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# A Dynamic Equilibrium for the Ross Arbitrage Model

JAMES A. OHLSON and MARK B. GARMAN\*

## I. Introduction

THE STRUCTURE OF SECURITY returns has been an object of constant attention in the financial economics literature. Several stochastic models for security returns have been proposed, and many of these subjected to strenuous empirical testing. Yet theory demands closure: security returns do not exist in isolation, but are intimately interconnected to all the other features of an economy in equilibrium. Of concern are the consistency of *contemporaneous* features, such as interest rates and prices, and importantly the *dynamic* characteristics, i.e., whether equilibrium is sustained over a multiperiod environment.

In this paper, we develop a consistent, dynamic equilibrium for the Ross arbitrage model of capital asset pricing. Our objective is to develop a multi-period model which is consistent with the Ross  $K$ -factor linear return generating process and its associated single-period equilibrium. Section 2 of the paper gives the initial description of the Ross model. In section 3 we construct a dynamic equilibrium. The state space is given by current dividends and these in turn are mapped into current prices. Section 4 demonstrates that additional state variables of an "informational" nature can also be introduced without damaging the essential structure of equilibrium. Several of the points developed herein are also discussed in Garman and Ohlson [1] where the focus centers on valuation in arbitrage-free markets generally; here we concentrate purely on the dynamic equilibrium nature of the Ross model in particular.

## II. The Arbitrage Model of Asset Pricing

Recently, Ross [5, 6, 7] has developed an arbitrage model of asset pricing. With some slight loss of generality, the approach can be described as follows. There is a countably infinite number of assets with random returns  $\tilde{R}_i$ ,  $i = 1, 2, \dots$ ; these returns are assumed to be governed by a " $K$ -factor" model

$$\tilde{R}_i = \alpha_i + \sum_{k=1}^K \beta_{ik} \tilde{\delta}_k + \tilde{U}_i \quad (i = 1, 2, \dots), \quad (1)$$

where  $\tilde{\delta}_1, \dots, \tilde{\delta}_K, \tilde{U}_1, \tilde{U}_2, \dots$  are independently distributed random variables with zero means and finite variances. As a parameterization, we will put  $\text{Var}(\tilde{\delta}_k)$

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$= 1, k = 1, \dots, K$ . (This means that capitalization-weighted beta's are unstandardized, i.e. do not necessarily add up to one). Suppose further that there is a risk-free asset with return  $R = 1 + r_F$ . Given certain additional mild regularity conditions, Ross then shows that in equilibrium one must have

$$\alpha_i - R = \sum_{k=1}^K \lambda_k \beta_{ik} \quad (i = 1, 2, \dots) \quad (2)$$

for some coefficients  $\lambda_1, \dots, \lambda_K$ . The latter coefficients reflect the risk-premia associated with the different commonality factors appearing in (1). The uncertainty associated with the residuals  $\tilde{U}_i$  vanishes due to the law of large numbers. The equality sign should actually be replaced by an approximate equality; the more general asymptotic relationship asserts that

$$\sum_{i=1}^{\infty} (\alpha_i - R - \sum_k \lambda_k \beta_{ik})^2 < \infty. \quad (3)$$

The above equilibrium result is derived under mild conditions on utilities, aggregate demands, and the sequence of parameters specifying the joint distributions of returns. There is no need to review those in any detail and the reader is referred to Ross [7]. It suffices to mention here that the derivations do not rely on the single-period setting generally required in the Sharpe-Lintner-Mossin capital asset pricing model.

### III. The $K$ -factor Model in a Dynamic Equilibrium

The introduction of a framework which permits a dynamic equilibrium analysis of Ross's  $K$ -factor model poses a number of problems. Prices and (time) sequences of prices are endogenous; this, of course, means that the joint distribution of returns at all points in time is also co-determined. In a multiperiod model with endogenous prices, one can therefore not assume *a priori* that the equilibrium returns process is generated by the  $K$ -factor model at all or any points in time. The price-sequence of a dynamic capital market equilibrium will be the outcome of preferences and beliefs and, more generally, states and the probabilistic structure of the state-evolution. Hence, in any dynamic model it becomes necessary to pose an analytical framework which allows for an endogenous derivation of prices and distributions of returns as functions of the states. The model must thereafter show that (1) obtains under appropriate specifications. The parameters  $\alpha_i, \beta_{ik}$  are generally endogenous functions of the states; they may or may not be state-independent. At least implicitly, the values of these parameters must be possible to derive. Similarly, the random terms appearing on the R.H.S. in equation (1) are endogenous and functions of uncertain states.

