The "Market Model" In Investment Management
Author(s): Andrew Rudd and Barr Rosenberg
Source: *The Journal of Finance*, Vol. 35, No. 2, Papers and Proceedings Thirty-Eighth Annual Meeting American Finance Association, Atlanta, Georgia, December 28-30, 1979 (May, 1980), pp. 597-607
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327421
Accessed: 28/11/2009 08:16

THE JOURNAL OF FINANCE • VOL. XXXV, NO. 2 • MAY 1980

The "Market Model" In Investment Management

ANDREW RUDD and BARR ROSENBERG*

CURRENT INVESTMENT PRACTICE, much influenced by the literature of the past two decades on capital asset pricing, relies heavily upon the concept of the "market portfolio." It is believed that *the* market portfolio of all outstanding risky securities is the most appropriate construct, but that an acceptable substitute for each class of securities (or sector of the market) is provided by a broadly representative index of that sector. Preferred indexes include as many securities as possible with large outstanding value (capitalization), and weight those securities in proportion to outstanding value (capitalization weighting). There are several choices for U.S. equity indexes (for example, the S&P 500, NYSE composite, and the Wilshire 5000), the Lehman Brothers Kuhn Loeb index for U.S. bonds, the Capital International Perspectives index for foreign equities, and comparable indexes for real estate and commodities are gradually being developed.

Within each sector, contingent upon the specification of the sectoral index and hence sector return, total investment returns are separated into a systematic component of return, arising from covariance with the sector return, and a residual component of return. This decomposition of total return has entered investment practice in a number of ways. "Index funds," which are passively managed portfolios offering returns purported to be highly correlated with a return on a broad-based sector index, provide a means to earn sector returns at a low management fee. In performance evaluation, historical performance is divided into two components: one associated with systematic return and one associated with residual return, with the mean and standard deviation of the latter indicative of active management. Many investment organizations establish policy toward the exposure to sector risk ("sector timing") as a separate aspect of investment, largely distinct from selection of positions on individual assets (asset selection); this distinction is implemented through control of the portfolio sectoral exposure ("beta"), on the one hand, and policies concerning residual positions, on the other. Finally, consequent upon academic arguments for the efficiency of the market portfolio in the basic capital asset pricing model (CAPM), some scholarly discussions have argued that index funds are a cost-effective and prudent means of investment.[1]

The criticisms which can be directed against these applications fall primarily into two categories, which we will call "omitted items" and "inappropriate weightings". The former arises because the investment problem within a given

* Graduate School of Business and Public Administration, Cornell University, and Schools of Business Administration, University of California, Berkeley.

[1] See, for example, Langbein and Posner [1, 2].

sector is addressed without attention to assets held in other sectors (omitted assets), and without attention to risky liabilities (omitted liabilities). For example, Long [3] has shown the effect of ignoring the value of the investor's own earning power. This problem is, at present, almost universal in institutional investment, since investment strategy and performance analysis for equity holdings are carried out in ignorance of positions in other asset sectors. Moreover, most long-term investment strategies are associated with a need to meet uncertain liabilities, but the exact probability distribution of the liabilities is little reflected in the investment decision.

The second category of criticisms focuses on the inappropriate weightings in the portfolio used to represent the sector. Richard Roll [6] has forcefully made the point that investment proportions in the market portfolio may be misspecified. Within one sector (all common equities, for example) a subgroup (the S&P 500, for example) may be designated as the representative portfolio. Many of the early studies of capital asset pricing used an equal-weighted, rather than capitalization-weighted, market portfolio. Now that market values are widely available, capitalization weighting is the norm. However, is it not clear that total outstanding capitalization is the correct weight. For instance, when one company's shares are owned in part by another company in the sector, those shares are ordinarily double-counted, since outstanding shares of both companies are assumed to be available for investment. Moreover, in a few cases, a large fraction of the company's shares are held by obligatory investors, who, for one reason or another, will not sell: should such shares be included in the sector portfolio?

The response to these criticisms, which in some cases was well enunciated before the criticisms were formalized in the academic literature, usually takes the form of a sensitivity analysis to the possible misspecifications.[2] In essence, it is argued that the applications to which the method is put will be little affected by the omitted items or inappropriate weights.

The purpose of this article is to take one step towards rigorous sensitivity analysis. To clarify the argument, we assume that there is one prominent common factor in a given sector (the prominent factor) which influences all securities to varying degrees. While not the only factor, we assume that it is easily the most important. We also assume that it is the only source of correlation between assets in that and other sectors. In other words, correlations between activities in two different sectors (stocks and bonds, for example) arise solely from the correlations between the prominent factors in the two sectors. To the extent that this condition is satisfied, we show that the optimization problem can be partitioned into subproblems, one for each sector, in which exposure to the sector's common factor plays the role of a single dominant risk parameter; the only one which gives rise to interactions between sectors. The overall multiple-sector problem can be solved as separate problems of small dimension, giving rise to decentralized guidelines for the sectoral subproblems.

