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Source: *The Journal of Finance*, Vol. 35, No. 3 (Jun., 1980), pp. 685-691
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327492
Accessed: 28/11/2009 08:17

THE JOURNAL OF FINANCE • VOL. XXXV, NO. 1 • MARCH 1980

General Risk Aversion and Attitude Towards Risk

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THIS PAPER PROPOSES A measure of attitude of individuals towards risk. This measure includes the characteristics of the existing measures as special cases. It enables one to predict how a proportional change in the individual's expected wealth affects his risk premium for a wide range of relationship between the mean wealth and its variance.

In their path breaking works, Pratt [3] and Arrow [1] developed two measures of risk aversion $A(w) = -\dfrac{U''(w)}{U'(w)}$ and $R(w) = -\dfrac{U''(w)}{U'(w)}.\ w = A(w)w$ which are generally referred to as absolute and relative risk aversion, respectively. $U(\cdot)$ is the von Neumann-Morgenstern utility function defined over an individual's terminal wealth, w, and assumed to be increasing and differentiable. When wealth is a random variable, the measures are defined over expected terminal wealth, $\bar{w}$, which includes the initial wealth and the expected value of the lottery (Pratt, p. 125).

These measures are useful in detecting how variations in the decision-maker's terminal wealth affect his attitude toward a lottery or a risky investment. This is reflected by the induced change in the decision maker's risk premium, $\Pi(w)$, which was defined by Pratt [3] such that $EU(w) = U(\bar{w} - \Pi(w))$. The risk premium may thus be interpreted as the maximum amount which the decision maker, who is engaged in a lottery, is willing to give up in order to obtain its expected value with certainty. Consequently, the monetary "worth" which the decision maker assigns to the lottery is measured by its "certainty equivalence", $\bar{w} - \Pi(\bar{w})$: It is the minimal amount for which he would be willing to exchange the lottery if he had it. In what follows, the focus will be on these measures of attitude towards risk.

For sufficiently "small" risk, Pratt (1964) showed that[1] $\Pi(\bar{w}) = -\dfrac{1}{2}\dfrac{U''(\bar{w})}{U'(\bar{w})}\sigma^2$, where $\sigma^2 = \text{Var}(w)$. Knowledge of the characteristics of the risk aversion measures (whether they are constant, decreasing, or increasing with wealth) enables one to predict the nature of the change in the decision maker's risk premium, even when the specific form of the utility function is unknown. This can be illustrated in the following two cases:

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[1] The risk premium clearly depends on the whole probability distribution of w or, by the approximation for small risks, on its mean and variance. Since we later define the varience as a function of the mean, we shall write from now on the risk premium as a function of $\bar{w}$.

(i) When the variance of wealth is a constant and unaffected by mean wealth, it is sufficient to know the sign of $A'(w)$ in order to predict the change in the risk premium as expected wealth changes. This is the case, e.g., when $w = w_0 + \bar{x} + (x - \bar{x})$, where x is a random variable with mean $\bar{x}$ and variance σ_x^2, and w_0 is the initial wealth.

(ii) When the risky part of wealth varies in exact proportion with expected wealth, the variance changes, too. Then, the sign of $R'(w)$ is sufficient to predict whether the risk premium changes proportionately by more or less than the mean wealth. For example, when $w = w_0(1 + x)$, a change in the initial wealth affects both the expected value and the variance of wealth, and $\Pi(\bar{w}) = -\frac{1}{2}\frac{U''(\bar{w})}{U'(\bar{w})} w_0^2\sigma_x^2 = \frac{1}{2} R(\bar{w}) \cdot \frac{\bar{w}}{(1 + \bar{x})^2} \sigma_x^2$, where $\bar{w} = w_0(1 + \bar{x})$. Then, the elasticity of the risk premium with respect to the mean wealth (through a change in w_0) is greater than, equal to, or smaller than unity as the relative risk aversion is increasing, constant or decreasing, respectively. Consequently, considering the average certainty equivalence of the risky wealth, we obtain that $\frac{d}{d\bar{w}}\left(\frac{\bar{w} - \Pi(w)}{\bar{w}}\right) \gtreqless 0$ when $R'(\bar{w}) \lesseqgtr 0$, respectively. Another example occurs when random wealth is $w = (w_0 + \bar{x}) + (x - \bar{x})$ as in (i) above, and both the safe and risky part of wealth are multiplied by the same factor, say α. Then $E(w) = \alpha\bar{w}$, $\mathrm{Var}(w) = \alpha^2\sigma_x^2$ and $R'(\bar{w}) \gtreqless 0$ implies that the elasticity of $\Pi(\bar{w})$ with respect to the multiplicative factor α is greater than, equal to, or smaller than unity.

