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An Analysis of Variable Rate Loan Contracts

JOHN C. COX, JONATHAN E. INGERSOLL, JR. and STEPHEN A. ROSS*

I. Introduction

THE 1970s heralded times of rapid and intense inflation accompanied by high and volatile interest rates. While there have been many economic consequences attributable to these factors, perhaps most strongly affected have been the various markets and institutions dealing with fixed income securities. One reaction has been the introduction of variable rate loans. Floating rate corporate notes were first introduced in 1974 when a total of $1.3 billion were sold. Variable rate mortgages probably originated in the 1930s in the United Kingdom and Canada. Until recently they were restricted in this country. In these contracts the interest due is determined by currently prevailing rates and not the rates which were in force at the time of issuance. The purpose of this design is to stabilize the price or value of the contract—immunizing it against changes in interest rates.

This paper presents a preliminary analysis of some aspects of variable rate loan contracts. Section II reviews a recently developed pricing model for the term structure of interest rates. Section III begins the discussion on variable rate loans by outlining the construction of various default-free contracts which offer perfect immunization. Section IV discusses some typical modifications found in existing variable rate contracts. In this section the price and risk of these modified contracts are determined in various examples. Finally Section V examines the changes required for variable rate contracts when there is the risk of default.

II. A Model for Analysis

In two recent papers,[1] we have developed a rational asset pricing model and applied it to the study of the term structure of interest rates. The major features of this model are outlined below. We denote the state of the economy by the vector of N state variables Y. Y includes all information relevant to investors and consequently uniquely determines the prices of all assets. The key assumption is that the evolution of the state of the economy has a continuous sample path. Intuitively this means that the characterization of the economy cannot change drastically in a short interval of time.

Specifically we assume:

A1) Trading in all assets takes place continuously in frictionless markets.

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[1] See Cox, Ingersoll, and Ross [2] and [3].

A2) The evolution of the state of nature which conditions the economy is governed by the vector equation for the N state variables

$$dY = \mu(Y, t)\, dt + H(Y, t)\, d\tilde{\omega}. \tag{1}$$

μ^i is the expected change in the i^{th} state variable. H is an $N \times N + 1$ matrix of responses to the Wiener process. HH' is the $N \times N$ nonsingular variance-covariance matrix of changes in the state variables. $d\tilde{\omega}$ is the increment of an $N + 1$ dimensional standardized Wiener process.

Under these assumptions the basic valuation equation for any security is

$$\tfrac{1}{2} \sum_{i,j} (HH')_{ij} F_{Y_i} F_{Y_j} + [\mu' - \lambda'(Y, t)] F_Y + F_t + \delta(Y, t) - r(Y, t) F = 0 \tag{2}$$

where $\delta(Y, t)$ is the possibly state-dependent cash flow or dividend paid to the security's owners, $r(Y, t)$ is the prevailing instantaneous spot rate of interest, and $\lambda(Y, t)$ is a vector of factor risk premiums. While the derivation of the valuation equation above requires no further assumptions, determining what the relevant state variables are and the form of the factor risk premiums requires an embedding equilibrium model. In [2] we provide one such equilibrium, and in [3] we discuss some problems which can arise when equilibrium modeling is ignored.

The first two terms in (2) are the risk-adjusted expected change in price and will be abbreviated here by the "risk-adjusted" differential generator $\hat{L}$ over the state variables. Equation (2) can then be written

$$\hat{L}[F] + F_t + \delta(Y, t)F - r(Y, t)F = 0. \tag{3}$$

If the security in question is not default-free but a claim against a limited-liability corporation, then, in general, one relevant state variable will be the value V of the firm. To emphasize this dependence we write the valuation equation as

$$\hat{L}[F] + F_t + \delta - rF + V\sigma'_{YV} F_{VY} + \tfrac{1}{2}\, \sigma^2(Y, V, t) V^2 F_{VV} + [rV - \Delta(Y, V, t)] F_V = 0 \tag{4}$$

where σ^2 is the variance rate of returns from investing in the firm as a whole, σ'_{YV} is the vector of covariances of this return with changes in the state variables and Δ is the aggregate dividends, coupons, etc. *paid out* to all claimants. In this case the vector Y includes all other state variables. We make a distinction between equations (3) and (4) because the values of specific firms need not typically affect the pricing of default-free assets.