Moreover, within each of the broadly defined sectors, it is reasonable to assume that almost all securities are exposed to the common factor with exposure

[2] For example, see Mayers [4] and Miller and Scholes [5].

coefficient of the same sign and similar magnitude, which ranges at most over an order of magnitude. This single common factor does appear to be by far the largest, although by no means the only, common factor. For this reason, almost all broadly defined capitalization-weighted indexes within the sector are highly correlated with one another. Since these alternative surrogates are so highly intercorrelated, comparison with any one of the surrogates, or the use of any one of the surrogates as a sectoral portfolio, can be shown to be robust.

Under these reasonable assumptions the applications of the market portfolio, with one exception, can be shown to be relatively insensitive to misspecification. These applications include: (i) the formulation of equilibrium or normal policy in terms of sectoral guidelines; (ii) identification of active management and performance through deviations from the sectoral portfolio; and (iii) sectoral index funds as a low fee management strategy. The one exception applies to the theoretical claim in the basic CAPM that the sectoral portfolio is an efficient portfolio; the claim of efficiency is simply not robust to misspecification, and it is certainly conceivable that the true "efficient portfolio" may diverge substantially from the sectoral one. However, in the absence of a means of identifying the true portfolio, it seems reasonable to employ broad-based sectoral portfolios to define normal investment policy and active management.

Effect of Omitted Assets and Liabilities

In order to analyze the effect upon portfolio selection of omitted investment opportunities, consider a complete environment where all the risky activities are partitioned into two mutually exclusive and exhaustive sets. The first set is that subset of all risky assets from which an individual investor constructs portfolios. We refer to this as the set of "included assets" or "included sector." The second set, referred to as the "omitted activities", contains all other risky investment activities not included in the first set; namely, the omitted assets and liabilities. To readily display the implications of omission, we further assume that a risk-free asset exists, which can be purchased or lent in unlimited amounts.

Let the vector of random excess returns (over the risk-free rate) on the included assets be given by $\mathbf{r}$ with mean excess return vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$.[3] Similarly, the random excess return vector, mean excess return vector and covariance matrix of the omitted activities are given by $\mathbf{r}_g$, $\mathbf{g}$ and $\mathbf{G}$, respectively. Finally, let the covariance between the included assets and omitted activities be given by:

$$\mathrm{Cov}[\mathbf{r}_g, \mathbf{r}] = \mathbf{C}.$$

Further, assume that investors are risk averse, single period, expected utility maximizers whose preference functions are of the form:

$$U = \mu_p - \lambda_v \sigma_p^2, \qquad \lambda_v > 0 \tag{1}$$

[3] A **tilde beneath** a character denotes a column vector or matrix; lower-case characters are column vectors, upper case characters are matrices; characters without tildes are scalars. A prime (′) denotes transposition.

where μ_p and σ_p^2 are the expectation and variance of portfolio excess return, respectively, and λ_v represents the disutility, in terms of certain excess return, that arises from one unit of variance.

If the restriction of constructing portfolios solely from the included assets is not enforced, then the solution to the mean/variance portfolio selection problem (when the investment holdings are unconstrained) is found by differentiating equation (1) and given by:[4]

$$\begin{pmatrix} \mathbf{h}_g \\ \mathbf{h} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{G} & \mathbf{C} \\ \mathbf{C}' & \mathbf{V} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{g} \\ \boldsymbol{\mu} \end{pmatrix} \tag{2}$$

where $\mathbf{h}_g$ and $\mathbf{h}$ are the proportional holding vectors for the activities in the omitted and included sets, respectively, and $\lambda \equiv \frac{1}{2}\lambda_v^{-1}$ is the "risk tolerance parameter."

We are concerned with the solution for the included assets, $\mathbf{h}$: in particular, under which conditions will it bear a simple relationship to the solution obtained by ignoring the omitted activities. If the mean/variance selection problem within the included sector is solved, then (as can be seen by deletion of the omitted activities from equation (2)) the optimal holdings are given by $\lambda \mathbf{V}^{-1}\boldsymbol{\mu}$.