It can be seen that the two measures of risk-aversion are particularly useful for two kinds of lotteries, i.e., when the risk is unaffected by the initial wealth, and when the risk varies in exact proportion with it. Then, the sign of $A'(\bar{w})$ or $R'(\bar{w})$ is sufficient to predict the direction of change in the absolute or average value, respectively, of the risk premium.

However, the world is not limited to these two forms of lottery. In reality, changes in the lottery or investment structure may bring about changes in the mean and variance of wealth in a different way than described by (i) and (ii) above. Then, a knowledge of the properties of $A(\cdot)$ or $R(\cdot)$ may be insufficient to predict the change in the risk premium. For example, suppose that a change in the lottery changes its mean and variance by the very same proportion. Then, in what proportion will the risk-premium change? Or, the mean and the variance can each change by a different proportion due to a change in the lottery structure; how will that affect the average risk premium or the average certainty-equivalence of the lottery? The existing measures do not give an immediate and complete answer to this question.

It is thus proposed that a single measure of risk averson be designed to be useful for a variety of lottery forms. It is suggested that changes in investment opportunities affect the risk premium, and thus the certainty equivalence, in two ways:

a) The expected value of terminal wealth may change, inducing a change in the subjective utility-related measure of attitude towards risk (or: the "price" of risk) $-U''(\bar{w})/U'(\bar{w})$.

b) The risk, measured by the variance of wealth, may change. This change depends on the lottery-structure, not on the individual's preferences, and is exogeneously given.

To summarize, the proportional change in the risk premium with respect to a proportional change in the expected wealth, i.e., the elasticity of the risk premium with respect to $\bar{w}$ can be decomposed into two elasticity-components as follows

$$\begin{aligned}\epsilon_{\Pi} &= \frac{d\Pi(\bar{w})/(\bar{w})}{d\bar{w}/\bar{w}} \\ &= \frac{d(-U''/U')/(-U''/U')}{d\bar{w}/\bar{w}} + \frac{d\sigma^2/\sigma^2}{d\bar{w}/\bar{w}} \\ &\equiv \Gamma(\bar{w}) + k(\bar{w})\end{aligned}$$

Then, to predict the change in the risk premium, it will be necessary to focus separately on its two factors and see how each of them is affected by the change in the lottery.

1) *The utility-based component of the risk premium.* $\Gamma(\bar{w})$, which is the elasticity of the utility-based component of the risk premium, is proposed as a general measure of change in risk aversion. As can be seen from Table I, there is a correspondence between some values of $\Gamma(\bar{w})$ with the well-known characteristics of $A(\bar{w})$ and $R(\bar{w})$. Yet, $\Gamma(\bar{w})$ gives a *continuum* of risk-aversion characteristics, of which those of the familiar two measures are only special cases.[2] (See Figure 1.) No longer are there two specific measures of risk aversion, which are particularly useful for two special forms of lottery; there is now a single measure which is derived from the utility function of the decision-maker and is useful for a wide range of lotteries.