These valuation equations can be used in two distinct fashions. First, given the payment functions δ and Δ the partial differential equation can be solved, subject to the appropriate boundary conditions, to give the price of the security and determine other characteristics about its value. This is the approach used in all past work in the area of contingent claims pricing, and here in Section IV. Alternatively the equation can be used to determine the payments which would be required to give the asset's value certain characteristics. We use this approach to design basis-risk-free contracts in Sections III and V.

III. Default-Free Variable Rate Loans

The simplest variable rate contract is a default-free loan with face value B making continuous coupon payments which are adjusted to the current level of the interest rate, i.e., $\delta(Y, t) = r(Y, t)B$. From (3), the value of this bond satisfies

$$\hat{L}[F] + F_t + rB - rF = 0 \tag{5}$$

subject to the maturity condition $F(Y, T, T) = B$. The differential generator includes only terms with derivatives so the immediately apparent solution to (5) is $F(Y, t, T) \equiv B$ independent of the current state. This answer should come as no surprise. The value of the bond is constant since the basis risk, the only source of uncertainty for this asset, has been eliminated by an instantaneous adjustment in the "coupon rate" to the prevailing rate of interest.

Note that it is the spot rate of interest which is used to determine the coupon payments regardless of the maturity of the contract. If any long term rate were used then the value of the bond would depend upon the functional relationship between the two rates. Furthermore, the bond's value would then depend on the stochastic state variables (except in the special case when the long and short rates were in constant proportion) and the immunization property of the simple variable rate bond would be lost.[2]

With continuous payments other types of repayment structures can also be easily immunized against basis risk. For example, under the variable rate equivalent to an annuity,[3] payments would be made according to the schedule

$$\delta(Y, t) = (1 - e^{-\rho(T-t_0)})^{-1}[r + (\rho - r)e^{-\rho(T-t)}]B \tag{6}$$

where t_0 is the time at which the annuity was contracted for and ρ is some arbitrarily selected reference rate governing the rate of principal repayment (e.g., $\rho = \infty$ corresponds to no repayment of principal prior to maturity as in the case above).

For this schedule of payments, the solution to (3) subject to $F(Y, T, T) = 0$ is

$$F(Y, t, T) = B(1 - e^{-\rho(T-t)})/(1 - e^{-\rho(T-t_0)}). \tag{7}$$

While this loan contract does not have a constant value, its value at any point of time is completely deterministic. Again basis risk has been perfectly eliminated.

If a bond pays coupons only at distinct points in time, the interest schedule required to eliminate basis risk becomes more complex because the spot rate varies continuously. Nevertheless, perfect immunization is still possible. If the bond pays coupons at times $t_1, t_2, \ldots, t_n$, the proper sum to be paid at time t_i to

[2] This should not be interpreted to preclude a correct use of a long rate to figure interest payments when the short rate is used to calculate interest charged with the difference being added to or subtracted from principal owed. See Cohn and Fisher [1] for a description and analysis of dual rate contracts.

[3] This variable rate loan is equivalent to an annuity because its value as given in (7) is identically equal to the value of a $T - t$ period annuity if the interest rate were constant at ρ.

eliminate basis risk is

$$\delta(t_i) = B\left[\exp\left(\int_{t_{i-1}}^{t_i} r(t)\, dt\right) - 1\right]. \tag{8}$$

The expression (8) will be recognized as simply the interest which would have been earned over the interval between coupon payments on an original sum of B by rolling over spot loans at the instantaneous rate.

