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Equilibrium Term Structure Models: Test Methodology*

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I. Introduction

IN THIS PAPER, it is shown that the Cox, Ingersoll and Ross (CIR)[1977] and Vasicek[1977] closed-form solutions for equilibrium default-free bond prices are particular cases of a more general solution. This solution is characterized by separability in maturity and "state variable" arguments of the bonds' semi-elasticity with respect to the state variables (or their instruments). The simple derivation of this result and several tests of the general solution are given in Section 2.

It is shown, in Section 3, that particular solutions for bond prices which are expressed as functions of state variable instruments impose restrictions on the time-series processes for these instruments. Hence, the solutions may be tested by testing the implied restrictions. To further the application of this test, an extended model which includes *two* instrumental variables—real per capita consumption and the nominal short rate of interest—is formulated and solved here.

In Section 4, a solution for the equilibrium bond price is obtained by numerical integration of the pricing function when stated in terms of per capita consumption. Two continuous-time processes for the consumption series—the lognormal and Ornstein-Uhlenbeck (O.U.)—are estimated by full information maximum likelihood methods. The results are used to illustrate deficiencies claimed here to exist in numerical methods.

Finally, it is shown in Section 5 that a test of the expectations hypothesis follows immediately from the "fundamental partial differential equation" on which closed-form solutions for equilibrium bond prices are based (see, for example, CIR[1977]). Tentative evidence tends to reject the expectations hypothesis.

All closed-form solutions for equilibrium bond pricing functions except the extended model referred to above and one given in CIR[1977] are single-state variable models. Whilst there appears little evidence to date that the dimensionality of the state variable vector cannot be treated as "small," tests based on single-state variable processes preclude, at the least, a satisfactory approach to the evaluation of the relative importance of nominal versus real factors in the determination of the nominal term structure. An alternative methodology has

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been developed which overcomes this limitation, but space prevents further discussion.

II. Separable Pricing Function Elasticities

Define the equilibrium bond pricing function as $F[W, \underline{Y}, \tau; t]$, where F is the nominal bond price; W is real wealth; $\underline{Y}$ is a vector of state or opportunity set shift variables, including "the" nominal price index; τ is term to maturity; and t is a point in time. In this section, it is shown that the class of solutions for F which are sufficient for separability of the semi-elasticity of F with respect to $[W, \underline{Y}]$ and τ is of a particularly simple form. It is further shown that the two closed-form solutions given by CIR[1977] and Vasicek[1977] are special cases of this class. Since these solutions are sufficient for separability, it follows that separability is necessary for the solutions to be valid.

In addition, the separability properties of three "additional" models are examined. The first is a model with two explanatory variables—real per capita consumption and the nominal rate of interest, a closed-form solution for which has been derived in Section 3. The second is a model containing an "extrapolative" element, given by Ingersoll, Skelton and Weil[1978]. The third is a model similar to the second, but with different assumptions about the behavior of the extrapolative component (which may be interpreted as a stochastic coefficient).

The CIR solution for the bond price $F_j(i, \tau_j; t)$ is:[1]

$$E(R_j/i) - i = \phi\epsilon_{F_j \cdot i}[i(W, \underline{Y}, t), \tau_j; t]\cdot \mathrm{var}(di/W, \underline{Y}, t) \tag{2.1}$$

where: $E(R_j/i)$ is the expected instantaneous rate of return on the bond; i is the instantaneously riskless rate of interest; $\epsilon_{F_j \cdot i}$ is the semi-elasticity $\dfrac{\partial \ln F_j}{\partial i}$; and ϕ is a risk parameter given by technology (except for log utility), and taste parameters.

It can be shown that the analogous expression when the bond pricing function is stated in terms of equilibrium per capita consumption $C^*(W, \underline{Y}, t)$ as $F[C^*, \tau_j; t]$ is:

$$E(R_j/i) - i = RRA\cdot\epsilon_{F_j \cdot C^*}[C^*(W, \underline{Y}, t), \tau_j; t]\cdot\frac{\mathrm{var}(dC^*/W, \underline{Y}, t)}{C^*} \tag{2.2}$$

where $\epsilon_{F_j \cdot C^*} \equiv \dfrac{\partial \ln F_j}{\partial C^*}$, and RRA is the Arrow-Pratt measure of relative risk aversion.

Equations (2.1) and (2.2) do not differ theoretically. The alternative instruments $i(W, \underline{Y}, t)$ and $C^*(W, \underline{Y}, t)$ appear, a priori, to each have relative advantages and disadvantages from an empirical viewpoint, but this cannot be discussed

Single-state instrumental variables such as $C^*(t)$ and $i(t)$ will not be uniquely invertible to a multiple state variable vector since they are given by dimension reducing transformations—hence conditioning in (2.1) is formally necessary.

here. All analysis will be stated in terms of $i(W, \underline{Y}, t)$. It may be applied mutatis mutandis to the $C^*(W, \underline{Y}, t)$ case.