At this point it is worth investigating the characteristics of $U(w)$ that corre-

Table I

The Relationship Between Γ and the Properties of the Absolute and Relative Measures of Risk Aversion

	$\Gamma < -1$	$\Gamma = -1$	$-1 < \Gamma < 0$	$\Gamma = 0$	$\Gamma > 1$
$A'(\bar{w})$	<0	<0	<0	$=0$	>0
$R'(\bar{w})$	<0	$=0$	>0	>0	>0

It is worth noting that a utility function whose general risk aversion measure falls in the range $-1 < \Gamma < 0$, satisfies Arrow's hypotheses. (1965, p. 35).

[2] Some writers suggested other general measures of risk aversion, which included the properties of $A(w)$ and $R(w)$. For example, Merton [2] suggested the HARA (hyperbolic absolute risk aversion) where $A(w) = 1\Big/\left(\frac{w}{1-\alpha}+\frac{b}{c}\right) > 0$, where a, b, and c are constants, satisfying some constraints. Different values of these constants give absolute and relative risk aversion measures which are increasing, constant or decreasing. As shown later, all these properties are obtained by a change of a single-parameter, Γ. The measure $\Gamma(\bar{w})$ also captures cases where the characteristics of the risk aversion measures change with $\bar{w}$; e.g., $A(\bar{w})$ is decreasing for small w and increasing for larger $\bar{w}$, and so for $R(\bar{w})$.

		−1	0	+	Γ
Absolute risk-aversion	decreases		constant		increases
Relative risk-aversion	decreases	constant			increases

Note: It can be seen that for a given value Γ° which corresponds to a constant value of one of the prevailing risk aversion measures, any $\Gamma <(>) \Gamma^\circ$ imply a decrease (increase) in that measure.

Figure 1. The Values of Γ and the Corresponding Properties of Absolute and Relative Risk-Aversion

spond to various values of Γ. Assume, for a moment, that Γ is a constant[3]. When $\Gamma = 0$, $U(w)$ is exponential (or linear) with constant absolute risk aversion. $\Gamma = -1$ corresponds to the logarithmic utility function $U(w) = \log w$, or the constant elasticity function $U(w) = \dfrac{1}{1-\alpha} w^{1-\alpha}$. When $\Gamma(\bar{w})$ varies, its bounds define the risk aversion characteristics of the utility function. For example, $U(w) = \log(w + a)$ implies $\Gamma(\bar{w}) = -\bar{w}/\bar{w} + a$, and $-1 < \Gamma(\bar{w}) < 0$ for $0 < \bar{w} < \infty$. By Table 1, $U(w)$, those $\Gamma(\bar{w})$ has these bounds exhibit decreasing absolute, and increasing relative risk aversion.[4]

2) The variance-component of the risk-premium $k(\bar{w})$ measures the point elasticity of the variance of wealth. Some special cases of $k(\bar{w})$, when k is a constant, are:
i) When $k = 0$, the probability distribution of the lottery is independent of the expected wealth and thus $w = \bar{w} + x$, where x is an additive random variable with zero mean. Here, the variance is constant and unaffected by changes in the mean wealth.
ii) When $k = 2$, the mean value of wealth and the lottery vary in the same proportion. For example, when $w = \bar{w} + x$, as above, and both mean wealth and the lottery are changed by the same proportion, say β, then $E(\beta w) = \beta\bar{w}$ and $\mathrm{Var}(\beta w) = \beta^2\mathrm{Var}(x)$ and the elasticity of $\mathrm{Var}(\beta w)$ with respect $\beta\bar{w}$ is 2. The same result is obtained in the familiar case of risky investment, when $w = w_0(1 + y)$, where w_0 is the initial wealth and y is the rate of return. Also, $k = 2$ when w has an exponential distribution.
iii) When $k = 1$, the whole wealth is composed of many small independent and

[3] Formally, when Γ is a constant, the assumption that $kw^{\Gamma(w)} = -U''(w)/U'(w)$ where k is some non-negative constant, implies a special class of utility functions:

$$U(w) = h \int \exp\left(-\frac{k}{\Gamma + 1} w^{\Gamma+1}\right) dw$$

where h is a non-negative constant. This utility function is increasing and concave. When $\dfrac{1}{\Gamma + 1}$ is an integer for values of Γ in the interval $(-1, 0)$, one can integrate $U(w)$ recursively, to obtain a function which is a product of a polynmial and exponential function.