The proof that this is the correct risk-eliminating payment relies upon Lemma 4 in Cox, Ingersoll and Ross [2]. There it is demonstrated that the formal solution to equation (2) can be written as

$$F(Y, t, T \equiv t_n) =$$

$$\hat{E}\left\{B \cdot \exp\left(-\int_t^T r(s)\, ds\right) + \sum_{i=1}^n \delta(t_i)\exp\left(-\int_t^{t_i} r(s)\, ds\right)\right\} \tag{9}$$

where $\hat{E}$ denotes expectation with respect to the transformed or risk-adjusted distribution. Substituting for the scheduled payments from (8) gives

$$F = B \cdot \hat{E}\left\{\exp\left(-\int_t^T r(s)\, ds\right) + \right.$$

$$\left.\sum_{i=1}^n \left[\exp\left(-\int_t^{t_{i-1}} r(s)\, ds\right) - \exp\left(-\int_t^{t_i} r(s)\, ds\right)\right]\right\}$$

$$= B \cdot \hat{E}\left\{\exp\left(-\int_t^{t_0} r(s)\, ds\right)\right\} = B \cdot \exp\left(\int_{t_0}^{t} r(s)\, ds\right) \tag{10}$$

where t_0 is the time of the last coupon payment. To determine the current value of the bond we need to know only the time path of the spot rate of interest since the last payment. The bond value equals its face value plus that portion of the next coupon already accrued independent of expectations about rates in the future.

In this case the price of the bond at all future points of time cannot be predicted perfectly. However, it is known that on each interest date just after the bond goes ex-coupon its price will be equal to par. Furthermore between coupon dates the price dynamics for this bond are

$$dF = rF\, dt \tag{11}$$

so the realized return is (locally) certain. It is in this sense that the basis risk has been eliminated for this bond.

While each of the above contracts eliminates basis risk completely, they might be impractical for several reasons. The costs of continuously monitoring the spot rate or even defining and measuring the instantaneous spot rate could be high.

Furthermore, in each instance cash balance planning would be hampered since an interest payment is uncertain until the day it is due. To reduce these problems each interest payment could be fixed in advance. At the two extremes we would have interest determined on the date of payment resulting in the zero-basis-risk bond above and interest determined on the date of issue resulting in an ordinary bond (if the coupons are all equal). In general the shorter is the interval between the interest fixing and payment dates, the smaller will be the basis risk of the contract.

Suppose that each interest payment is determined Δ units of time prior to the date it is paid and that interest payments occur at equal intervals of τ units of time. One logical choice at time $t_i - \Delta$ for the level of payment at time t_i is $\delta(t_i) = [\exp(R\tau) - 1]B$ where R is the contemporaneous τ period interest rate.[4] The value of this payment at the time its level is determined is

$$f_i(Y, t_i - \Delta) = B \cdot P(Y, t_i - \Delta, t_i)[1/P(Y, t_i - \Delta, t_i - \Delta + \tau) - 1] \tag{12}$$

where $P(Y, t, T)$ is the time t state Y present value of one dollar at time T.

As in (10) the value of this contract at time t can be written as

$$F(Y, t, T) = \hat{E}\left\{B \cdot \exp\left(-\int_t^T r(s)\,ds\right) + \textstyle\sum_{i=1}^n f_i(t_i - \Delta) \cdot \exp\left(-\int_t^{t_i-\Delta} r(s)\,ds\right)\right\}. \tag{13}$$

In general the functional form of F and the basis risk of this bond cannot be determined until the state variables and their dynamics are specified. However in at least one interesting special case the problem simplifies and this is not required.

If the size of each payment is determined at the time the preceding payment is made (i.e., $\Delta = \tau$), then from (12) $f_i = B[1 - P(Y, t_{i-1}, t_i)]$ for $i \geq 2$ and[5]

$$F(Y, t, T) = B \cdot P(Y, t, t_1)/P(Y_0, t_0, t_1) \tag{14}$$

[4] This is obviously not the only logical choice. If the interest payment is determined some time after the point at which it has begun to accrue (i.e., $\Delta < \tau$), then one might prefer to use the already accrued interest at the spot rate as in (6) plus an accrual for the remainder of the period at the current Δ period rate. On the other hand, if payment is determined prior to the time at which it begins to accrue ($\Delta > \tau$), one might prefer to use the expected or forward τ period rate. Both of these alternate choices have some of the practical drawbacks mentioned previously.