Using the partitioned form of the matrix inverse,[5] the lower half of equation (2) can be rewritten as:

$$\mathbf{h} = \lambda(\mathbf{V} - \mathbf{C}'\mathbf{G}^{-1}\mathbf{C})^{-1}(\boldsymbol{\mu} - \mathbf{C}'\mathbf{G}^{-1}\mathbf{g}) \tag{3}$$

It is immediately clear that if $\mathbf{C} = \mathbf{O}$ (i.e., the included and omitted activities are uncorrelated), then this solution is identical to the solution of the sectoral problem.[6] In addition, a continuity argument would suggest that if $|C_{ij}|$ was "small" for all i, j, then the solution to the relevant component of the total problem would be "close" to the solution of the sectoral problem.

A more interesting and realistic structural form of $\mathbf{C}$ occurs under the assumption that there exists a single prominent factor which influences the random excess returns of the included assets. Let the return generating process for the included sector be given by:

$$\mathbf{r} = f\mathbf{q} + \boldsymbol{\rho} \tag{4}$$

where $\mathbf{q}$ is the vector of asset loadings onto the factor, f, and $\boldsymbol{\rho}$ is the vector of asset nonfactor returns which are uncorrelated with the return on the prominent factor. Notice that there is no assumption that the factor, f, accounts for *all* the correlation among returns; the nonfactor returns, $\boldsymbol{\rho}$, may covary due to other less prominent factors. Assume further that an asset's loading onto this factor is the only characteristic which influences its covariance with the omitted activities. In other words, the only (non-negligible) commonality in the returns of the included

[4] We assume that the complete variance matrix is of full rank.

[5] See Thiel [7], p. 18.

[6] If the investor's risk aversion depends upon the solution to the total problem, then the constant of proportionality, λ, must be obtained from the overall solution.

and omitted activities arises from a dependency upon the factor. Under these assumptions, the covariance matrix, $\mathbf{C}$, must be of the form:

$$\mathbf{C} = \mathbf{d}\mathbf{q}' \tag{5}$$

where the ith element of $\mathbf{d}$ represents the covariance between the ith excluded activity and the factor. If this relation is substituted into equation (3), the optimal holdings are given by:

$$\mathbf{h} = \lambda(\mathbf{V} - c_1\mathbf{q}\mathbf{q}')^{-1}(\boldsymbol{\mu} - c_2\mathbf{q}) \tag{6}$$

where $c_1 = \mathbf{d}'\mathbf{G}^{-1}\mathbf{d}$ and $c_2 = \mathbf{d}'\mathbf{G}^{-1}\mathbf{g}$. In this case, the impact of the omitted activities causes the solution to differ by the terms involving the loading vector, $\mathbf{q}$. This is more clearly seen when the inverse is expanded, provided $c_1\mathbf{q}'\mathbf{V}^{-1}\mathbf{q} \neq 1$, to give:[7]

$$\mathbf{h} = \lambda(\mathbf{V}^{-1}\boldsymbol{\mu} + c_3\mathbf{V}^{-1}\mathbf{q}) \tag{7}$$

where $c_3 = -(\mathbf{q}'\mathbf{V}^{-1}\mathbf{q} - c_1^{-1})^{-1}(\mathbf{q}'\mathbf{V}^{-1}\boldsymbol{\mu} - c_2\mathbf{q}'\mathbf{V}^{-1}\mathbf{q}) - c_2$.

Expressed in words, the optimal holdings can be separated into two terms. The first depends upon the vector of expected excess returns and is identical to the solution to the induced sectoral problem ($\lambda\mathbf{V}^{-1}\boldsymbol{\mu}$). The second term is a multiple (λc_3) of the vector $\mathbf{V}^{-1}\mathbf{q}$.

Extension to the Case of Fixed Liabilities

Normally, some or all liability holdings are fixed, and hence do not enter the problem as decision variables. This variant of the problem can be modelled by redefining the set of omitted activities to contain only the omitted decision variables, assets and liabilities, and introducing a third set of activities, which are the fixed liabilities. Let $\mathbf{h}_l$ be the proportional liability holdings, $\mathbf{H}$ the covariance matrix between the omitted activities and fixed liabilities, and $\mathbf{J}$ the covariance matrix between the included assets and the fixed liabilities.

The solution to the total problem is found in a similar manner to that described above and is given by:

$$\begin{pmatrix} \mathbf{h}_g \\ \mathbf{h} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{G} & \mathbf{C} \\ \mathbf{C}' & \mathbf{V} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{g} - \mathbf{H}\mathbf{h}_l \\ \boldsymbol{\mu} - \mathbf{J}\mathbf{h}_l \end{pmatrix} \tag{2'}$$

Hence, the explicit inclusion of the liabilities causes an effective reduction in the mean return of assets which are positively correlated with the liabilities, and vice versa.