To develop the separability test, take the ratio of the equilibrium "term" premium on bond j to that on a bond k with maturity τ_k, at a given point in time t_0:

$$\frac{E(R_j/i) - i}{E(R_k/i) - i} = \frac{\epsilon_{F_j \cdot i}}{\epsilon_{F_k \cdot i}} \tag{2.3}$$

(2.3) follows since ϕ, $\text{var}(di)$, and $i(t_0)$ are common to both bonds cross-sectionally. If the semi-elasticity $\epsilon_{F \cdot i}[i(W, \underline{Y}, t), \tau; t_0]$ is separable in $i(W, \underline{Y}, t)$ and τ:

$$\frac{E(R_j/i) - i}{E(R_k/i) - i} = \frac{\epsilon_{F_j \cdot i}}{\epsilon_{F_k \cdot i}} = \frac{g(i)\,|_{W(t_0), \underline{Y}(t_0)} \cdot f(\tau)\,|_{\tau_j}}{g(i)\,|_{W(t_0), \underline{Y}(t_0)} \cdot f(\tau)\,|_{\tau_k}} = \frac{f(\tau_j)}{f(\tau_k)} \tag{2.4}$$

Obviously a *sufficient* condition for (2.4) is that:

$$\epsilon_{F \cdot i} \equiv \frac{\partial \ln F}{\partial i} = f(\tau) \tag{2.5}$$

$$\Rightarrow \quad \ln F = f(\tau) \cdot i + \text{k where k is an arbitrary constant in } i \tag{2.6}$$

$$\Rightarrow \quad F = \exp[\text{k} + f(\tau) i] \tag{2.7}$$

Now consider the two closed form solutions given by CIR and Vasicek. The CIR[1977, (51)] solution is:

$$F = A(\tau) \cdot e^{-B(\tau) \cdot i} \tag{2.8}$$

Then set $\text{k} \equiv \ln A(\tau)$, $f(\tau) \equiv -B(\tau)$ in (2.7) to obtain (2.8). For the Vasicek[1977] solution, set:

$$\frac{F_i}{F} = \frac{1}{\alpha}(1 - e^{-\alpha\tau}) \equiv -f(\tau) \tag{2.9}$$

where α is the constant coefficient on the "elastic random walk" drift for $i(t)$.

The Vasicek solution in (2.9) and the CIR solution in (2.8) are special cases of a general solution which holds for log utility and a single-state variable stochastic process for $i(t)$ of the form:

$$di = (a_1 + b_1 i)\, dt + \sqrt{a_2 + b_2}\, dZ(t) \tag{2.10}$$

where $Z(t)$ follows a standardized Wiener process.[2] After the following section on estimation, the separability properties of three additional models are discussed.

[2] The statement appears to be an "i.f.f." except in a few pathological cases. The drift term in (2.10) can be slightly elaborated. Note that for a dogmatic prior on (2.10), a test of the separability property becomes a test of the utility assumption.

Estimation

Define:

$$\frac{E(R_j - i)}{E(R_{j-1} - i)} \equiv \frac{\mu_j}{\mu_{j-1}} \equiv \theta_j \tag{2.11}$$

where j is an index of maturity—if there are k default-free bonds, $j = 2, \ldots, k$. The vector $\underline{\theta}' = [\theta_k, \theta_{k-1}, \ldots, \theta_2]$ is to be estimated.

Two cases are considered: (i) the θ_j, $j = 2, \ldots, k$ are *constant* through time; and (ii) the θ_j are *stochastic* through time. Case (i) allows only proportional shifts in the term structure, as defined in terms of holding period returns.[3] More importantly, it assumes away the problem, since an invariant ratio will not be correlated with stochastic state variables by definition. Case (ii) allows θ_j to change—constancy may then be tested as a special case.

If $\underline{\theta}$ is constant, the maximum likelihood (*ML*) estimate of $\theta_j \equiv \frac{\mu_j}{\mu_{j-1}}$, $j = 2, \ldots, k$, is simply the ratio of sample means. Estimation of the vector $[\mu_j, \mu_{j-1}, \ldots, \mu_1]$ is a "k-means" problem, with:

$$(R_j - i)_t = \bar{\mu}_j + e_{jt} \qquad j = 1, \ldots, k \tag{2.12}$$

If $E(e_j, e_p) = 0, p \neq j$, it is well-known that for $k \geqslant 3$, the James-Stein estimator of $\underline{\mu}$ dominates the *ML* estimator in the sense that the latter is inadmissible. In the single-state variable case, $E(e_j, e_p) \neq 0$, and a Stein-like estimator for vector observations is needed (see Efron and Morris[1972]).

It is also the case that the *ML* estimator is inappropriate for quadratic (and other) loss functions since finite sample moments do not exist. Zellner[1975] has suggested a minimum expected loss (*MELO*) estimator, for the relative squared loss function, in such a case. Space does not permit a fuller explanation.

When $\underline{\theta}$ is allowed to be stochastic, the methodology has to be altered. Consider (say) θ_{kt}:

$$\theta_{kt} \equiv \frac{E_t(R_k - i)}{E_t(R_{k-1} - i)} \equiv \frac{E_t(R'_{k,t})}{E_t(R'_{k-1,t})} = \frac{R'_{k,t}\left(1 - \dfrac{e_{k,t}}{R'_{k,t}}\right)}{R'_{k-1,t}\left(1 - \dfrac{e_{k-1,t}}{R'_{k-1,t}}\right)} \quad (R'_{k,t}, R'_{k-1,t} \neq 0) \tag{2.13}$$

where $R'_{k,t} = E_t(R'_{k,t}) + e_{k,t}$ with $E[e_{k,t}/E_t(R'_{k,t})] = 0$. Two means of estimation in (2.13) are considered. The first uses logs. From (2.13):

$$\ln\left(\frac{R'_{k,t}}{R'_{k-1,t}}\right) = \ln \theta_{k,t} + \ln\left(1 - \frac{e_{k-1,t}}{R'_{k-1,t}}\right) - \ln\left(1 - \frac{e_{k,t}}{R'_{k,t}}\right) \tag{2.14}$$

$$\equiv \theta'_{k,t} + \eta_t \tag{2.15}$$

The hypothesis to be tested is that $\theta'_{k,t}$ is unrelated to the state variable vector

[3] It does allow a wider class of shifts in the term structure as conventionally stated in terms of yield to maturity.