[5] To derive (14) substitute f_i into (13) and use

$$P(Y, t, T) = \hat{E}\left[\exp\left(-\int_t^T r(s)\,ds\right)\right]$$

to get

$$F = B \cdot P(Y, t, t_n) + B \textstyle\sum_{i=2}^n \left\{P(Y, t, t_{i-1}) - \hat{E}\left[P(Y, t_{i-1}, t_i) \cdot \exp\left(-\int_t^{t_{i-1}} r(s)\,ds\right)\right]\right\} + B \cdot P(Y, t, t_1)[1/P(Y_0, t_0, t_1) - 1].$$

Using the properties of conditional expectations the second term in the sum above is equal to $P(Y, t, t_i)$. Cancelling like terms then gives (15).

where t_0 is the time of the last payment and Y_0 the state at that time. As was the case for the last bond examined, this bond will always sell for par on each ex-coupon date. Unlike the previous bond, however, the dynamics of this bond are not locally certain and some basis risk remains.

The fraction $B/P(Y_0, t_0, t_1)$ is constant; therefore, during the interval between coupon payments this bond's price is strictly proportional to the price of a pure discount bond maturing on the next coupon date. Obviously then it has the same level of basis risk as this discount bond. While the basis risk has not been eliminated as before, it has been reduced. For long term bonds the reduction can be substantial.

IV. Variable-Rate Loans with Partially Adjustable Rates

We saw in the previous section that the basis risk of loan contracts could be entirely eliminated through a proper adjustment in the coupon payment. Even in the case where certain factors make it desirable to know the amount of each interest payment some time before it is to be made we saw that basis risk could be substantially reduced. In spite of the achievable reduction in risk, many variable rate contracts explicitly limit their own immunization ability by certain restrictions on the coupon payments. For example, some contracts provide for variable payments with a guaranteed minimum and/or maximum. Other contracts include options such as the right to convert to a fixed rate loan.

In this section we analyze several of these bond contracts to determine the potential immunization lost. To derive analytical solutions and numerical results we require a specific model for the state variables. The model we use is a simple one, but many of the qualitative properties should carry over to more complex models.

In particular we assume that: There is a single relevant state variable, Y, which has a stochastic evolution as given in (1) of $dY = hY^{3/2}d\tilde{\omega}$. The mean and variance of return on each physical investment opportunity is proportional to Y. All investors have Bernoulli logarithmic utility of consumption.

Under these assumptions:[6] The instantaneous spot rate of interest will be a sufficient statistic for determining the price of all default-free bonds whose coupons depend only upon this interest rate. The stochastic evolution of r will be

$$dr = sr^{3/2}\, d\tilde{\omega}. \tag{15}$$

The factor risk premium for any default-free bond will be

$$\lambda(Y, t) = \lambda r^2 \tag{16}$$

which is just the covariance of changes of the interest rate with returns on the market portfolio. The valuation equation (2) becomes

$$\tfrac{1}{2}\, s^2 r^3 F_{rr} - \lambda r^2 F_r + F_t + \delta(r, t) - rF = 0. \tag{17}$$

[6] See Cox, Ingersoll, and Ross [3] for a similar derivation with a different stochastic process for the state variable. As a technical requirement we assume that $s^2 + \lambda < 1$ to assure bounded prices. This issue is discussed further in the appendix.

For our measure of basis risk we choose the ratio $\phi(r, t) \equiv -F_r/F$. On a cross-section of bonds this measure is proportional to the standard deviation of returns since by Ito's lemma $\mathrm{var}[dF/F] = s^2r^3(-F_r/F)^2$. This feature lends intuitive appeal to the measure; furthermore, in single state variable models such as this variance or standard deviation is a valid measure of risk in the Rothschild-Stiglitz sense.[7]

In the following examples we confine our attention to perpetual loan contracts with time-independent payments. This substantially simplifies the analysis because bond values do not depend upon time explicitly and (17) reduces to an ordinary differential equation. The two homogeneous solutions to (17) (i.e., with $F_t = \delta = 0$) are

$$F(r) = \begin{cases} A_1 r^{\gamma} \\ A_2 r^{\eta} \end{cases} \tag{18}$$

where

$$\gamma \equiv [(\lambda + s^2/2) - [(\lambda + s^2/2)^2 + 2s^2]^{1/2}]/s^2 \equiv b_1 - b_2 < -1$$

$$\eta \equiv b_1 + b_2 > 0.$$

A discussion of the properties of each of the examples is postponed until after all of them are developed. Plots of the modeled prices are presented in Figure 1.

