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Hedging and Joint Production: Theory and Illustrations

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I. Introduction

THE TRADITIONAL VIEW of hedging in futures markets depicts an agent with a predetermined position in a single cash good as seeking to avoid price risk by taking the equal and opposite position in a futures contract involving that same cash good (see eg. Hieronymus (1971)). This *routine hedge* has long been recognized as only a simple form of hedging practice. The literature on futures markets includes several attempts to give a more complete account of hedging (see, Working (1962), Telser (1956), Johnson (1960), and Stein (1961)). Recently, that literature has been extended, in several ways. Anderson and Danthine (1978) and Rolpho (1978) treat the case where the cash good production relations are uncertain. Anderson and Danthine (1979) stress the fact most hedging decisions are akin to what are called *cross hedges* in the trade; that is, they involve a cash good that differs in type, grade, location, or delivery date from that specified in the futures contract. Consequently, the hedger faces the risk that the difference between the cash and futures price (the basis) will vary. We argued that fact of basis risk means that hedges involving portfolios of futures may be preferable to those involving only a single futures. To that end we presented the theory of optimal hedging with one cash good and many futures markets.

The purpose of the present paper is two-fold. First, we generalize our previous treatment of cross hedging to the case of *multiple cash goods* and multiple futures. The primary reason for this is that in certain markets the hedging participants are likely to hold an array of cash goods. This is the case notably for the interest rate futures where many participants hold portfolios of debt instruments. Second, we illustrate the theory through examples. The first is the case of the storage of agricultural goods. It is emphasized that price uncertainty implies the output decision should be made jointly even if the technical production relations are separable. The second case depicts optimal use of Treasury bond and bill futures for an agent holding a portfolio of debt. In contrast with the storage illustration in which the coefficients of the optimal hedge are found by direct statistical analysis of the cash and futures prices, the optimal positions in interest rate futures are found indirectly by using the theory of the term structure of interest rates and empirical estimates of the yield curve.

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II. Theory

In this section we present the basic theory of an individual agent's choice of position sizes for many cash goods and many futures contracts. For ease of exposition and explicitness of results we do not hesitate to make simplifying assumptions when these do not impair the features of the model that we seek to emphasize. Many of the results could be obtained under weaker assumptions at the expense of longer, less transparent derivations.

2.1 *The General Hedging Problem*

There are two trading dates. At time 0 the agent chooses the amount, y_i, to be sold at time 1 of each of m cash goods. Let y be m (column) vector of these cash positions and p be the vector of cash good prices p. The cash good production relations are represented by a cost function $c(.)$. $c(y)$ is interpreted as the cost, valued at time 1, incurred by producing the cash positions y. Implicitly the cost function is parameterized by the prices of inputs. We treat $c()$ as non-stochastic reflecting the assumption that production possibilities are certain. Let ∇c be the vector of partial derivatives c_i and $\nabla^2 c$ be the matrix of the second partials c_{ij}. We assume that $\nabla c \geq 0$ and that $\nabla^2 c$ is non-negative definite which implies $c_i \geq 0$ and $c_{ii} \geq 0$ for all i (non-negative and non-decreasing marginal costs).

In addition to dealing in cash goods the agent may trade n futures which differ with respect to deliverable good, delivery place, or delivery date. Let f_i be the size of the long position in the i'th future that the agent opens at time 0. (A short position is represented if $f_i < 0$). This position will be closed out by an offsetting trade at time 1. Let f be the n column vector of futures positions. Let p^f_{it} be the price of the i'th futures that prevails at time t and p^f_t be the n vector of future prices.

At time 0 the period 1 prices are random variables $\tilde{p}$ and $\tilde{p}^f_1$ for which the agent has a joint subjective probability distribution. Let $\bar{p}$ and $\bar{p}^f_1$ denote the vectors of expected prices and let Σ be the variance/covariance matrix of ($\tilde{p}'$, $\tilde{p}^{f\prime}_1$) which we partition as,

$$\Sigma = \begin{bmatrix} \Sigma_{00} & \Sigma_{01} \\ \Sigma_{10} & \Sigma_{11} \end{bmatrix}$$

where Σ_{00} is mxm, Σ_{11} is nxn, and $\Sigma_{01} = \Sigma'_{10}$ is mxn.

