference), then $U^3(w)$ must be positive.

Theorem 2: Consistent risk aversion, strict consistency of moment preference and positive preference for positive skewness imply negative preference for the



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fourth statistical moment (kurtosis) i.e.: (A2), (A3) and $U^3(w) > 0$ imply $U^4(w) < 0 \; \forall w$.

Proof:
(1) Assume $U^4(w) > 0$ or $U^4(w) = 0$
(2) The mean value theorem states that for $w_2 > w_1 \; \exists \; \bar{w}\epsilon(w_1, w_2)$ for which $U^2(w_2) = U^2(w_1) + U^3(\bar{w})(w_2 - w_1)$.
(3) By (1) we have $U^2(w_2) \geq U^2(w_1) + U^3(w_1)(w_1 - w_1)$.
(4) For $w_2 \geq w^* = w_1 + \dfrac{-U^2(w_1)}{U^3(w_1)}$. $U^2(w_2) \geq 0$ and $U^2(w_2) > 0$ for $w_2 > w^*$
(5) This contradicts (A2) and by (A3) yields $U^4(w) < 0$.

Q.E.D.

In general, similar proofs can easily be constructed to show that the general and common assumptions of positive marginal utility, consistent risk aversion together with strict consistency of moment preference imply:[2]

$$U^n(w) = 0 \; \forall w \quad \text{if } n \text{ is odd and}$$

$$U^n(w) = 0 \; \forall w \quad \text{if } n \text{ is even.}$$

The meaning of this can be seen in the context of the well-known application

[2] The arguments above may also be used to prove that: $U^n(w) \to 0$ as $w \to \infty \; \forall n$.

of Taylor's expansion (1). The sign of $U^2(\mu)$ is known but the signs of $U^n(\mu)$ needed to determine $E(U)$ when higher moments are considered are more difficult to deal with. Especially important has been the sign of $U^3(\mu)$ needed to account for the nature of the influence of skew (μ_3) on expected utility. Reasonable economic arguments have been used to justify that $U^3(\mu)$ is positive. The results here reinforce that assertion of preference for positive skew as an inherent result of risk aversion. Further this paper gives the direction of preference for each moment when the direction of preference for each moment is known on some a-priori grounds. For example, if the sign of $U^4(\mu)$ is consistent, it is negative indicating that μ_4 is looked upon in much the same way as σ^2. Perhaps this indicates that μ_4 is not a very important consideration as diversification reduces μ_4 at a considerably greater proportional rate than it reduces σ^2 while both effects are desired. Hopefully, these results will aid in studying this and other such prospects.

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