Proceeding as before, the holdings in the included sector are given by:

$$\mathbf{h} = \lambda(\mathbf{V} - c_1\mathbf{q}\mathbf{q}')^{-1}(\boldsymbol{\mu} - \mathbf{J}\mathbf{h}_l - c_2^*\mathbf{q}) \tag{6'}$$

where $c_2^* = \mathbf{d}'\mathbf{G}^{-1}(\mathbf{g} - \mathbf{H}\mathbf{h}_l)$. Thus, the impact of the omitted activities again causes the solution to differ by the terms involving the loading vector $\mathbf{q}$. Further, under the "single factor of correlation" assumption, $\mathbf{J} = \mathbf{q}\mathbf{b}'$, where the ith

[7] We have used the following matrix identity, which may be confirmed directly, $(\mathbf{V} - c_1\mathbf{q}\mathbf{q}')^{-1} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{q}(\mathbf{q}'\mathbf{V}^{-1}\mathbf{q} - c_1^{-1})^{-1}\mathbf{q}'\mathbf{V}^{-1}$. The result then follows by substitution into equation (6).

element of $\mathbf{b}$ represents the covariance between the ith liability and the factor. Hence, the holdings in the included sector can be rewritten as:

$$\mathbf{h} = \lambda(\mathbf{V} - c_1\mathbf{q}\mathbf{q}')^{-1}(\boldsymbol{\mu} - c_2''\mathbf{q}) \tag{6''}$$

where $c_2'' = c_2^* - \mathbf{b}'\mathbf{h}_l$. In this case, the holding vector is of the same form as obtained in equation (6), differing only in the weight placed on the loading vector. Again the inverse can be expanded to show that the optimal holdings can be separated into two components, one of which is proportional to the expected excess return vector while the other is proportional to the vector of covariances with the factor. That is:

$$\mathbf{h} = \lambda(\mathbf{V}^{-1}\boldsymbol{\mu} + c_3^*\mathbf{V}^{-1}\mathbf{q}) \tag{7'}$$

where $c_3^* = -(\mathbf{q}'\mathbf{V}^{-1}\mathbf{q} - c_1^{-1})^{-1}(\mathbf{q}'\mathbf{V}^{-1}\boldsymbol{\mu} - c_2''\mathbf{q}'\mathbf{V}^{-1}\mathbf{q}) - c_2''$.

The Distinction Between Global and Sectoral Problems

Most owners of large pools of funds now apportion them among multiple managers. Managers have responsibilities in one or several sectors, and are given instructions as to the goals and scope of activity within each sector, and as regards movement of funds across sectors. This is a form of decentralized decision making whereby the owner establishes normal (or equilibrium) goals and positions for the total fund, and the individual investment managers actively manage their components so as to produce (hopefully) superior returns relative to those obtainable from passive investment in the normal position.

The owner, who typically lacks the facilities to collect and apply timely information, usually determines the normal position for a long-term investment horizon. Some of the factors in the decision are the expectations for long-term rates of return and variance of return, the decision makers' attitudes toward risk and reward, the nature of the liabilities to be met, and constraints perceived to rule out variously defined "imprudent practices".

The optimal normal portfolio requires the specification of long-term normal or equilibrium expectations for returns. The vectors of mean returns for securities, and hence for the underlying sectoral factors, are such as expected to prevail in equilibrium. Let these vectors be denoted by $\boldsymbol{\mu}^0$ and $\mathbf{g}^0$. Then the normal position for the included assets, $\mathbf{h}^0$, optimal for this expectation under the simplifying assumptions, is found from equation (7'):

$$\mathbf{h}^0 = \lambda(\mathbf{V}^{-1}\boldsymbol{\mu}^0 + c_3^*\mathbf{V}^{-1}\mathbf{q}) \tag{8}$$

where c_3^* depends linearly upon $\mathbf{q}'\mathbf{V}^{-1}\boldsymbol{\mu}^0$ and $\mathbf{d}'\mathbf{G}^{-1}\mathbf{g}^0$.

Since the prominent factor is the principal source of risk within the sector, and the only source of correlation between sectors, it would be reasonable (under an assumption of homogeneous investors) to postulate that it constitutes systematic risk which is rewarded in aggregate. In this case, the mean excess returns are in proportion to the factor exposure; that is $\boldsymbol{\mu}^0$ is some multiple of $\mathbf{q}$, say $\boldsymbol{\mu}^0 = c_4\mathbf{q}$, and the solution simplifies to $\mathbf{h}^0 = \lambda(1 + c_3^* c_4^{-1})\mathbf{V}^{-1}\boldsymbol{\mu}^0$.