With this notation the agent's period 1 net revenue is the random variable,

$$\tilde{\pi} = \tilde{p}'y - c(y) + (\tilde{p}^f_1 - p^f_0)'f \tag{1}$$

In Anderson and Danthine (1979) we pointed out for the case of a single cash good ($m = 1$) the net revenue relation (1) was capable of representing the choices of a broad range of futures markets participants. If $y_1 > 0$ we have a "short hedger" such as a farmer or storage company. A "long hedger" such as an exporter who enters a forward sale agreement is represented by $y_1 < 0$; then $c(y_1) < 0$ is understood as minus the revenue derived from the forward sale. Finally, $y_1 = 0$ corresponds to a speculator. In the more general form ($m \geq 1$) the relation (1) is capable of modelling an extended range of participants. We can have multiple long cash positions, multiple short cash positions, or mixtures of the two. The case where an input is fixed at time 0 but input price is uncertain is represented as taking a short position in some cash good. The general cost function $c(\cdot)$ can represent a broad class of technical forms of joint production. Many interesting special cases emerge by making the appropriate separability assumptions for c. In particular, the case of no technological jointness of production occurs with an additive cost function,

$$c(y) = \Sigma_{i=1}^{m} c^i(y_i).$$

We assume that the agent selects y and f so as to maximize a function of the mean and variance of net revenue. Our choice of mean/variance analysis is dictated by our desire to find tractable, closed form expressions that focus on the essential points we wish to make. Further justification for this preference function as well as statement of conditions when this is compatible with expected utility analysis is contained in Anderson and Danthine (1979) and Rolpho (1978) among others. The problem is to

$$\max_{y,f}(\bar{\tilde{\pi}} - \tfrac{1}{2}\lambda \operatorname{var} \tilde{\pi}) \tag{2}$$

subject to net revenue relation (1).[1] Here the degree of risk aversion is taken to be a positive constant, λ.

The first order necessary conditions for a maximum are,[2]

$$\bar{p} - \nabla c(y) - \lambda(\Sigma_{00}y + \Sigma_{01}f) = 0 \tag{3}$$

$$(\bar{p}_1^f - p_0^f) - \lambda(\Sigma_{11}f + \Sigma_{10}y) = 0 \tag{4}$$

These conditions are sufficient in view of the fact that $\nabla^2 c$ and Σ are non-negative definite.

2.2 *Optimal Futures Positions*

The vector of optimal futures positions given y is unique if and only if the covariance matrix of futures prices, Σ_{11}, is nonsingular. Assuming this, equation (4) implies,

$$f = \frac{1}{\lambda}\Sigma_{11}^{-1}(\bar{p}_1^f - p_0^f) - \Sigma_{11}^{-1}\Sigma_{10}y \tag{5}$$

As in Anderson and Danthine (1979) it is interesting to view the optimal futures position as decomposable into two parts. The first is $1/\lambda\,\Sigma_{11}^{-1}(\bar{p}_1^f - p_0^f)$ is the *pure speculation* in the sense that this is the optimal position for a speculator ($y = 0$). As we would expect this speculative portfolio depends upon the expected price changes as filtered by the variances and covariances of futures prices and by

[1] Since we impose no wealth constraint this implicitly assumes the availability of financing of futures positions at a constrant rate reflected in p_0^f.

[2] We omit reference to bounds on the range of y. If inequality constraints apply then equation (3) is appropriate for an interior solution.

attitudes toward risk aversion. The remaining term, $-\Sigma_{11}^{-1}\Sigma_{10}y$, is the *pure hedge*. The pure hedge in a future, say the j'th, is the sum of m terms. Each of these is proportional to the cash position y_i, with a coefficient of proportionality equal to the coefficient corresponding to p_j^f in the theoretical multiple regression of p_i on p^f. Since multiple regression coefficients determine the pure hedge it is clear that the *partial correlation* of p_i and p_j^f given the remaining futures prices is a better indication of the usefulness of the j'th future is hedging the i'th cash good; this contrasts with the conventional view that the simple correlation is an indicator of hedging effectiveness. For a further discussion of the optimal futures position and the relation to the existing literature on hedging, see Anderson and Danthine (1979).