We shall hypothesize that in equilibrium

$$\tilde{R}_{it+1} = \alpha_{it} + \sum_{k=1}^K \beta_{ikt} \tilde{\delta}_{kt+1} + \tilde{U}_{it+1}, \quad t = 0, 1, 2, \dots, \quad (1')$$

is the dynamic analogue of the Ross single-period model. Then it is clear that  $\alpha_{it}, \beta_{ikt}$  are functions of the state that obtains at time  $t$ ; in addition, the outcomes of  $\tilde{\delta}_{kt+1}$  and  $\tilde{U}_{it+1}$  must be related to the outcome of the state at time  $t + 1$ . The

parameters which govern the state-evolution process will affect the explicit nature of the aforementioned functions.

A formal dynamic framework will thus require the introduction of states and prices as endogenous functions of these states. Let the vector  $\tilde{Z}_t$  denote the state at time  $t$ , and let  $\Pr\{\tilde{Z}_t \leq z_t | z_{t-1}\} \equiv F(z_t | z_{t-1})$  be the conditional distribution function of state-evolution, so that a Markovian environment is assumed. Hence, it is assumed that:

A1: *Investors have homogenous Markovian beliefs,  $F$ , and agree upon the prices and dividends that will obtain for any given (current or future) state  $z_t$ .*

The assumption entails that the price of the  $i^{\text{th}}$  asset at time  $t$ ,  $P_{it}$ , can be written as a function of  $z_t$ ; i.e.,  $P_{it} = P_i(z_t)$ . Of course, the solution is endogenous and must be related to  $F$ . This follows since the (conditional) distribution of returns  $\tilde{R}_{it+1} \equiv [\tilde{P}_{it+1} + \tilde{D}_{it+1}]/P_{it} = [P_i(\tilde{Z}_{t+1}) + D_i(\tilde{Z}_{t+1})]/P_i(z_t)$  is induced by the (conditional) distribution of states, where  $d_{it} = D_i(z_t)$  is the dividend paid by asset  $i$  at time  $t$ . The notation indicates that dividends can be viewed in two equivalent ways: either as a function of the state-description or as part of state-description.

The risk-free rate and the risk-premiums are economy-wide parameters. Any assumption regarding their stochastic behavior is an assumption about aggregate preferences and vice-versa. As a consequence, one might as well make assumptions about these directly rather than specifying the preferences of a consensus investor. The arbitrage-free nature of an economy is sufficient to assure that such specifications are sustained by some (generally state-dependent) consensus utility function.<sup>1</sup> A particularly simple case is one in which  $R, \lambda_1, \dots, \lambda_K$  are assumed to be state-independent constants. This leads to the following assumption.

A2: *Equilibrium asset prices are determined by*

$$\alpha_i(z_t) - RP_i(z_t) = \sum_{k=1}^K \lambda_k b_{ik}(z_t) \quad (\text{all } i \text{ and } z_t), \quad (2')$$

where  $R, \lambda_1, \dots, \lambda_K$  are exogenous and state-independent, and

$$\alpha_i(z_t) \equiv E[P_i(\tilde{Z}_{t+1}) + \tilde{D}_{it+1} | z_t] = P_i(z_t) \alpha_i(z_t) \quad (4)$$

$$b_{ik}(z_t) \equiv \text{Cov}[P_i(\tilde{Z}_{t+1}) + \tilde{D}_{it+1}, \tilde{\delta}_{kt+1} | z_t] = P_i(z_t) \beta_{ik}(z_t). \quad (5)$$

It is important to note that (2') lacks content unless it is demonstrated at some point in the analysis that (1') is consistent with equilibrium. Further, equation (2'), combined with (4) and (5) is a system of functional equations which must be solved for  $P_1(z_t), P_2(z_t), \dots$  given  $F, R, \lambda_1, \dots, \lambda_K$ . (Equation (2') could have been relaxed by introduction of an approximate norm, as in (3). However, if (2') is satisfied exactly, then one solution has been provided.)