[4] This utility function was suggested by Rubinstein [4] as a "premier model of financial markets".

identically distributed random variables, $z_i (i = 1, 2, \ldots, n)$, with mean μ and variance σ^2. Then, $w = \sum_{i=1}^{n} z_i$, $E(w) = n\mu$ and $\text{Var}(w) = n\sigma^2$. Here, the variance of wealth is proportional to the expected value and $k = 1$. Also, the Poisson distribution gives $k = 1$ and so does the binomial distribution (see case 4 in the example below).

There may be cases where k varies with $\bar{w}$. Consider e.g., a case in which terminal wealth is $w = a + bx$, where b is the amount invested and x is one plus the random rate of return. Then $w = a + b\bar{x}$, $\sigma^2 = b^2\text{Var}(x)$, and $k = 2 + \dfrac{2a}{b\bar{x}}$. (When $a = 0$, this case collapses to (ii) above).

Finally, the information about the effect of a change in wealth on the risk-premium is summarized by the sum of $\Gamma(\bar{w})$, the utility-based elasticity measure and $k(\bar{w})$, the lottery-based elasticity measure, as demonstrated in Table II for some special cases.

The proposed measure enables one to analyze the effect of a wide variety of lottery forms on the decision maker's risk premium, even when the relation between the resulting expected value and variance of wealth do not fall in the two categories covered by the prevailing measures. In addition, the suggested measure does not only tell the direction of change in the risk premium when the lottery changes, as the prevailing measures do, but also the rate of that change.

Examples

The above results may now be applied to some commonly used utility functions and distribution functions of the uncertain wealth. The question to be answered is how the risk-premium changes in response to a change in wealth for the given functions. The usefulness of the general risk aversion measure is seen in light of

Table II

The Properties of the Risk Premium for Some Given Lottery Structures, Measured by k, and Risk-Aversion Properties, Measured by Γ

The Variance Elasticity	The General Measure of Risk Aversion			
	$\Gamma = -2$	$\Gamma = -1$	$\Gamma = 0$	$\Gamma = 1$
$k = 0$	$\Pi \alpha \frac{1}{\bar{w}^2}$ $\frac{CE}{\bar{w}}$: increasing	$\Pi \alpha \frac{1}{\bar{w}}$ $\frac{CE}{\bar{w}}$: increasing	Π = constant $\frac{CE}{\bar{w}}$: increasing	$\Pi \alpha \bar{w}$ $\frac{CE}{\bar{w}}$: constant
$k = 1$	$\Pi \alpha \frac{1}{\bar{w}}$ $\frac{CE}{\bar{w}}$: increasing	Π = constant $\frac{CE}{\bar{w}}$: increasing	$\Pi \alpha \bar{w}$ $\frac{CE}{\bar{w}}$: constant	$\Pi \alpha \bar{w}^2$ $\frac{CE}{\bar{w}}$: decreasing
$k = 2$	Π = constant $\frac{CE}{\bar{w}}$: increasing	$\Pi \alpha \bar{w}$ $\frac{CE}{\bar{w}}$: constant	$\Pi \alpha \bar{w}^2$ $\frac{CE}{\bar{w}}$: decreasing	$\Pi \alpha \bar{w}^3$ $\frac{CE}{\bar{w}}$: decreasing

(α = proportional to, $\frac{CE}{\bar{w}}$ = average certainty equivalence).

the fact that none of the prevailing risk-aversion measures enables a straightforward answer to this question, as does the proposed measure.