It is useful to note that the pure hedge position in a future given a vector of optimal cash positions may be obtained by summing the optimal pure hedges with respect to each of the cash positions alone. However it should be kept in mind that the optimal position in a cash good treated in conjunction with other cash good differs from that when the cash good is treated in isolation. Furthermore, since a speculative component would be present in the total futures position for each single cash good hedger, total futures for an agent dealing in cash goods is not the sum of the m single-cash-good optimal futures positions.

Finally, we note that in certain applications it may be useful to consider the cash position as predetermined and to select futures positions so as to minimize risk. It is easy to see that what we have called the pure hedge is formally equivalent to the position that minimizes Var $\tilde{\pi}$ subject to (1). However, in this case the predetermined cash position y need not be optimal. Generally, the optimal cash and futures positions are chosen simultaneously.

2.3 *Optimal Cash Positions*

The optimal production decision for many cash goods differs in important ways from the case of a single cash good. The firm with many cash goods must generally select outputs simultaneously. As we would expect from the neoclassical theory of the firm under certainty, one reason for this is that there may be a jointness of production opportunities or of input prices so that the marginal cost of a good may depend on the outputs of all goods. The second, less obvious, reason is that the stochastic characteristics of output prices generally mean that the risk premium applicable to a given output depends upon the levels of output of all cash goods.

To investigate the optimal cash goods decision we insert the optimal futures position (5) into the first order condition (3) and rearrange to obtain,

$$\bar{p} - \Sigma_{01}\Sigma_{11}^{-1}(\bar{p}_1^f - p_0^f) - \lambda(\Sigma_{00} - \Sigma_{01}\Sigma_{11}^{-1}\Sigma_{10})y = \nabla c(y) \tag{6}$$

This expression can be interpreted as stating that for each good the marginal cost should be equated to a planning price corrected by a risk premium. The vector of marginal costs $\nabla c(y)$ is in general a function of output levels of all goods. This represents the possible jointness of the production process or the case where inputs are used in more than one production process and where input prices depend upon amounts purchased. The vector of planning prices is $\bar{p}$ −

$\Sigma_{01}\Sigma_{11}^{-1}(\bar{p}_1^f - p_0^f)$; thus for a given good the planning price is a linear combination of the expected price of the cash good and the current and expected prices of all futures. The weights applied to futures prices are taken from the theoretical multiple regression of the cash price on all futures prices. In the case where expected futures prices "explain" cash price expectations, i.e., $\bar{p} = \Sigma_{01}\Sigma_{11}^{-1}\bar{p}_1^f$, the planning price is a function of observed futures prices only.

The vector of risk premia, $\lambda(\Sigma_{00} - \Sigma_{01}\Sigma_{11}^{-1}\Sigma_{10})y$, depends upon output levels, variances and covariances of all prices (cash and futures), as well as the degree of risk aversion. It is useful to write out the risk premium applied for a single production process, say the k'th,

$$\lambda(\Sigma_{i=1}^{m} y_i \operatorname{cov}(p_k, p_i) - \Sigma_{j=1}^{n}\beta_{kj}\Sigma_{i=1}^{m} y_i \operatorname{cov}(p_j^f, p_i))$$

where β_{kj} is jth coefficient in the multiple regression p_k on all futures prices. Clearly the risk premia for different production processes will generally differ. Further, the risk premium for a process k will generally depend upon the production levels of all cash goods not just the k'th. This form of stochastic jointness production decisions is present in all but extreme special cases (e.g., $n = m$, $\operatorname{Cov}(p_k, p_i) = 0$ $(i \neq k)$, $\beta_{kj} = 0$ $(j \neq k)$ and $\operatorname{Cov}(p_j^f, p_i) = 0$ $(i \neq j)$). It should be emphasized that our result shows, in a world of uncertain output prices, optimal output choices must be made simultaneously even if there is no technical jointness in production so that the cost function of the multiproduct firm is additive. This is a natural result of applying a portfolio approach to the production decision.

Anderson and Danthine (1979) studied the effect on output of changing the degree of risk aversion, the correlation of cash and futures prices, and the number of futures markets available. Space does not permit a similar discussion for the present case of multiple cash positions. Suffice it to say that the same issues arise in the present case. In particular, the presence of speculative opportunities means that the increased correlation of cash and futures prices (as through the introduction of additional futures) need not increase the size of the cash positions. Furthermore, new complications arise from the fact of multiple cash goods. Thus the effect of a decrease of risk aversion is generally ambiguous, whereas in the case of single cash good, this generally increases the size of the cash position.