Examination of (2') reveals that the simplest possible solutions must be linear; i.e.,  $P_i(z_t)$  and  $\alpha_i(z_t), b_i(z_t)$  are linear. A particularly simple specification of  $F$  is thus as follows.

<sup>1</sup> See Ross [6].

A3: The state at time  $t$  is given by  $z_t \equiv (d_{1t}, d_{2t}, \dots)$  and the state-dynamics is specified by

$$\tilde{D}_{it+1} = (1 + \theta_i + \sum_{k=1}^K \omega_{ik} \tilde{\delta}'_{kt+1} + \tilde{\epsilon}_{it+1}) d_{it}, \quad (i = 1, 2, \dots) \quad (6)$$

where  $\tilde{\delta}'_{1t}, \dots, \tilde{\delta}'_{Kt+1}, \tilde{\epsilon}_{1t}, \tilde{\epsilon}_{2t}, \dots$  are cross-sectionally and intertemporally independent with zero means and finite variance. The parameters are scaled such that  $\text{Var}(\tilde{\delta}'_{kt+1}) = 1$ , for all  $z_t$ , and  $\Pr\{\tilde{D}_{it+1} > 0\} = 1$ .

The framework above then entails that all productive decisions are exogenous. It may further be noted that  $z_t$  is the information set at time  $t$  and this set is equivalent to the state-space. The dividends obey a  $K$ -factor process across securities. In A3 the state-space is comprised only of current dividend payments, but this restriction is solely for purposes of illustration; section 4 considers the case of an enlarged state-space.

We now show the consistency of our three assumptions by linking together their parameters.

PROPOSITION. Suppose that assumptions A1, A2, and A3 simultaneously hold. Then there exist unique linear price solutions  $P_i(z_t)$ ,  $i = 1, 2, \dots$  such that the returns are governed by a  $K$ -factor model in which  $\tilde{\delta}_{kt} = \tilde{\delta}'_{kt}$  and

$$\alpha_i(z_t) = \alpha_i = (1 + \theta_i)\Gamma_i$$

$$\beta_{ik}(z_t) = \beta_{ik} = \omega_{ik}\Gamma_i$$

$$\tilde{U}_{it} = \Gamma_i \tilde{\epsilon}_{it}$$

where  $\Gamma_i \equiv R(1 + \theta_i - \sum_{k=1}^K \lambda_k \omega_{ik})^{-1}$ . Prices will be finitely positive if it is further assumed that  $R$  and  $\lambda_k$  are chosen such that  $R\Gamma_i^{-1}$  and  $r_F - (\theta_i - \sum_k \lambda_k \omega_{ik})$  are positively bounded away from zero. (The latter avoids any "growth paradox" situations.) Hence, the distribution of returns is state-independent and stationary.

*Proof:* Consider a linear trial solution  $P_i(z_t) = P_{it} = B_i d_{it}$  for all  $i$ . Such a solution will imply that returns follow a  $K$ -factor model:

$$\begin{aligned} (\tilde{P}_{it+1} + \tilde{D}_{it+1})/P_{it} &= (B_i + 1)B_i^{-1}(1 + \theta_i) + (B_i + 1)B_i^{-1} \sum_k \omega_{ik} \tilde{\delta}'_{kt+1} \\ &\quad + (B_i + 1)B_i^{-1} \tilde{\epsilon}_{it+1} \\ &\equiv \alpha_i + \sum_k \beta_{ik} \tilde{\delta}_{kt+1} + \tilde{U}_{it+1}, \end{aligned}$$

where

$$\alpha_i = (B_i + 1)B_i^{-1}(1 + \theta_i),$$

$$\beta_{ik} = (B_i + 1)B_i^{-1}\omega_{ik},$$

$$\tilde{\delta}_{kt+1} = \tilde{\delta}'_{kt+1},$$

and

$$\tilde{U}_{it+1} = (B_i + 1)B_i^{-1}\tilde{\epsilon}_{it+1}$$

Next we must solve for  $B_i$  such that (2'), (4), and (5) are satisfied, i.e.