(i.e. to opportunity set shifts), whilst the alternative hypothesis is that it is a function of these state variables or their instruments. As a specific alternative, take (say) per capita consumption as an instrument, and consider:

$$\tilde{\theta}_{k,t} = \tilde{\alpha}_t + b\tilde{C}_t + \tilde{u}_t \tag{2.16}$$

where $E[\tilde{u}_t/a_t, C_t] = 0$. Substituting (2.16) into (2.15):

$$\ln\left(\frac{R'_{k,t}}{R'_{k-1,t}}\right) = a_t + bC_t + (u_t + \eta_t) \tag{2.17}$$

In (2.17), it must be assumed that η_t is negligible, which is the case if: (*a*) $e_{k,t}/R'_{k,t} \simeq e_{k-1,t}/R'_{k-1,t}$ and/or (*b*) $e_{k,t}/R'_{k,t}$, $e_{k-1,t}/R'_{k-1,t}$ are small. Both (*a*) and (*b*) are plausible. Given conditions (*a*) and/or (*b*), a second means of estimation is given by the "unlogged" version of (2.17).

Under the null hypothesis of separability, $b = 0$. (2.17) is a regression equation with a stochastic intercept. The estimation of these equations (termed adaptive regression models by Cooley and Prescott[1973a] [1973b] [1976]), has been studied and programmed by Ansley[1979]. For estimation purposes, $\{a_t, C_t\}$ must be uncorrelated and a stochastic process for $\{a_t\}_t$ must be specified. Under the null hypothesis, the former is true by construction. Conceptually, the stochastic process for a_t can be related to its origin in the process governing the stochastic coefficients of the state variable process by using the closed-form solution for the bond price. Practically however, this usually turns out to be analytically intractible. Hence, the process for $\{a_t\}_t$ is taken to be a general linear ARIMA (Box-Jenkins[1976]) process. Results are not given here.

Additional Closed-Form Models

To illustrate the application of the above tests, three additional models are now briefly considered. The first is a model given in Ingersoll, Skelton and Weil [1978]. The bond price is defined there in terms of the instantaneous riskless rate $i(t)$, which is assumed to follow the stochastic process:

$$di(t) = [\xi + \kappa(\mu - i)]\, dt + \sigma\, dZ(t) \tag{2.18}$$

where:

$$d\xi(t) = -\beta\xi\, dt + \eta[di - E(di)] = -\beta\xi\, dt + \eta\sigma\, dZ(t) \tag{2.19}$$

Thus, an elastic random walk drift for $i(t)$ is augmented by an extrapolative "velocity" $\xi(t)$ which is perfectly corrected with $i(t)$. $\xi(t)$ can represent a stochastic coefficient e.g. if the "central tendancy" μ shifts according to $\tilde{\mu}(t) = \mu + \tilde{\delta}(t)$. $\tilde{\xi}(t) \equiv \kappa\tilde{\delta}(t)$.

In this model, the ratio of risk premia on bonds of adjacent maturity is:

$$\frac{E(R_{j-i}) - i}{E(R_{j-1}/\mathrm{i}) - \mathrm{i}} = \frac{\lambda(\mathrm{i}, \xi)[\epsilon_{F_j \cdot i}\,\sigma + \epsilon_{F_j \cdot \xi'}]}{\lambda(\mathrm{i}, \xi)[\epsilon_{F_{j-1} \cdot i}\,\sigma + \epsilon_{F_{j-1} \cdot i}]} \tag{2.20}$$

where $\lambda(i, \xi)$ is a risk compensation term not depending upon the particular bonds in question. The closed-form solution in this case is (see Ingersoll, Skelton and

Weil[1978, (33)]):

$$F[i, \xi, \tau] = A_2(\tau) \cdot e^{-B_2(\tau)i} \tag{2.21}$$

where:

$$A_2(\tau) = \exp[g(\tau)\xi + h(\tau)]$$

$$B_2(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa\tau})$$

$g(\tau)$, $h(\tau)$ are functions not involving $i(t)$ or $\xi(t)$.

Then $\epsilon_{F \cdot i} = -B_2(\tau)$, $\epsilon_{F \cdot \xi} = g(\tau)$, so that with σ and η constant, the ratio of term premia will be *constant*—only proportional shifts in the term structure, as defined here in terms of expected holding period premia, can take place.