Case 1: An individual invests his wealth w in a project whose return has the Poisson distribution with parameter λ. Then, $E(w) = \mathrm{Var}(w) = \lambda$. Let the individual's utility function be of constant risk aversion, $U(w) = -e^{-\gamma w}$, $\gamma > 0$. The risk premium Π can be found explicitly:

$$EU(w) = -e^{\lambda(e^{-\gamma}-1)} = -e^{-\gamma(\lambda-\Pi)} = U(E(w) - \Pi)$$

$$\text{Then, } \Pi = \lambda(\gamma - 1 + e^{-\gamma})/\gamma$$

This implies that the individual's risk-premium is proportional to $\bar{w}$. But the same result could be obtained from Table II, observing that here $\kappa = 1$, $\Gamma = 0$.

Case 2: Let the utility function be $U(w) = \dfrac{1}{1-\alpha} w^{1-\alpha}$, $\alpha \neq 1$, α being the constant proportional risk aversion. Let w have exponential distribution, $f(w) = \lambda e^{-\lambda w}$, $\lambda > 0$ and $w > 0$, whose first two moments are λ^{-1} and λ^{-2}, respectively. Again,

$$EU(w) = E\left(\frac{1}{1-\alpha} w^{1-\alpha}\right) = \frac{1}{1-\alpha}(E(w) - \Pi)^{1-\alpha} = U(E(w) - \Pi);$$

$$\Pi = \lambda^{-1}[1 - (\Pi(2-\alpha))^{1/(1-\alpha)}]$$

The proportionality of the risk premium to the mean could have been obtained much easier from Table II, observing that $k = 2$ and $\Gamma = -1$.

Case 3: When $w \sim N(\bar{w}, \sigma^2)$, $k = 0$, and the change in the risk premium with respect to $\bar{w}$ depends only on Γ, i.e., on the individual's utility function. Then Γ can be applied to Table II to find how Π changes with respect to $\bar{w}$.

Case 4: The proposed method can, e.g., tell the attitude of an individual whose wealth is invested in many projects with random, independent and indentically distributed returns. If he increases his investment in additional same-type projects, the mean and variance of his random wealth will increase by the same proportion ($k = 1$). How will his risk premium change? This depends also on his general attitude towards risk captured by $\Gamma(\bar{w})$, his general risk aversion. Then, $\Gamma(\bar{w}) + k$ will indicate how his risk premium would change as the number of projects he undertakes changes. If $\Gamma(\bar{w}) < 0$ as Arrow hypothesized, then (by Table II) his risk premium will increase proportionately by less than the increase in the number of such projects. For example, consider a risk-averse farmer whose wealth is n eggs. Each egg which hatches with probability of p, $0 < p < 1$, yields \$1. He will be willing to sell his eggs for any price greater than his certainty equivalence $m = np - \Pi(np)$. If his risk aversion is such that $\Gamma(\bar{w}) < 0$, the average minimal required price for his pre-hatched eggs is increasing with n.

One can extend this result to the case presented by Samuelson [5], where an individual is presented with a number of identical lotteries whose outcomes depend on a toss of a coin and whose expected value is positive. Then, when $\Gamma < 0$ the risk-premium increases more slowly than the mean wealth since $k + \Gamma < 1$. This implies that the certainty equivalence increases with the number of such lotteries, and it might be that an individual's certainty equivalence which is negative for a small number of these lotteries will turn positive when the number of lotteries increases. For $\Gamma > 0$, the opposite may occur: A smaller number of such lotteries may be more desirable, since the risk premium is increasing faster than the mean return from these lotteries, and the average certainty equivalence is decreasing. Equivalently, the change in the per-lottery monetary "worth", the average certainty equivalence, depends on Γ. Thus, if there is a fixed per-unit charge to play such a lottery (this charge may be, of course, zero), the player whose $\Gamma < 0$ may be willing to play if he is allowed at least a certain number of lotteries, i.e., there may be a lower limit on the number of the lotteries he is willing to take. If $\Gamma > 0$, there would be an upper limit on the number of such lotteries. As Samuelson correctly predicted, constant absolute risk aversion implies that a rejection (or acceptance) of one lottery leads to a rejection (or acceptance) of any number of such lotteries.

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