$$a_i(z_t) = (B_i + 1)(1 + \theta_i)d_{it}$$

$$b_{ik}(z_t) = (B_i + 1)\omega_{ik}d_{it}.$$

(2') now becomes

$$(B_i + 1)(1 + \theta_i)d_{it} - RB_id_{it} = (B_i + 1)(\sum_k \lambda_k \omega_{ik})d_{it},$$

which will be satisfied with probability one if and only if

$$B_i = (1 + \theta_i - \sum_k \lambda_k \omega_{ik}) / (r_F - (\theta_i - \sum_k \lambda_k \omega_{ik})).$$

(Here we make use of the fact that the denominator is assumed to be bounded away from zero.) It is also easily shown that  $P_{it} = B_id_{it}$  is a unique<sup>2</sup> linear solution; any trial solution of the form

$$P_{it} = A_i + B_id_{it} + \sum_{j \neq i} C_{ij}d_{jt}$$

substituted into (4), (5), and (2') will reveal that (2') will not hold unless  $A_i = C_{ij} = 0$ .

The case  $K = 1$  deserves some further analysis. As Ross points out, the arbitrage model is now structurally equivalent to the capital asset pricing model. We shall examine some implications of the equilibrium solution as it pertains to the behavior of the investment opportunity set and its aggregate characteristics.

Under mild regularity conditions, easily satisfied in the present setting, the market-portfolio will be sufficiently "well-balanced" and the return on this portfolio will be perfectly correlation with  $\delta_{t+1}$ . That is, if the (triangular)<sup>3</sup> sequence of market-weights  $\{w_1(z_t), w_2(z_t), \dots\}$  is "well-balanced," it is implied that the law of large numbers holds and  $\sum_{i=1}^n w_i(z_t)\tilde{U}_{it+1} \rightarrow 0$  as  $n \rightarrow \infty$ . In the limit the residuals are unimportant, so  $\tilde{\delta}_{t+1}$  must be perfectly correlated with the return on the market portfolio. More precisely, let  $\tilde{R}_{mt+1}$  denote the return on the market-portfolio over the interval  $[t, t + 1]$ ; it now follows that

$$E[\tilde{R}_{mt+1} | z_t] - R = \lambda \sqrt{\text{Var}[\tilde{R}_{mt+1} | z_t]}, \quad (7)$$

since  $\text{Var}[\tilde{\delta}_{t+1} | z_t] \equiv 1$  and  $(\sum_{i=1}^{\infty} w_i(z_t)\beta_i)^2 = \text{Var}[\tilde{R}_{mt+1} | z_t]$ . The relative market-values are state-dependent since prices are state-dependent; hence, ruling out degenerate cases,  $E[\tilde{R}_{mt+1} | z_t]$  and  $\text{Var}[\tilde{R}_{mt+1} | z_t]$  are also state-dependent. In other words, equation (7) implies that the capital market line is constant although the location of the market-portfolio on the line will change from one period to the next. Figure 1 illustrates the situation.

The result may appear to be somewhat paradoxical if not inconsistent. If the

<sup>2</sup> In a finite-horizon economy, terminal boundary conditions will preclude the existence of "speculative bubbles," and the valuation function will then be unique. Specifically, an appropriately constructed liquidating dividend (random variable) entails a unique solution as given in the proposition.

<sup>3</sup> The sequence is triangular since  $w_i(z_t) = P_i(z_t) / [\sum_{j=1}^n P_j(z_t)]^{-1}$ , where  $i = 1, 2, \dots, n$  and  $n = 1, 2, \dots$ . Thus better but more cumbersome notation would be  $w_{in}(z_t)$ .