To pursue this model, the second example alters it by assuming that $\xi(t)$ follows a process which is orthogonal to $i(t)$. Instead of (2.19) let $\xi(t)$ follow (say) a non-stationary random walk with unit variance:

$$d\xi(t) \equiv dZ_2(t)$$

$$dZ(t) \cdot dZ_2(t) = 0$$

For this model,

$$\frac{E(R_j/i) - i}{E(R_{j-1}/i) - i} = \frac{\lambda_1(i, \xi)\epsilon_{F_j \cdot i} + \lambda_2(i, \xi)\epsilon_{F_j \cdot \xi}}{\lambda_1(i, \xi)\epsilon_{F_j \cdot i} + \lambda_2(i, \xi)\epsilon_{F_{j-1} \cdot \xi}} \tag{2.22}$$

where $\lambda_1(i, \xi)$ and $\lambda_2(i, \xi)$ are equilibrium risk premia (given below). If λ_1 and λ_2 are *linear* in $i(t)$ and $\xi(t)$, the closed form solution for the bond price F is in the general form given in (2.21).

Suppose now that the test given in (2.17) is implemented. Then what can be learned about the economics assumed in the model solution? Three outcomes in (2.17) are possible: (i) The joint hypothesis that $b = 0$ *and* a_t is constant cannot be rejected. This outcome would be consistent with (2.20) rather than (2.22) except under circumstances given below. (ii) The hypothesis $b = 0$ cannot be rejected, but the hypothesis that a_t is constant is rejected. *If* the time series process $\{i(t), \xi(t)\}$ is modelled properly, such a result would be possible in (2.22) only if at most:

$$\lambda_1(i, \xi) = \lambda_{10} + \lambda_{11}\xi$$

$$\lambda_2(i, \xi) = \lambda_{20} + \lambda_{21}\xi \tag{2.23}$$

If (2.23) held, θ_{jt} would be stochastic, but not a function of state variables so that separability would hold. But, from the general equilibrium model, where J (W, $\underline{Y}$, t) is the indirect utility function:

$$\lambda_1(i, \xi) = \frac{-J_{WW}}{J_W}\operatorname{cov}(dW, di) + \frac{-J_{Wi}}{J_W}\operatorname{var}(di) + \frac{-J_{W\xi}}{J_W}\operatorname{cov}(di, d\xi) \tag{2.24a}$$

$$\lambda_2(i, \xi) = \frac{-J_{WW}}{J_W}\operatorname{cov}(dW, d\xi) + \frac{-J_{Wi}}{J_W}\operatorname{cov}(di, d\xi) + \frac{-J_{W\xi}}{J_W}\operatorname{var}(d\xi) \quad (2.24b)$$

If the time series process assumed for the solution leading to (2.22) held—it is important that this can be tested *directly* by maximum likelihood estimation of the process and examination of the diagnostics—then var(di), cov(di, $d\xi$), var($d\xi$) are all constant. Given this and an estimate of cov(dW, di), cov(dW, $d\xi$), the restriction (2.23) implies a restriction on the risk aversion and hedging terms: $-J_{WW}/J_W$, $-J_{Wi}/J_W$, $-J_{W\xi}/J_W$. Finally, note that if λ_1 and λ_2 are constant, θ_j would be constant.

The third possible outcome—a_t stochastic and $b \neq 0$—is the most difficult. This result would indicate separability does not hold and/or (2.23) does not hold. It is only the former which is inconsistent with the solution.

As a final example, consider the two variable model developed in Section 3, where C^*(t) denotes real per capita consumption, and $i(t)$ is the nominal riskless interest rate. For this model:

$$\frac{E(R_j/i) - i}{E(R_{j-1}/i) - i} = \frac{\epsilon_{F_j \cdot C^*} \cdot \mathrm{ARA} \cdot \sigma^2 + \epsilon_{F_j \cdot i} \cdot \mathrm{ARA} \cdot \rho_{12} s \sigma}{\epsilon_{F_{j-1} \cdot C^*} \cdot \mathrm{ARA} \cdot \sigma^2 + \epsilon_{F_{j-1} \cdot i} \cdot \mathrm{ARA} \cdot \rho_{12} s \sigma} \quad (2.25)$$

where ARA is the coefficient of absolute risk aversion. If it is assumed that ARA is constant, θ_j is constant. Allowing the parameters of the joint time series process of $\{i(t), C^*(t)\}$ to become stochastic would allow non-proportional shifts in the term structure by injecting a "third" state variable into (2.25). It is feasible to derive solutions for such a case, since the solution technique leading to (2.25) can be applied to any multivariate random walk process for the state variable instruments. However, as in the earlier examples, it remains the case that it is difficult to determine analytically the process for θ_j from that for the parameters of this random walk process.

III. Time Series Restrictions Implied by the Equilibrium Models

Suppose a closed form solution $F(i, \tau; T)$ for the equilibrium bond price exists. At a point in time, consider an observation vector of the prices of discount bonds across the maturity spectrum. Then, since $i(t)$ is given for all securities, this cross-section of bond prices implies through the model $F(i, \tau)$, a restriction on the time series process for $i(t)$, $t \epsilon [0, \Upsilon]$, where Υ is the sample period. Hence, one way of testing the validity of the model is to test the restricted versus the estimated unrestricted time series process specified for the closed form solution.

The test is applied to two models here. The first is the CIR model assuming log utility and a square root process for the instantaneous nominal rate of interest $i(t)$. The second is a model derived here which contains both per capita real consumption and the nominal instantaneous rate of interest as arguments of the bond pricing function, but has to assume constant absolute risk aversion. It is obvious that the methodology of this section requires a closed form solution be available. However, note that five different models were discussed in Section 2—only two are considered here since they are sufficient for illustration.