III. Illustrations

We now turn to examples with a view to making the implications of the preceding theory concrete. For agents considering a portfolio of futures for hedging purposes an important question is how to obtain estimates of the parameters needed to compute the optimal position sizes. We suggest two approaches to this question—direct statistical estimation using cash and futures price series or computation of parameters from relations estimated for the cash prices alone. While trying to make these examples realistic, we abstract from some institutional details. Furthermore, the statistical estimates we employ are quite rough; clearly, more serious statistical work would be required if one were to actually apply either of the approaches.

3.1 *Grain Storage*

A grain storage company commonly has more than one kind of cash good that could be held in its facilities. When cash prices are uncertain a charge for risk must be taken into account in choosing amounts to be stored. The analysis of section 2.3 shows that this charge depends on the range of cash goods available, the amounts of those goods in storage, as well as the characteristics of available futures markets. To illustrate how this occurs consider a *hypothetical* storage company which can store oats and/or barley. Suppose this firm also may take positions in the Chicago Board of Trade oats and corn futures. The conventional view point holds that the firm's oats cash position might be hedged routinely in oats futures whereas the barley position can only be cross hedged in either the corn or oats futures. In contrast we will apply the results of section 2 to find optimal hedges involving both oats and corn futures and to find the optimal cash positions. To do so we must specify the cost function, and the first and the second moments of the joint distribution of prices.

Our cost function takes a particularly simple form,

$$c(y_1, y_2) = (1 - \alpha)q_1y_1 + (1 - \alpha)q_2y_2 = (1 - \alpha)q'y$$

where q_i is the cash price of good i prevailing at the time the cash position is acquired, $i = 1$ for oats $i = 2$ for barley, and α is a constant $(0 \leq \alpha \leq 1)$. Thus the cash goods are acquired at a discount from the then prevailing cash price where the discounts are the same for the two goods. Note this additive cost function abstracts from the interdependence of marginal costs that we might think plausible when storage capacity is fixed in the short run. We assume the firm naively adopts static expectations. That is, $\bar{p}_i = q_i (i = 1, 2), \bar{p}^f_{j1} = p^f_{j0}$ ($j = 1$ for oats futures and $j = 2$ for corn futures). This is equivalent to saying the firm never preceives an incentive to speculate. Applying these assumptions to equation (6) shows the optimal cash position to be,

$$y = \frac{\alpha}{\lambda} (\Sigma_{00} - \Sigma_{01}\Sigma_{11}^{-1}\Sigma_{10})^{-1}q \tag{7}$$

The second moments of the price distribution are estimated directly.[3] The data are the average monthly Minneapolis cash prices of oats and barley and average monthly closing prices of the Chicago Board of Trade oats and corn futures for March 1976 delivery. All prices are in dollars per bushel. The sample covered

[3] Variances and covariances were estimated by their sample analogues. Regression coefficients were obtained by ordinary least squares. We emphasize that before actually applying statistical estimates for hedging purpose, estimates should be carefully checked for possible misspecification. We view the possibility of non-constancy of the parameters over time as potentially a most serious problem. This is particularly the case since an historical time series of futures involves an instrument of constantly decreasing time to delivery.

June 1975 through February 1976; all data were obtained from the Chicago Board of Trade *Statistical Annual* (various years). The results are given in Table 1.

The optimal cash position sizes can be computed using equation (7). Table 2 contains the optimal cash positions for various values of λ assuming $\alpha = .1$ (cash goods acquired at ten percent discount) and cash prices were at the sample period averages (q_1 = \$1.652 and q_2 = \$2.37). The optimal futures positions (which in view of the static expectations assumption coincide with the pure hedge positions) were computed using equation (5) and the results of Table 1.

Notice that despite the fact that the producer expects to make a ten percent gain on either cash good, the position size in cash barley is more than ten times that of cash oats. This follows from the fact that cash barley can be more effectively hedged with the available futures than can cash oats. (Compare R^2's). This is true even though one of those futures calls for delivery of oats. Next notice that the firm chooses to hedge its long position in oats and barley by taking a short position in corn futures and a *long* position in oats futures. This is due to the negative partial correlation of barley price and oats futures price given corn futures and the large barley position relative to the oats position.