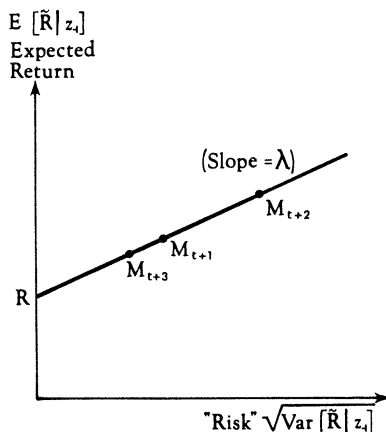


Figure 1. State-dependent market portfolios on a state-independent capital market line.

distribution of returns is state-independent, then the set of efficient risky portfolios must be state-independent. Hence, with a fixed risk-free rate, the tangent of the efficient frontier which intercepts  $R$  must also be constant across periods. Now, if each investor has preferences which are functions of the mean and standard deviation of returns, as is implied by Ross [5], then the tangent on the efficient frontier of risky securities must be at a point where the expected return and standard-deviation of returns of the market-portfolios are located. Hence,  $E[\tilde{R}_{mt} | z_{t-1}]$  and  $\text{Var}[\tilde{R}_{mt} | z_{t-1}]$  are state-independent in sharp contradiction to (7). However, the "paradox" is easily resolved: there are an infinite number of assets, and the efficient frontier of risky assets will be flat. In other words, the  $\tilde{U}_i$ 's can be neglected due to efficient diversification, and all efficient risky portfolios are perfectly correlated, implying a frontier which is a straight line.

A direct examination of the derivation which leads to (7) reveals that the capital-market line will be constant regardless of whether the distribution is state-independent or not. (Section 4 develops the case in which the equilibrium return-distribution is state-dependent). Provided that returns follow a one-factor model (1), it is implicit from assumption A2 that the situation will always be as depicted in Figure 1. The analogous result for the  $K$ -factor model is a temporally constant capital market hyper-plane. It is thus important to note that the commonality-factors of the returns,  $(\delta_{1t}, \dots, \delta_{Kt})$ , are induced by (and equal to) the commonality factors which governed the dividend processes,  $(\delta'_{1t}, \dots, \delta'_{Kt})$ . The joint factors in (1) were not due to shifts in aggregate preferences as would have been reflected by (stochastic) shifts in the risk-free rate and/or the risk-premiums. It is therefore unclear whether or not one of the factors in (1) can in fact be identified as an unexpected change in the risk-free rate; we have not yet constructed such a model. This would seem to be an important area for future research.

Some additional comments are now in order. First, it is worthwhile to note that the standardized beta-coefficients will have a state-dependent distribution. This obtains even if the ex-ante opportunity set is state-independent. Hence, in case of a one-factor model,  $\text{Cov}[\tilde{R}_{it}, \tilde{R}_{mt} | z_{t-1}] / \text{Var}[\tilde{R}_{mt} | z_{t-1}]$  will be state-dependent. Secondly, the latter also implies that if there are more than two distinct

factors, then the CAPM will never hold. Ross [6, Section IV] discusses the conditions under which a two-factor (or  $K$ -factor) model is identical to the CAPM. His analysis shows that contemporaneous restrictions must be imposed on the parameters (means, variances) characterizing the basis-portfolios<sup>4</sup> and the market portfolio. In the dynamic model developed here, these parameters are stochastic across periods (due to changing relative values), and the required constraint cannot be satisfied at all points in time with probability one. Thirdly, the analysis here shows that the arbitrage model avoids a paradox discussed in Rosenberg and Ohlson [4] and in Ross [8], namely, the incompatibility between a stationary investment opportunity set and portfolio-separation. The reason is simple: portfolio-separation plays no essential role in the arbitrage model.