For log utility and the following process for the instantaneous (short) rate of interest:

$$di = \kappa(\mu - i)\,dt + \sigma\sqrt{i}\,dZ(t) \tag{3.1}$$

the price $F(i, \tau)$ of the bond is given by (3.2) which is taken (copied) from CIR [1977, (51)]:

$$F(i, \tau) = A(\tau)\cdot e^{-B(\tau)i} \tag{3.2}$$

where:

$$A(\tau) \equiv \left[\frac{2\gamma\exp[(\kappa+\lambda+\gamma)\cdot\tau/2]}{(\gamma+\kappa+\lambda)(\exp[\gamma\tau]-1)\ +2\gamma}\right]^{2\kappa\mu/\sigma^2}$$

$$B(\tau) \equiv \left[\frac{2[\exp(\gamma\tau)-1]}{(\gamma+\kappa+\lambda)(\exp[\gamma\tau]-1)+2\gamma}\right]$$

$$\gamma \equiv (\kappa+\lambda)^2 + 2\sigma^2$$

Then:

$$\frac{F(i, \tau_1)}{F(i, \tau_2)} = \frac{A(\tau_1)}{A(\tau_2)}\cdot e^{[B(\tau_2)-B(\tau_1)]i} \tag{3.3}$$

Given λ, plugging in τ_1 and τ_2 in (3.3) imposes a restriction on the region (κ, μ, σ^2) under the null hypothesis that the model $F(i, \tau)$, as specified, is correct. If prices can be observed for bonds of k maturities, then $(k - 1)$ ratios similar to the *LHS* of (3.3) exist. If the model is correct, all should imply the same region for (κ, μ, σ^2), and it should not be possible to reject the restrictions imposed on (κ, μ, σ^2) in (3.1).

The same methodology can be applied to a second model in which both per capita *real* consumption and the *nominal* rate of interest constitute instrumental variables. Such a model overfits one based on either variable alone, and widens the class of term structure behavior which can potentially be explained. A particular form of such a model for which a closed-form solution has been found follows.

Under the assumption that equilibrium gross supplies of basis financial assets equal zero, it follows from the first order necessary condition for a representative consumer's intertemporal optimization problem that the equilibrium risk compensation $(\beta F - iF)$ on a bond of maturity τ_j is given by (see CIR [1977, (13)]):

$$\begin{aligned}\beta F - iF = F_W&\left[\frac{-J_{WW}}{J_W}\operatorname{var}(dW) + \textstyle\sum_i \frac{-J_{WY_i}}{J_W}\operatorname{cov}(dW, dY_i)\right]\\ &+ \textstyle\sum_i F_{Y_i}\left[\frac{-J_{WW}}{J_W}\operatorname{cov}(dY_i, dW) + \textstyle\sum_j \frac{-J_{WY_i}}{J_W}\operatorname{cov}(dY_i, dY_j)\right]\end{aligned} \tag{3.4}$$

Suppose the state variable vector is one-dimensional, and denote the single state variable Y_1 as x. Then the two "instrumental" variables for nominal bond pricing are taken to be equilibrium real consumption per capita, $C^*(W, x, t)$, and the equlibrium nominally riskless rate of interest, $i(W, x, t)$.

Let F' denote the bond price in terms of the underlying variables, i.e. $F'(W, x, t)$. Then the composite function $F[C^*(W, x), i(W, x)] = F'(W, x)$ (t is suppressed) must be obtained. The composite derivatives are:

$$F'_W = F_{C^*}C^*_W + F_i i_W$$

$$F'_x = F_{C^*}C^*_x + F_i i_x$$

Substituting in (3.4) and using:

$$\frac{-J_{WW}}{J_W} = \frac{-U_{C^*C^*}}{U_{C^*}} C^*_W$$

$$\frac{-J_{Wx}}{J_W} = \frac{-U_{C^*C^*}}{U_{C^*}} C^*_x$$

where U(C*,t) is the utility function, gives:

$$\beta^j F^j = iF^j + \frac{-U_{C^*C^*}}{U_{C^*}} F^j_{C^*}\text{var}(dC^*)$$

$$+ \frac{-U_{C^*C^*}}{U_{C^*}} F^j_i\{C^*_W i_W \text{var}(dW) + C^*_W i_x \text{cov}(dW, dx)$$

$$+ C^*_x i_W \text{cov}(dx, dW) + C^*_x i_x \text{var}(dx)\} \qquad (3.5)$$

where $\beta^j \equiv E(R^j/i)$ is the instantaneous equilibrium expected return on bond j— it is conditional on the instantaneous riskless rate in the sense that it is set relative to that rate.

$$\Rightarrow \quad \beta^j = i + \frac{-U_{C^*C^*}}{U_{C^*}}\left[\frac{F^j_{C^*}}{F}\text{var}(dC^*) + \frac{F^j_i}{F}\text{cov}(di, dC^*)\right]$$

or

$$\beta^j = i + RRA\left[\frac{F^j_{C^*}}{F}\frac{\text{var}(dC^*)}{C^*} + \frac{F^j_i}{F}\text{cov}\left(di, \frac{dC^*}{C^*}\right)\right] \qquad (3.6)$$