These results stand in obvious contrast to the practice of routine hedging. They also differ importantly from the results one would obtain by applying the rules for optimal choice of a single futures to the data of the present example. For example, suppose the firm were to hedge a cash position in the futures most highly correlated with that good. By calculating the appropriate simple regression we find the pure hedge proportions would be short .4 bushels oats futures for every long position of 1 bushel of cash oats. For every long position in barley of one bushel the firm would go short 1.7 bushels of corn futures. Furthermore, application of equation (7) for the case $m = n = 1$ to the present data reveals the

Table 1

	Cash		Futures	
	Oats	Barley	Oats	Corn
1. Covariances				
Cash Oats	.00142	.01020	.00208	.00538
Cash Barley	.01020	.16324	.02079	.08396
2. Multiple Regressions				
Cash Oats (R^2 = .594)			.36779	.01420
Cash Barley (R^2 = .866)			−.30836	1.75987

Table 2

Optimal Position Sizes (thousand bushels)

	Cash		Futures	
λ	Oats	Barley	Oats	Corn
.000005	225	3516	1001	−6191
.000010	113	1758	500	−3095
.000015	75	1172	334	−2063

optimal position in cash oats would be almost 30 times that of cash barley.[4] This completely reverses the pattern seen when cash positions were chosen jointly and dramatically demonstrates how the stochastic nature of output prices introduces an important jointness into the production decision even though the production technology is separable.

3.2 *Debt Portfolios and Interest Rate Futures*

The emergence of interest futures has been recognized as having important potential for agents facing unwanted interest rate risk. However, the traditional tenant of routine hedging has limited practical appeal to those who hold positions in a variety of debt instruments, for some of their positions may effectively hedge others. In this section we see how the theory of section 2 can serve as a framework for the approach to financial futures by an agent dealing in portfolios of debt. Specifically we consider the problem of hedging portfolios of default risk free, non-callable coupon bonds of a variety of maturities using futures contracts calling for delivery of bonds of this type though of possibly different maturities.

Our approach will differ from that taken in the preceding example in two ways. First, we will take the cash position as given and determine the futures position which minimizes the variance of total net revenues. While the spirit of our theoretical treatment is to stress the simultaneous determination of cash and futures positions, the more restricted formulation is not without interest. For many agents may have a satisfactory way of choosing positions in the cash markets but may not have a similar ease with the interest rate futures. Further, the high liquidity and low transactions costs of organized futures markets may lead an agent to revise futures positions more frequently than cash positions. The second way the approach of this section differs from that of 3.1 is in the way the optimal hedge proportions are estimated. One could attempt direct statistical estimation using joint observations of the relevant bond price and bond futures price series. However, this has an important drawback in that the compilation of quality data of the full range of bond prices is a difficult task. A more economical approach is possible in recognizing that all the cash instruments are related through the term structure of interest rates which obeys a degree of regularity. Furthermore interest rate futures will be related to the cash instruments by arbitrage considerations. In this spirit we work from an estimated model of the yield curve and derive the implied estimates of the pure hedge proportions of interest rate futures.

Before proceeding further, we need to recall some concepts from the theory of the term structure of interest rates, (see Richard (1978) for a discussion of this). Let $B(t, m)$ be the price at time t of a default risk free discount bond which will pay \$1 at time m. The link between bond prices and interest rates is given in the relation $B(t, m) = \exp - (R(t, m)(m - t))$ where $R(t, m)$ is the rate continuously

[4] The simple regression coefficient of oats on oats futures is .403; of barley on corn futures is 1.681. Combining these coefficients and the relevant coefficients from Table 1 we find the optimal position sizes for an agent with $\lambda = 10^{-5}$ are 28.3 million bushels of oats and 1.07 million bushels of barley.