The above development can also be related to preference-based theories of equilibrium. The "consensus" investor whose demands for risky assets are consistent with a constant capital market line when  $K = 1$  has the following state-dependent "derived" expected utility function at time  $t$ :

$$\begin{aligned} \max_p E[V(R_p(\tilde{z}_{t+1}); \tilde{z}_{t+1}) | z_t] \\ = \max_p G \left[ E(\tilde{R}_{pt+1}) - \frac{\lambda}{2} \text{Var}(\tilde{R}_{pt+1}) \{ \text{Var}(\tilde{R}_{mt+1} | z_t) \}^{-1/2} \right] \end{aligned}$$

where  $G$  is some monotonically increasing function and  $p$  denotes a portfolio. Due to state-independence, the return distribution of any portfolio  $p$  does *not* depend upon  $z_t$ . However, the distribution of  $\tilde{R}_{mt+1}$  does depend upon  $z_t$ , the weights being determined by price levels. An analogous derived utility function may be shown consistent with the  $K > 1$  case. However, the existence of this "derived" expected utility function is of only limited interest. The function has, of course, not been deduced in the context of an investor who maximizes his expected utility of future consumption-streams. It is therefore unclear what substantive characteristics any such utility-of-consumption-streams function has; in particular, there appears to be no reason to expect it to be of the usual additively-separable variety. In sum, the present treatment has not included any extended specification of a consensus investor's preferences for consumption-streams such that the equilibrium derived will in fact be sustained. Although difficult, it may not be impossible to do so, and future research might profitably be directed towards this issue.

Finally, it should be mentioned that the equilibrium pricing has not been accomplished by any *a priori* assumptions about the (possibly stochastic) parameters in the  $K$ -factor model (1). These have been derived endogenously, and, as a consequence, the theory has been enriched. The strategically important perspective has been one of appreciating that sequential prices and returns can be derived once assumptions have been made regarding the macro-parameters of interest rates and risk-premia. No such similar extension of the traditional (discrete-time) CAPM appears possible if the return on (and value of) the market

<sup>4</sup> By Ross's definition, these portfolios have returns which are perfectly correlated with their respective commonality factors  $\delta_{kt+1}$ ; in addition, they are uncorrelated with each other. For details, see Ross [6].

portfolio and the riskfree rate are to be derived endogenously from direct assumptions pertaining to investors utilities. For example, the introduction of utilities must force an endogenous derivation of the riskless rate  $R$ , creating considerable analytical difficulties if  $R$  turns out to be state-dependent. From this perspective, there is a logical incompleteness remaining in much previous work on this subject (Leroy [2], Ohlson [3], Stapleton and Subrahmanyam [9]). These papers purport to derive utility-based dynamic equilibria, and yet they assume that  $R$  is constant. This is, of course, incomplete since  $R$  must also be endogenously determined once preferences are introduced. The more direct approach, which has been taken in this paper, is simply one of recognizing that *every* non-arbitrage path of asset prices can be sustained by some (possibly state-dependent) aggregate utility function. (See Ross [6]). However, if assets with positive dividends are to possess finitely positive prices, care must be taken to assure that interest rates and risk-premia are consistent with the chosen state-dynamics.

#### IV. The Case of an Augmented State Space

Let  $z_t(i) \equiv (x_{1t}(i), \dots, x_{n-1t}(i), d_t(i))$  and let  $z_t \equiv (z_t(1), z_t(2), \dots, z_t(i), \dots)$ . The new variables  $x_{jt}(i)$ ,  $j = 1, 2, \dots, n - 1$  therefore represent "information" not captured in dividends alone. A general linear model of the state-dynamics is then given by

$$\tilde{Z}_t(i) \equiv \eta(i) + [I + \Theta(i) + \tilde{\Gamma}_t(i)]z_{t-1}(i) + \tilde{\nu}_t(i)$$

where  $i = 1, 2, \dots$  (assets) and

$\eta(i)$  = a vector of  $n$  constants

$I = n \times n$  identity matrix

$\Theta(i) = [\theta_{lj}(i)]$  = a  $n \times n$  matrix of constants

$\tilde{\nu}_t(i)$  = a vector of  $n$  random variables

$\tilde{\Gamma}_t(i) = [\tilde{\epsilon}_{ljt}(i)]$  = a  $n \times n$  matrix of random variables.