The *PDE* corresponding to (3.6) which is to be solved for the equilibrium bond price F is as follows:

$$F_{C^*}\left[E(dC^*) - RRA\cdot\frac{\text{var}(dC^*)}{C^*}\right] + F_i\left[E(di) - RRA\cdot\text{cov}\left(di, \frac{dC^*}{C^*}\right)\right]$$

$$-F_\tau - iF + \frac{1}{2}F_{C^*C^*}\text{var}(dC^*) + \frac{1}{2}F_{ii}\text{var}(di) + F_{C^*i}\text{cov}(di, dC^*) = 0 \quad (3.7)$$

Assume firstly that the stochastic processes for $i(t)$ and $C^*(t)$ are of the simple single-state O.U. form:

$$dC^*(t) = (a_1 + b_1C^*(t))\,dt + \sigma\,dZ_1(t) \qquad (3.8a)$$

$$di(t) = (a_2 + b_2 i(t))\,dt + s\,dZ_2(t) \qquad (3.8b)$$

(3.8) will be elaborated somewhat below. Further assume that *absolute* risk aversion, denoted ARA, is constant. This last assumption is quite restrictive (see, for example, Arrow's[1964] discussion), but it *can* be relaxed in some cases.[4] Substituting these assumptions in (3.7) gives:

$$\tfrac{1}{2} F_{C^*C^*}\sigma^2 + F_{C^*}[a_1 + b_1 C^* - \text{ARA}\ \sigma^2] + F_i[a_2 + b_2 i - \text{ARA}\ \rho_{12} s\sigma] + \tfrac{1}{2} F_{ii} s^2 + F_{C^*i}\rho_{12} s\sigma - iF - F_\tau = 0 \quad (3.9)$$

(3.9) has the following solution:

$$F(C^*, i, \tau) = e^{B(\tau)C^* + D(\tau)i + A(\tau)} \quad (3.10)$$

Now it may be verified that, for the case in which the time series process for $\{C^*(t), i(t)\}$ is restricted as in (3.8), the terminal condition $F[C^*(T), i(T), \tau = 0; T] = 1.0$ (T is the maturity date of the bond) requires $B(\tau) = 0$, so that (3.10) collapses to the one-variable case. Hence, it is interesting to relax restriction (3.8) as follows:

$$dC^*(t) = [a_1 + b_1 C^*(t) + d_1 i(t)]\, dt + \sigma\, dZ_1(t) \quad (3.11a)$$

$$di(t) = [a_1 + b_2 C^*(t) + d_2 i(t)]\, dt + s\, dZ_2(t) \quad (3.11b)$$

The solution (3.10) applies for the more general process (3.11) where:

$$B(\tau) = A_1 e^{r_1\tau} + A_2 e^{r_2\tau} + b_2/(d_1 b_2 - b_1 d_2)$$

r_1, r_2 are roots of the characteristic equation:

$$r^2 + (d_2 + b_1)r + (d_1 b_2 + b_1 d_2)$$

$$A_1 = \frac{-r_2 b_2}{(r_1 - r_2)(d_1 b_2 - b_1 d_2)}$$

$$A_2 = \frac{-r_1 b_2}{(r_1 - r_2)(d_1 b_2 - b_1 d_2)}$$

$$D(\tau) = \left(\frac{1 - b_1}{b_2}\right)\left(A_1 e^{r_1\tau} + A_2 e^{r_2\tau}\right)$$

$A(\tau)$ is a maturity dependent function.

It is apparent from (3.9) that *any* number of instruments can be included without altering the form of (3.10), so long as they conform to what might be termed a "multivariate" elastic random walk.

As in the earlier example, the restrictions on the coefficients of the time-series process (3.11) implied by a vector of bond prices across the maturity spectrum, if the model (3.10) holds, can be tested. For example, a likelihood ratio test uses the

[4] For example, if the instantaneous variance of $C^*(t)$ is $[\sigma\sqrt{C^*}]^2$ in (3.11a) and the instantaneous variance of i(t) is $[s\sqrt{C^*}]^2$ in (3.11b), it is possible to obtain a solution (not necessarily unique) in the same form for the case of constant *relative* risk aversion (*RRA*). Such a multivariate process for (3.11) does seem plausible.

restricted and unrestricted full-information maximum likelihood estimates which can be computed directly by the Wymer program DISCON.

It is interesting to note that the test methodology of this section is a particular application of a more general procedure. In rational expectations models, cross-equation restrictions implied by the models with exogenous shocks are tested (see, for example, Hansen and Sargent[1979]). Just as those restrictions are critically dependent upon the process assumed to generate the exogenous shocks, the restrictions on *the parameters* of the time-series process for the state variable instrument(s) implied by the pricing function solution are critically dependent upon the form of that process (i.e. the serial correlation functions) assumed in the solution.

IV. Numerical Integration

In this section, the following "fundamental" partial differential equation (*PDE*) for the equilibrium bond price is integrated numerically:

$$-i(C)F + F_C\left[E(dC) - RRA\cdot\frac{\mathrm{var}(dC)}{C}\right] + \frac{1}{2}F_{CC}\,\mathrm{var}(dC) + \delta = F_\tau \qquad (4.1)$$

where F, i, τ and RRA are defined as previously, and δ allows a (continuous) coupon payment on the bond if applicable. $C(t)$ is alternately taken as equilibrium real and nominal per capita consumption.[5] If $C(t)$ is fitted as a single-state Markov process as here, the process for neither variable will be invertible to that for the underlying state variable vector, and various rationalizations are possible for the use of real or nominal consumption (see Breeden[1978] for the general statement of models in terms of equilibrium consumption).