compounded to maturity. The forward rate of interest for funds between two future dates m_1 and m_2 is the $FR(t, m_1, m_2)$ defined implicitly in the relation

$$B(t, m_2) = \exp - (R(t, m_1)(m_1 - t) + FR(t, m_1, m_2)(m_2 - m_1))$$

An important concept is the instantaneous forward rate $FR(t, m; m)$ defined as the limit of $FR(t, m, m_2)$ as m_2 goes to m. It can be shown that

$$FR(t, m, m) = \frac{-B_2(t, m)}{B(t, m)} \tag{8}$$

where B_2 is the partial derivative of B with respect to m. The yield curve can be interpreted as the path of $FR(t, m, m)$ as m varies. The importance of relation (8) is that it shows that we can obtain a bond price from the yield curve by integration. Given this link between the yield curve and discount bonds, we can in turn link these to the price of a default risk free coupon bond. In the absence of default risk coupon bonds are linear combinations of discount bonds. For example, the price $B(t, m, c, P)$ of a bond pay coupon c every period until maturity when it also pays the principal P is given by,

$$B(t, m, c, P) = c\, \Sigma_{i=t+1}^{m} B(t, i) + P\, B(t, m) \tag{9}$$

Finally, we can extend the link to interest rate futures by an arbitrage argument. Let $F(t, d, m - d, c, P)$ be the price at time t of a futures contract calling for delivery at time d of a bond with coupon c, Principal P and time to maturity $m - d$. A long position in this future opened at time t and held until delivery would (abstracting from margins) generate no cash flow at time t and $B(d, m, c, P) - F(t, d, m - d, c, P)$ at time d. An equivalent trade in the cash market would involve going long at time t one (c, P) bond maturing at time m and at the same time shorting $B(t, m, c, P)/B(t, d, c, P)$ of the bonds of the same coupon and principal but with maturity date d. This generates no cash at time t and $B(d, m, c, P) - (B(t, m, c, P)/B(t, d, c, P))\, P$ at time d. (This treatment ignores a slight discrepancy of the net coupon payments on this forward transaction.) A comparison of the futures and forward transactions shows they involve no initial investment and have similar risks. Consequently, in the absence of transactions costs profitable arbitrage between the cash and futures market will be possible unless

$$F(t, d, m - d, c, P) = \frac{B(t, m, c, P)}{B(t, d, c, P)} P. \tag{10}$$

This is no arbitrage relation establishes an important link between bond prices (and interest rates) and interest rate futures. The relation can be expected to hold only as an approximation since in reality there are transactions costs and the futures position would generate random cash flows over the holding period. Rendelman and Carabini (1979) present evidence suggesting that the former is a more important source of error than the latter. In what follows we make use of (10) while recognizing that the quality of this approximation may vary across applications.

If one has a workable model of the yield curve equations (8), (9), and (10) allow the construction of the optimal hedge proportions for interest rate futures. We illustrate this for an agent who believes that the equilibrium yield curve depends upon two random variables—the instantaneous long and short rates—and that other rates tend to lie along an exponential curve linking these two rates. Yield curve models based on a small number of instrumental rates have recently received a theoretical foundation in the analysis of Cox, Ingersoll, and Ross (1978). While it would be interesting to apply one of their models to the present problem we prefer first to employ the simpler exponential yield curve. Ayres and Barry (1979) present statistical estimates of such curves and argue the model has good explanatory power. If we take Ayres' and Barry's exponential form as the functional specification for the instantaneous forward rate, we have,

$$FR(t, m) = \alpha_1(1 - \alpha_3(m - t))e^{-\alpha_3(m-t)} + \alpha_2$$

where α_2 is the long rate, $\alpha_1 + \alpha_2$ is the short rate, and α_3 determines the speed with which the rates approach α_2 for higher maturities. Then, substituting this into (8) and integrating yields the following bond price expression,

$$B(t, m) = \exp - (\alpha_1 e^{-\alpha_3(m-t)} + \alpha_2)(m - t). \tag{11}$$

Ayres and Barry argue that when maturities are expressed in years this curve with $\alpha_3 = .5$ gives a good approximation to the yield curve for U.S. government bonds. Accepting this estimate, we complete the model by assuming a stochastic specification of $\alpha_1 + \alpha_2 = .005\ x_1$ and $\alpha_2 = .0018\ x_2$ where x_1 and x_2 are distributed Chi Square with 18 and 50 degrees of freedom respectively. It may be verified that this implies that the short rate has a mean of .09 and standard deviation of .03. Similarly the long rate has a mean of .09 and standard deviation of .018. Admittedly these specifications are somewhat arbitrary. However, relative size of the standard errors seems plausible. A skewed shape receives some theoretical foundation from Cox, Ingersoll, and Ross (1978) who conclude for one of their models the interest rate has a noncentral Chi Square distribution. Here we simply approximate a noncentral Chi Square using the Chi Square as suggested by Scheffé (1959).