On the other hand, a perception of nonhomogeneous investors, differing for example in their treatment by the tax law, might lead to assumed equilibrium rewards which are not in proportion to $\mathbf{q}$, in which case the full complexity of equation (7′) applies.

Active Management

Now suppose that one or more active managers are hired, whose responsibility is to seek superior returns by active deviations from the normal portfolio. These returns are obtained as the consequence of research, which produces a nonconsensus vector of expected excess returns, $\boldsymbol{\mu}$. The central question is the extent to which the availability of such expectations modifies the solution given in equation (7′). When the return vector $\boldsymbol{\mu}$ is substituted into equation (7′) and the difference between the modified and normal solution (equation (8)) is taken, the active deviation $\mathbf{h}^A$, is found to be:

$$\mathbf{h}^A = \lambda\mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}^0) - \lambda(\mathbf{q}'\mathbf{V}^{-1}\mathbf{q} - c_1^{-1})^{-1}\mathbf{q}'\mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}^0)\{\mathbf{V}^{-1}\mathbf{q}\} \tag{9}$$

This difference between the optimal holdings, consequent upon the research of the active manager(s), is the optimal "active" portfolio. It consists of two terms. In order to understand the meaning of these, observe that the efficient (generalized least squares) estimate for the mean of the return on the prominent factor, given the forecasts for the individual securities, $\boldsymbol{\mu}$, is:

$$\hat{f} = (\mathbf{q}'\mathbf{V}^{-1}\mathbf{q})^{-1}\mathbf{q}'\mathbf{V}^{-1}\boldsymbol{\mu} \tag{10}$$

Thus, the term $\mathbf{q}'\mathbf{V}^{-1}\boldsymbol{\mu}$ determines the judgment for the prominent factor. If the manager's forecast for the expected factor return coincides with the owner's equilibrium forecast, then this term is equal to $\mathbf{q}'\mathbf{V}^{-1}\boldsymbol{\mu}^0$. In this case, the second term in equation (9) is zero and the holdings in the active portfolio simplify to:

$$\mathbf{h}^A = \lambda\mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}^0) \tag{11}$$

Notice that the loading of this active portfolio onto the factor, $\mathbf{q}'\mathbf{h}^A = 0$, so that "sectoral timing" is absent. This case may therefore be termed "pure selectivity" since the effect of the forecasts is to change the asset holdings without modifying the portfolio exposure to the factor from its normal position. The solution is independent of the existence of the other sectors and liabilities and is exactly the same as is obtained when all other items are omitted. For the case of pure selectivity, omitted items cause no error under the assumptions which we have made.

This is not surprising because the single factor of correlation assumption, together with unconstrained optimization of a linear mean/variance utility function, imply that a set of forecasts which do not effect the expected return on the prominent factor, have no effect upon the optimal solutions in other sectors. Moreover, the adjustment to the solution in the included sector implied by selective forecasts for the included sector is independent of the situation in other sectors.

When the requirement of pure selectivity is relaxed, the active portfolio is further adjusted by a multiple of $\mathbf{V}^{-1}\mathbf{q}$. The formula for the adjustment need not concern us here in detail.[8] The key point is that the adjustment of the portfolio exposure to the prominent factor depends upon the characteristics of solutions in the omitted sectors. The optimal response to the active factor forecast, which can be termed "sector timing", is modified by the correlated information from other sectors. Thus, the treatment of sector timing should be carried out with attention to the multiple-sector problem.

Misspecification of the Sector Portfolio

We now turn to the second concern of the paper: the question as to the importance of misspecifying the weights within the sector portfolio. In order to focus the discussion, it is convenient to hypothesize that the true sector portfolio is derived as the optimal normal portfolio, under the assumption of homogeneous investors. Specifically, the assumptions of the usual (Sharpe-Lintner-Mossin) CAPM are presumed to apply to the universe of all (included and excluded) assets. This specifically rules out the case of risky liabilities, which we have allowed for previously, and is thus an inconvenient simplification. However, it is useful in this section, since it provides us with a clear and endogenous solution to the normal portfolio.

Our first concern is to determine the relationship between a surrogate for the sector portfolio, composed from the included assets, and the prominent factor. Let the holding vector $\mathbf{h}_s$ describe the proportional asset weights in a broad-based index, s, which is designed to capture the performance of the sector. The "prominent" factor generating process (equation (4)) implies that the included assets covariance matrix can be decomposed as:

$$\mathbf{V} = \mathbf{q}\mathbf{q}'\sigma_f^2 + \mathbf{R}^* \tag{12}$$

where σ_f^2 is the variance of the factor and $\mathbf{R}^*$ is the covariance matrix of the nonfactor returns. (If the nonfactor returns were entirely specific returns, $\mathbf{R}^*$ would be a diagonal matrix. Typically, however, the influence of other factors causes $\mathbf{R}^*$ to have nonzero off-diagonal entries.) Without loss of generality, the factor can be normalized so that $\mathbf{h}_s'\mathbf{q} = 1$.