The data for per capita consumption is taken from the *MPS* Quarterly Econometric Model data file. The consumption variable used here includes state and federal government non-capital expenditures. Two steps are necessary prior to numerical integration in (4.1): (a) Estimation of the time series process for $C(t)$, $t\epsilon[0, \Upsilon)$; and (b) estimation of the parameters of $i(C)$. A direct test of the expectations hypothesis arises from (*b*), as explained in Section 5.

Two continuous-time processes were estimated for the integral[6] of per capita consumption: an Ornstein-Uhlenbeck process and a lognormal process. The former was estimated by using the Wymer continuous-time program DISCON, and the latter by logarithmic transformation. Both give full-information maximum likelihood estimates of the continuous-time model parameters. The coefficients of these models are assumed constant which implies, via the forward equation, that the probability density of future $C(t)$ *conditional upon* current

[5] The numerical program used can accommodate only one space variable. It is a modified version of one provided me by Jeff Skelton, based on the PDPARB and PDPARC subroutines of the M.I.T. Information Processing Center. Backward differences are used, and the $(R - 1)$ linear finite difference equation system (for R interior increments in C), which is tridiagonal, is solved by the Thomas algorithm.

[6] Since consumption is a flow variable which is never observed in discrete-time, the model is transformed to be based on the quarterly integral.

$C(t_0)$ is fixed. If these coefficients are allowed to be stochastic (i.e. functions of exogenously specified "state" variables), the processes become *vector*-Markov and the logic in using $C(t_0)$ as an instrumental variable no longer seems to hold. Further details are omitted. The function $i(C)$ is linearized and estimated by two-stage-least-squares (*2SLS*) regression. Inserting point estimates into (4.1) gives a linear parabolic *PDE* which, with appropriate boundary conditions, can be solved for $F(C, \tau)$. The terminal condition is $F(C, \tau = 0) = 1.0$. The boundary conditions were set as: $F(C = 0, \tau) = 0.0$, and $F(C = C_U, \tau) = 1.0$. Detailed reasoning is omitted. Data with which to compare model prices is from the *CRSP* Government Bond Returns file, and consists of all six and twelve month Treasury Bills and 1 to 10 year Treasury Notes maturing at the end of a quarter. The first Notes were those maturing on 9/30/74, with the first quote given on 3/29/73 for delivery on 4/2/73.

Table 1 contains the model solutions for the Ornstein-Uhlenbeck and lognormal models fitted to $C(t)$, $t \epsilon [t_0, \Upsilon]$. The estimated linear approximation to $i(C)$ is given at the top of the table. In the table, three quarters for which actual price data was available were selected randomly, whilst the forth (1953/*I*) was selected to judge the sensitivity of the solution to C.

In all cases, *RRA* is not estimated separately. Its value is initially set to one (the log utility value). Then, in several cases in Table 1, the value of *RRA* for which the solution value of F equals the observed price is reported.

When the level of nominal per capita consumption is used as the instrumental variable, both time series models appear to do "well" for six-month Bills. The lognormal model gives uniformly a slightly higher solution. For maturity dates beyond six months, the estimated model tapers the term structure too steeply. Similarly for real consumption levels. When nominal consumption differences are used, the observed prices are spanned in all cases by the O.U. model and lognormal model solutions.

The Test Significance of the Numerical Method

It is claimed there are methodological problems with this numerical method. The intuition is straightforward: It is well-known (e.g., CIR[1977], Garman[1979]) that equilibrium asset prices follow a fundamental *PDE* in the diffusion case. Since states of nature are not traded, particular assumptions about tastes and technology are needed to put flesh on the skeleton. All such assumptions will be imbedded in the equilibrium *PDE*. The argument is simply that tests can be constructed which are based on the *PDE*, and, more importantly, have known econometric properties, whilst numerical integration of the *PDE* adds nothing to the economics, but severely complicates or completely obscures the econometrics.

The above results serve to illustrate the point. They indicate $\frac{\partial F}{\partial C} < 0$, so that the "improved" results at longer maturities when consumption differences are used can be attributed to the lower values for the instrument. Yet the reason $\frac{\partial F}{\partial C} < 0$ is that the consumption drift is proportional to C (or ΔC), and, for a fixed

Table 1

Numerical Solutions for $F(C, \tau)$

(a) Model based on Quarterly Integrals of per capita *Nominal* Consumption.

Estimated $i(C_t)$: $\hat{\imath}_t = 0.00024 + 0.00001332\ C_t$

(1) Period and Observed C_t	(2) Maturity (Qtrs.)	(3) Actual Price	(4) O.U. Model	(5) Lognormal Model
1) 1974/IV	2	0.9645	0.9668	0.9673
$C = 1244.71$	3	—	0.9340	—
	4	0.9327	0.9015	0.9039
	8	0.9993	0.8318	0.8423
	(Coupon)			
2) 1975/IV	2	0.9717	0.9638	0.9643
$C = 1634.0$	4	0.9414	0.8926	0.8954
	8	—	0.8648	0.8771
3) 1973/IV	2	0.9625	0.9709	0.9709
$C = 1101.033$				
4) 1953/I	2	—	0.9887	0.9880
$C = 414.384$	4	—	0.9121	0.9661