Suppose further that the random variables above are governed by  $K$ -factor models:

$$\tilde{\epsilon}_{ljt}(i) = \sum_{k=1}^K \omega_{ljk}(i) \tilde{\delta}'_{kt} + \text{pure } \sim \text{noise}_{ljt}$$

$$\tilde{\nu}_{jt}(i) = \sum_{k=1}^K \omega'_{jk}(i) \tilde{\delta}'_{kt} + \text{pure } \sim \text{noise}_{jt}.$$

In the model above it may be useful to think of  $x_{jt}(i)$  as representing an associated numerical characteristic of asset  $i$  at time  $t$ ; e.g., accounting numbers such as "net earnings," "shareholders' equity," or "gross margin," disclosed at time  $t$ . These quantities predict future dividend flows, and so are potentially important in the valuation-function. The general linear model as specified therefore describes the stochastic evolution of these state variables ( $x_{jt}(i)$ ) in terms of  $K$  commonality factors ( $\delta'_{kt}$ ,  $k = 1, \dots, K$ ) and pure noise terms. The commonality factors are common to all characteristics  $j$  and, as well, all assets  $i$ . It follows that

if the  $P_{it}$ 's are linear in the state variables, then so are the payoffs,  $\bar{P}_{it+1} + \bar{D}_{it+1}$ , and thus the returns,  $\bar{R}_{it+1}$ . All asset returns now have  $K$  factors in common, and the remaining factors determining returns are uncorrelated across assets. A valuation solution which is linear will therefore be consistent with a  $K$ -factor model for all assets in all states.

Using equations (2') the preceeding analysis can be made precise. From these one derives after some manipulation a general linear solution

$$P_{it} = B_0(i)x_{1t}(i) + \dots + B_{n-1}(i)x_{n-1t}(i) + B_n(i)d_t(i)$$

and, further, return-processes which have  $K$ -factor structures; the parameters in (1') are endogenous. The solutions  $B_0(i)$ ,  $\dots$ ,  $B_n(i)$  are therefore solved via the linear system

$$\begin{bmatrix} m_{00}(i) & \dots & m_{0n}(i) \\ \text{---} \\ \text{---} \\ \text{---} \\ m_{n0}(i) & \dots & m_{nn}(i) \end{bmatrix} \begin{bmatrix} B_0(i) \\ \cdot \\ \cdot \\ \cdot \\ B_n(i) \end{bmatrix} = \begin{bmatrix} f_0(i) \\ \cdot \\ \cdot \\ \cdot \\ f_n(i) \end{bmatrix} \quad (i = 1, 2, \dots)$$

where

$$m_{00}(i) = r_F$$

$$m_{j0}(i) = 0 \quad (j = 1, \dots, n)$$

$$m_{0j}(i) = -(\eta_j(i) - \sum_k \lambda_k \omega'_{jk}(i)) \quad (j = 1, \dots, n)$$

$$m_{jj}(i) = r_F - (\theta_{jj}(i) - \sum_k \lambda_k \omega_{jjk}(i)) \quad (j = 1, \dots, n)$$

$$m_{\ell j}(i) = -(\theta_{j\ell}(i) - \sum_k \lambda_k \omega_{j\ell k}(i)) \quad (j, \ell = 1, \dots, n; \ell \neq j)$$

$$f_j(i) = -m_{jn}(i) \quad (j = 0, \dots, n-1)$$

$$f_n(i) = 1 + r_F - m_{nn}(i)$$

Additional mild regularity conditions will ensure a unique linear solution. Although the system of equations solved for above is directly derived from the functional equations (2'), it should be mentioned that the solution can also be obtained as a special case of a more general valuation theorem derived by Garman and Ohlson [1]. Finally, a lengthy but direct calculation will reveal that  $\alpha_i(z_t)$  and  $\beta_{ik}(z_t)$  are generally state-dependent. From (4) and (5) it is readily seen that these latter functions are ratios of linear functions. Hence, we can provide no simple characterization of their stochastic processes.

## V. Summary

The Ross [5, 6, 7] arbitrage model of capital asset pricing has been shown to be consistent with at least one complete, dynamic equilibrium. This equilibrium involves state-independent opportunity sets and constant risk "premiums". Its simplest form entails only dividends as the information set. However, we see that

the state space and corresponding dynamics may be augmented in an arbitrary fashion without endangering our capability to derive explicit equilibrium solutions.

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