Under the CAPM, expected excess returns are proportional to the regression coefficients on the true market portfolio. Let $\boldsymbol{\beta}^*$ be the vector of the true beta for the included assets, which is given by the relevant partition of the vector, $\mathbf{V}^*\mathbf{h}_M/\sigma_M^2 = (\mathbf{C}'\mathbf{h}_E W_E + \mathbf{V}\mathbf{h}_I W_I)/\sigma_M^2$, where $\mathbf{V}^*$ is the complete variance matrix; $\mathbf{h}_M$, $\mathbf{h}_E$, $\mathbf{h}_I$ (each normalized to sum to unity) are the holding vectors for the true market portfolio and for the sector portfolios composed from the omitted activities and included assets, respectively; W_E and W_I are the proportion of value in the two sets so that $\mathbf{h}_M' = (W_E\mathbf{h}_E'\colon W_I\mathbf{h}_I')$; and σ_M^2 is the variance of the true market portfolio return. Recall that under the single prominent factor assumption $\mathbf{C} = \mathbf{d}\mathbf{q}'$, so that the true beta vector for the included assets reduces to:

[8] Any forecast $\boldsymbol{\mu}$, can be rewritten as $\boldsymbol{\mu} = \boldsymbol{\mu}_s + \bar{\mathrm{f}}\mathbf{q}$ where $\boldsymbol{\mu}_s \equiv \boldsymbol{\mu} - \bar{\mathrm{f}}\mathbf{q}$ is the selective prediction, and $\bar{\mathrm{f}}$ is the prediction for the prominent factor. By substitution into equation (9) and a little matrix algebra, the active holdings are found to be:

$$\mathbf{h}^A = \lambda\mathbf{V}^{-1}(\boldsymbol{\mu}_s - \boldsymbol{\mu}_s^0) - \lambda(\mathbf{q}'\mathbf{V}^{-1}\mathbf{q} - c_I^{-1}(\hat{\mathrm{f}} - \mathrm{f}^0)\{\mathbf{V}^{-1}\mathbf{q}\}$$

$$\boldsymbol{\beta}^* = a_1\mathbf{q} + a_2\mathbf{R}^*\mathbf{h}_I \tag{13}$$

where a_1 and a_2 are constants.

Current practice tends to identify the return on the misspecified sector portfolio, $\mathbf{h}_s$, with the prominent factor *f*. Let $\boldsymbol{\beta}_s$ be the regression coefficients with respect to the misspecified sector portfolio, then current practice would assert (implicitly or explicitly) that:

(i) since $\boldsymbol{\mu}^0$ is proportional to $\boldsymbol{\beta}^*$, then $\boldsymbol{\mu}^0$ is proportional to $\boldsymbol{\beta}_s$.
(ii) since $\mathbf{h}_I$ is the optimal normal sector portfolio, then so is $\mathbf{h}_s$.
(iii) exposure to the prominent factor, which is proportional to $\mathbf{q}$, is proportional to $\boldsymbol{\beta}_s$.
(iv) the distinction between prominent factor and other returns, appropriately determined by $\mathbf{V}^{-1}\mathbf{q}$, can be identified by $\mathbf{V}^{-1}\boldsymbol{\beta}_s$.

Evidently, the key questions are the extent to which $\boldsymbol{\beta}^*$ and $\mathbf{q}$ are proportional to $\boldsymbol{\beta}_s$, and whether the risk/reward properties of $\mathbf{h}_s$ approximate those of $\mathbf{h}_I$.

No conclusions can be drawn without some empirical evidence on the subject, and exact conclusions require that $\mathbf{h}_I$ and $\boldsymbol{\beta}^*$ be known. However, indirect evidence for U.S. equities can be obtained from the fact that all the broad-based candidates for the sector portfolio seem to be highly correlated. For instance, sample estimates for the correlation between four equity indexes were obtained by using five years on monthly data ending October 31, 1979. These coefficients are given below:

	A1000	NYSE	FRMSU
S&P 500	0.988	0.994	0.981
A1000		0.995	0.997
NYSE			0.994

where A1000 is the (capitalization-weighted) index composed of the 1000 largest companies, and FRMSU is a (capitalization-weighted) index composed of approximately 5300 companies of interest to institutional investors.