(b) Model based on Quarterly Integrals of per capita *Real* Consumption

Estimated $i(C_t)$: $\hat{\imath}_t = -0.013304 + 0.000028131\ C_t$

(1) Period and Observed C_t	(2) Maturity (Qtrs.)	(3) Actual Price	(4) O.U. Model	(5) Lognormal Model
1) 1974/IV	2	0.9645	0.9577	0.9578
$C = 1012.75$	4	0.9327	0.8771	0.8784
	8	0.9993	0.7866	0.79188
2) 1975/IV	2	0.9717	0.9561	0.9563
$C = 1043.29$	4	0.9414	0.8725	0.8738
	8	—	0.8284	0.8336
3) 1973/IV	2	0.9625	0.9578	0.9580
$C = 1010.12$				
4) 1953/I	2	—	0.9758	0.9760
$C = 671.23$	4	—	0.9284	0.9290

(c) Model based on Quarterly Differences of *Nominal* per capita Consumption.

Estimated $i(C_t)$: $\hat{\imath}_t = 0.017464 - 0.00064217\ \Delta C_t$

(1) Period and Observed C_t	(2) Maturity (Qtrs.)	(3) Actual Price	(4) O.U. Model	(5) Lognormal Model
1) 1974/IV	2	0.9645	0.9775	0.9443
$\Delta C = 26.115$	3	—	0.9738	0.9238
	4	0.9327	0.9721	0.9135[2]
	8	0.9993	1.0165[1]	0.9676
2) 1975/IV	2	0.9717	0.9771	0.9450[3]
$\Delta C = 29.171$	4	0.9414	0.9700	0.9130[4]
3) 1973/IV	2	0.9625	0.9775	0.9443[5]
$\Delta C = 26.427$				
4) 1953/I	2	—	0.9955	0.9469
$\Delta C = 6.425$	4	—	0.9951	0.9312

Approximate Value of RRA to set numerical solution equal to actual price:

[1] 4.2; [2] 2.3; [3] 7.0; [4] 3.0; [5] 4.2.

terminal condition, the higher the drift, the lower F. Thus, it is clear that *any* model with a positive drift will describe a downward sloping term structure. It follows that it is extremely important for the statistical tests to be powerful against points "close" in the "model" space. But the problem is that it is methodologically difficult to even perform hypothesis tests for the numerical techniques—at best, a lot of computation is required. In the sampling theory case, it is presumably possible to "grind out" a sampling distribution for the model price F by "simulating" over permutations of the parameter estimates from their sampling distributions. In the Bayesian case, it is not obvious how the parameters of F would be integrated out, since the solution itself has no closed form. The point estimate of F will not be unbiased if the model solution is highly non-linear in the parameters when unbiased estimates of these parameters are inserted in (4.1). In short, once estimation risk is taken into account (as it obviously must be for hypothesis testing), little is known or knowable.

Introduction of more than one space variable will be of qualified advantage in the numerical analysis. Clearly, there will be more degrees of freedom for the model, giving it the potential to "pick up" turns etc. in the term structure. However, ipso facto, the problems of hypothesis testing will be accentuated.

Other problems occur. Everything above assumes stationarity in the parameters. Yet there is evidence that this may not be the case, and further that the model solution is "relatively sensitive" to any non-stationarity—in Table 1(c), the observations prior to 1961/*I* were excluded in estimation as a hatchet solution to the problem of negative differences in estimating the lognormal model. This produced the gap between the O.U. and lognormal model solutions. Measurement errors in consumption will, in general, cause bias and inconsistency in the results, yet it is difficult to do anything econometrically about these problems when working with numerical methods.

V. The Expectations Hypothesis

As indicated in S.4, a test of the Expectations Hypothesis may be based on the function $i(C)$ in (4.1). If $i(C)$ is constant in C, then $F_C = F_{CC} = 0$ solves (4.1). But this is the solution implied by the expectations hypothesis (or one interpretation thereof). This result is briefly discussed.

Suppose $i(t)$ is distributed independently of C (or the state variable *vector* in the multiple state variable case), e.g., $i(t) = i_0 + \epsilon(t)$, where $\epsilon(t) \perp C(t)$, $E[\epsilon(t) \cdot \epsilon(t - \delta)] = 0 \; \forall \; \delta$. Semantically, $\epsilon(t)$ can be regarded as an element of the jointly distributed state variable vector, with the special requirement that its marginal and conditional densities be equal. The serial independence of $\epsilon(t)$ (assuming normality) implies that, even for utility functions other than Bernoulli logarithmic, there will be no "hedging" compensation—as in Fama [1970]. The orthogonality condition means that there will be no systematic "one-period" risk either—hence risk compensation is zero, and the expectations hypothesis, in the sense being used here, obtains.

The function $i(C)$ has been studied by three interdependent techniques: (i)

Box-Jenkins [1976] cross-correlation analysis; (*ii*) cross-spectral estimation; and (*iii*) estimation of a regression for a linear approximation to the function. Since theory suggests i_t and C_t will be jointly endogenous, a simultaneous equation technique is required. Unfortunately, space does not permit a fuller explanation or results to be given. In general, the tests indicate (loosely) a "significant" relationship between i_t and C_t does exist.

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Discussion by

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MARSH'S PAPER ON equilibrium term structure models and tests of them has

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