For purposes of comparison we first determine the prices of a consol bond with constant coupons paid at the rate ρB and of a callable consol with constant call price K and coupons paid at the rate ρK.

Example A: *Consol.* A particular solution to (17) for constant payments $\delta = \rho B$ is

$$F(r) = \rho B/ar \tag{19}$$

where $a \equiv 1 - \lambda - s^2$. For the consol this is the complete solution as well since the boundary conditions require that the two homogeneous solutions vanish.[8] The basis risk for this bond is

$$\phi(r) = -F'/F = 1/r. \tag{20}$$

Example B: *Callable Consol.* The same particular solution applies. When the interest falls to a sufficiently low level it will benefit the issuer to call the bond and refinance. Denote this point by $\hat{r}$. Then we require that $F(\hat{r}) = K$ and $F(\infty) = 0$. These two conditions determine the constants A_1 and A_2. Applying them gives

$$F(r) = \frac{\rho K}{ar} + K\left(1 - \frac{\rho}{a\hat{r}}\right)\left(\frac{r}{\hat{r}}\right)^{\gamma}, \qquad r \geq \hat{r}. \tag{21}$$

Since $\hat{r}$ is at the discretion of the debtor, he will choose that refunding policy that

[7] See Merton [6, appendix 2] for a proof of this equivalence in the context of the option pricing model. A similar proof would apply here.

[8] See the appendix for a complete discussion of the technical details of the examples.

minimizes the value of the contract, namely

$$\hat{r} = \frac{\rho(1+\gamma)}{a\gamma}. \tag{22}$$

The basis risk of this contract is

$$\phi(r) = -\frac{\gamma}{r} + \frac{(1+\gamma)\rho K}{ar^2 F(r)}. \tag{23}$$

We now turn to the valuation of variable rate contracts. When variable rate notes were first introduced in 1974, they were usually redeemable at par on demand. This feature was sold as "protection" for the investor if interest rates declined. Actually this option has no value on a default-free bond because it will always sell for par. More recently, floating rate notes have guaranteed a minimum rate of interest payment—either explicitly or through the right to exchange the bond for a fixed rate note. In certain instances, particularly when the debtor is an individual, such as in a variable rate mortgage, there is a ceiling rather than a floor rate.

Example C: Variable Rate Consol. As we saw in the previous section a variable rate bond will always sell for its face value B. This remains true if the bond is callable by the issuer at a price $K \geq B$ or, redeemable by the holder at a price $K \leq B$ because it will never pay to call or redeem it. The basis risk of both the callable and noncallable variable rate bonds is zero.

Example D: Variable Rate Convertible. This bond pays variable interest at the rate r until such time as its owner exercises his option to exchange it for a straight consol paying ρ. As in Example B this conversion will take place when the interest rate falls to a sufficiently low level $\hat{r}$. At this point $F(\hat{r}) = \rho B/a\hat{r}$ the value of the straight consol. The other boundary condition is $F(\infty) = B$. B is also a particular solution to (18) with variable rate payments. Using the same two-step procedure as in Example B we can determine the value of the variable rate convertible

$$F(r) = B - \frac{B}{1+\gamma}\left(\frac{r}{\hat{r}}\right)^{\gamma}, \qquad r \geq \hat{r}, \tag{24}$$

and the conversion point $\hat{r}$ which is as given in (23). Note that since $\gamma < -1$, this bond is always more valuable than an ordinary floating rate note.