In addition, we have studied more than twenty aggregate portfolios held by owners of funds employing multiple management of equities (of which either all or the greater proportion was actively managed). Typically, these portfolios are correlated with the NYSE at the 0.98 level or higher. This fact is particularly important because these aggregates are the result of *active* management, so that there is no explicit attempt to match capitalization weights. Of course, the active decisions of multiple managers tend to average out to some extent, so that the aggregate more closely approximates outstanding security proportions than do the constituent portfolios. Nevertheless, this very high degree of correlation suggests that the behavior of the equity sector index is little dependent on the exact weights.

To clarify the implications of this, it will be useful to demonstrate the algebraic relationships between $\boldsymbol{\beta}^*$, $\mathbf{q}$, and $\boldsymbol{\beta}_s$ in order to show that high correlations among sector portfolios with differing asset holdings suggests proximity of these vectors.

The correlation coefficient between the sector portfolio, *s*, and the factor is given by:

$$\mathbf{h}_s'\mathbf{q}\sigma_f^2/(\mathbf{h}_s'\mathbf{q}\mathbf{q}'\mathbf{h}_s\sigma_f^2 + \mathbf{h}_s'\mathbf{R}^*\mathbf{h}_s)^{1/2}\sigma_f = \sigma_f/(\sigma_f^2 + \mathbf{h}_s'\mathbf{R}^*\mathbf{h}_s)^{1/2}$$

This expression can be clarified by expanding as a Taylor series. Write the nonfactor variance as $\mathbf{h}_s'\mathbf{R}^*\mathbf{h}_s \equiv \sigma_r^2$, then the correlation coefficient is:

$$1 - \tfrac{1}{2}(\sigma_r/\sigma_f)^2 + \tfrac{3}{8}(\sigma_r/\sigma_f)^4 - \ldots.$$

Hence, if the sector portfolio has zero nonfactor risk, then the correlation is unity. For broad-based sector portfolios, where $\sigma_r \ll \sigma_f$, the correlation will be close to unity.

The regression slope coefficients (betas) of asset returns computed relative to the sector portfolio are given by:

$$\boldsymbol{\beta}_s = \mathbf{V}\mathbf{h}_s/\sigma_s^2 \tag{14}$$

where $\sigma_s^2 = \mathbf{h}_s'\mathbf{V}\mathbf{h}_s$. Using the decomposition of the covariance matrix (equation (12)), the beta vector can be expressed as:

$$\boldsymbol{\beta}_s = (\mathbf{q}\sigma_f^2 + \mathbf{R}^*\mathbf{h}_s)/\sigma_s^2 \tag{15}$$

and $\sigma_s^2 = \sigma_f^2 + \mathbf{h}_s'\mathbf{R}^*\mathbf{h}_s$. Notice that $\mathbf{h}_s'\boldsymbol{\beta} = 1$ so that the portfolio has unit beta.

Comparing equations (13) and (15), it is seen that $\boldsymbol{\beta}_s$ will be proportional to $\mathbf{q}$ and $\boldsymbol{\beta}^*$ if (i) the second terms $\mathbf{R}^*\mathbf{h}_s$ and $\mathbf{R}^*\mathbf{h}_I$ are negligible relative to $\mathbf{q}$; or (ii) these terms are themselves nearly proportional to $\mathbf{q}$; or (iii) these two tendencies combine to produce a negligible difference. Under any of these circumstances $\boldsymbol{\beta}^*$ will be proportional to $\boldsymbol{\beta}_s$. Proportionality between $\boldsymbol{\beta}^*$ and $\boldsymbol{\beta}_s$ is the easier condition to obtain since it will also occur for certain values of $\mathbf{R}^*\mathbf{h}_s$ and $\mathbf{R}^*\mathbf{h}_I$, even if these values cause $\boldsymbol{\beta}_s$ to differ from $\mathbf{q}$.

Finally, the component of residual variance of the sectoral portfolio, s, relative to the true sector portfolio, $\mathbf{h}_I$, which arises from nonfactor risk is:

$$[(\mathbf{h}_I'\mathbf{R}^*\mathbf{h}_I)(\mathbf{h}_s'\mathbf{R}^*\mathbf{h}_s) - (\mathbf{h}_s'\mathbf{R}^*\mathbf{h}_I)^2]/\sigma_I^2$$

This variance is a measure of the inferiority of the surrogate for the sector relative to the true sector portfolio. Again this will be small if $\mathbf{R}^*\mathbf{h}_I$ approximates $\mathbf{R}^*\mathbf{h}_s$.