Using this model of the yield curve we are able to find the optimal pure hedge positions in financial futures for an agent holding a portfolio of coupon bonds. Specifically we assume the agent holds positions in 2, 5, and 10 year Treasury bonds paying an *annual* coupon of \$8000 and principal of \$100,000. We assume two futures are available—one calling for delivery in 1 year of \$1 million in 90 day Treasury bills; the other calling for delivery in one year of a 25 year Treasury bond with \$8000 annual coupon and \$100,000 principal. The optimal hedge proportions are found as follows. One hundred realizations of α_1 and α_2 are obtained by stochastic simulation of the distributions assumed above. Given these, one hundred observations of the prices of each of the three bonds and two futures are obtained using (9), (10), and (11) as required. Using these data the optimal hedge proportions are obtained by an ordinary least squares regression of each cash price on the two futures prices. These results are reported in Table 3 as are the results for the regressions of cash on a single futures.

Table 3

Regressions of Simulated Bond Prices on Simulated Bond and Bill Futures

	Constant	Futures: 90 day bill	Futures: 25 year bond	R^2
1. 2 year bond	−937.0	1.065	−.0723	.989
	−602.0	.747		.921
	0.87		.099	.447
2. 5 year bond	−825.0	.923	.184	.996
	−1600.0	1.733		.915
	62.0		.333	.921
3. 10 year bond	−742.0	.807	.467	.997
	−2706.0	2.86		.824
	33.7		.598	.978

Accepting the term structure model of this section as valid, the risk minimizing position sizes of futures are given by the slopes of regressions. For example, for an agent who is long one hundred 10-year bonds the pure hedge position is to be short 81 Treasury bill contracts and short 47 Treasury bond contracts. Notice that as we would expect the longer the time to maturity of the cash bond to be hedged the greater the size of the T bond futures position and the smaller the size of the T bill futures position. Notice also that optimal hedge proportions when using a single futures differ sharply from those when both futures are employed. Judging from the R^2 we confirm the intuitive notion that for a 2 year bond the T bill futures are preferable if only one futures is to be employed whereas for the 10 year bond the T bond futures are preferable. This would be a relevant concern if cash positions are small so that the discreteness of futures contracts was an issue. Finally, recall that agents with portfolios of the three cash bonds find the total pure hedge position in futures by adding the pure hedges corresponding to each cash position.

IV. Conclusion

It should be stressed that the two examples presented above are meant to illustrate our theory and to suggest methods for its application. The estimates themselves are only rough and should not be taken as guides for actual hedging practice or as a test of the effectiveness of our model. The latter will depend on the robustness of our assumptions as well as the quality of the regression estimates. Our future research will try to give additional information on both of these issues.

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DISCUSSION

GORDON C. RAUSSER*: The principal results of Anderson and Danthine can be summarized as follows: (1) in general, cash and future positions should be determined simultaneously; (2) the optimal hedge should be viewed as a sum of risk-minimizing position and a speculative position; (3) basis risk is not only important for physical goods which are equivalent to goods defined by prespecified futures contracts but as well for those goods for which no futures market exists; and (4) the allocation signal to be used by hedger in selecting among alternative production levels is a linear combination of his spot market price expectations and the relevant futures prices. These results, aside from the clarifications needed to incorporate basis risk, follow immediately from conventional portfolio theory.

To be sure, the paper's distinguishing feature is the explicit recognition of basis risk in the context of conventional portfolio theory. This recognition provides one of the more interesting implications of the A-D analysis in terms of existing literature. Specifically, in the absence of basis risk, the risk premium disappears; and the relevant price signal to be used by the hedger does not include the spot price expectations component. This result has been noted recently by Holthausen and Feder *et al.*, as well as previous work of A-D. All risk-averse firms in the market key their production decisions to the futures price; thus, there is a separation of the real production decision and the futures market position decision. The incorporation of basis risk, however, eliminates this separation.[1]

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[1] Note that, if the relevant covariance matrix among the multiple cash and futures prices is singular, we have an equivalence to the absence of basis risk; and, once again, we have a separation of physical and futures market position decisions.