Next consider the correlation between two different broad-based sector portfolios, s and t, with asset holdings $\mathbf{h}_s$ and $\mathbf{h}_t$. The correlation coefficient between these two portfolios is given by:

$$(\mathbf{h}_t'\mathbf{q})(\sigma_f^2/\sigma_s\sigma_t) + \mathbf{h}_t'\mathbf{R}^*\mathbf{h}_s/\sigma_s\sigma_t.$$

The nonfactor covariance between the two indexes will be small under the single prominent factor assumption (i.e., $\sigma_f^2 \gg \mathbf{h}_t'\mathbf{R}^*\mathbf{h}_s$). Moreover, since s and t are both broad-based sector portfolios, they will approximately coincide in their larger values (i.e., the holdings for larger companies). This suggests that $\mathbf{h}_t'\mathbf{R}^*\mathbf{h}_s \sim \mathbf{h}_s'\mathbf{R}^*\mathbf{h}_s \sim \mathbf{h}_t'\mathbf{R}^*\mathbf{h}_t = K$, say, so that the correlation coefficient is similar to:

$$\frac{(\mathbf{h}_t'\mathbf{q})\sigma_f^2 + K}{\sqrt{(\mathbf{h}_t'\mathbf{q})^2\sigma_f^2 + K}\,\sqrt{\sigma_f^2 + K}} \sim 1.$$

Finally, notice that the difference between the betas computed from s and t will be small. This difference in betas, $\Delta\boldsymbol{\beta}$, is:

$$\Delta\boldsymbol{\beta} = \mathbf{q}\sigma_f^2(1/\sigma_s^2 - \mathbf{q}'\mathbf{h}_t/\sigma_t^2) + \mathbf{R}^*(\mathbf{h}_s/\sigma_s^2 - \mathbf{h}_t/\sigma_t^2)$$

If the terms $\mathbf{R}^*\mathbf{h}_s$ and $\mathbf{R}^*\mathbf{h}_t$ are approximately equal, then the beta vectors will be approximately proportional, with $\Delta\boldsymbol{\beta} = (\mathbf{q}'\mathbf{h}_t - 1)\boldsymbol{\beta}_s$. When the beta vectors are proportional, the decomposition between portfolio systematic and residual return is invariant to the specification of the sector portfolio.

Conversely, the fact that there is such a high empirical correlation among broad-based sector portfolios with weighting similar to capitalization, suggests that either $\mathbf{R}^*\mathbf{h}$ is consistently small relative to $\mathbf{q}$, or consistently proportional across sector portfolios. This tends to confirm approximations to $\boldsymbol{\beta}^*$ by $\boldsymbol{\beta}_s$, and the derivative assertion that the residual variance of $\mathbf{h}_s$ is small relative to $\mathbf{h}_I$.

As we have remarked above, this does not necessarily prove that $\mathbf{q}$ is proportional to $\boldsymbol{\beta}_s$. However, that is the most likely circumstance in view of the fact that no interesting discrepancies have been shown in empirical research known to us.

REFERENCES

1. John Langbein and Richard Posner. "Market Funds and Trust Investment Law," *American Bar Foundation Research Journal,* (Winter 1976), pp. 1–34.
2. ———. "Market Funds and Trust Investment Law II," *American Bar Foundation Research Journal,* (Winter 1977), pp. 1–43.
3. John Long. "Consumption Investment Decisions and Equilibrium in the Secunties Market," in M.C. Jensen, ed., *Studied in the Theory of Capitol Markets,* (New York: Praeger, 1972)
4. David Mayers. "Nonmarketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset," *Journal of Business,* (April 1973), pp. 258–267.
5. Merton Miller and Myron Scholes. "Rates of Return in Relation to Risk: A Re-examination of Some Recent Findings," in M. C. Jensen, ed., *Studies in the Theory of Capital Markets,* (New York: Praeger, 1972).
6. Richard Roll. "A Critique of the Asset Pricing Theory's Tests," *Journal of Financial Economics,* (March 1977), pp. 129–176.
7. Henri Theil. *Principles of Econometrics,* (New York: John Wiley and Sons, Inc., 1971).

DISCUSSION

WALT MCKIBBEN*: This excellent paper provides a basis for evaluating the robustness of the "Market Model"—as currently applied by investment managers—to misspecification of the market portfolio.

This issue is vitally important to those practitioners who rely on systems using some base portfolio or index as a norm against which to study performance, policy quantification and monitoring, risk analysis, and comparative characteristics of portfolios.

I believe it would be fair to say that many regular users are reasonably comfortable with current applications where results have been intuitively credible over a period of years. For other applications or extensions, however, we are uneasy and would like more evidence.

* FMR Investment Management Service, Boston.