Example E: Guaranteed Minimum Variable Rate Consol. Like the convertible above this bond "protects" its owner from a sharp decrease in the interest rate by providing a floor rate, $\delta(r) = \text{Max}(\rho B, rB)$. Unlike the convertible, the payments become variable again after a rise in the interest rate. To price this bond we must determine separate solutions in the two interest rate regions, above and below ρ, and then match the value and first derivative at the boundary. In the lower and upper regions the particular solutions are the consol value $\rho B/ar$ and the variable rate bond value B. The entire solution is

$$F(r) = \begin{cases} B + \dfrac{(1-a)\,\eta + 1}{a(\eta-\gamma)}\left(\dfrac{r}{\rho}\right)^{\gamma} B & r \geq \rho \\ \dfrac{\rho B}{ar} + \dfrac{(1-a)\gamma + 1}{a(\eta-\gamma)}\left(\dfrac{r}{\rho}\right)^{\eta} B & r \leq \rho. \end{cases} \tag{25}$$

Example F: Guaranteed Maximum Variable Rate Consol. For this bond $\delta(r) = \mathrm{Min}(\rho B, rB)$. We could use the same two-region solution method as in previous example; however, it is simpler to construct an equivalent portfolio. A long position in one consol and one variable rate bond and a short position in the guaranteed minimum variable rate bond of Example E pays coupons of $\delta(r) = \rho B + rB - \mathrm{Max}(rB, \rho B) = \mathrm{Min}(\rho B, rB)$. Therefore, the value of this bond is equal to the value of the portfolio

$$F(r) = \begin{cases} \dfrac{\rho B}{ar} - \dfrac{(1-a)\eta + 1}{a(\eta-\gamma)}\left(\dfrac{r}{\rho}\right)^{\gamma} B & r \geq \rho \\ B - \dfrac{(1-a)\gamma + 1}{a(\eta-\gamma)}\left(\dfrac{r}{\rho}\right)^{\eta} B & r \leq \rho. \end{cases} \tag{26}$$

Figure 1 plots the values of these bond contracts. The parameter values selected were $s^2 = 0.4$ and $\lambda = -0.3$. The former corresponds to an annualized standard deviation of changes in the interest rate of 171 basis points at $r = 9\%$ and 200 basis points at $r = 10\%$. Over the ten years 1967–1976 the actual standard deviation of the 91 day T-bill rate was 185 basis points. For this model the expected rate of return on any bond is $r + \lambda r^2 F_r/F$. For a consol this is $r(1 - \lambda)$

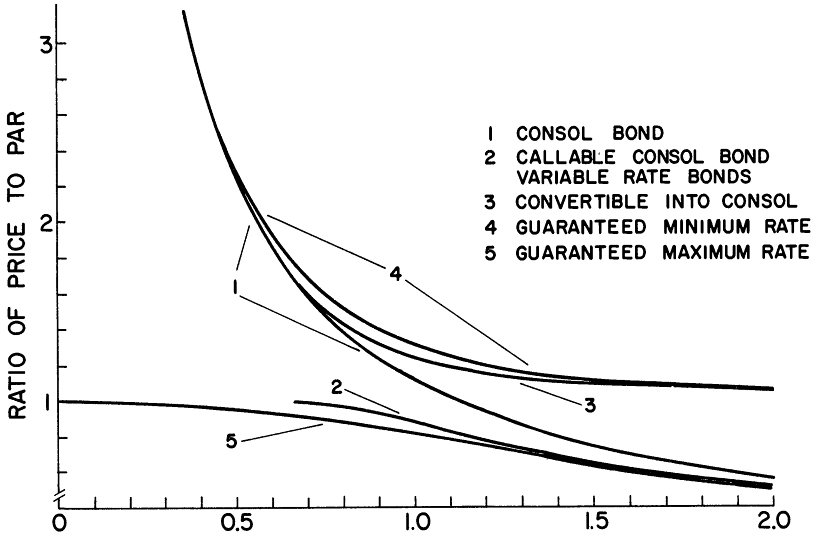

Figure 1.

which for the selected value is 130% of the prevailing spot rate. From the Ibbotson and Sinquefield [4] data series the premium on 20 year bonds averaged 33% higher than the spot rate. For scaling purposes the abscissa is the ratio of the spot rate to the "coupon" rate, ρ.

The first point of interest in Figure 1 is that none of the modified variable rate contracts ever sells for par. This might lead to some difficulties in connection with their issuance.[9] Secondly, the callable consol would appear to be a close substitute for the guaranteed maximum variable rate loan. The guaranteed minimum variable rate loan and the variable rate convertible also appear to be close substitutes.

Figure 2 plots the relative risk, $-\rho F_r/F$, for each bond on a logarithmic scale.[10] At the point of equality of the coupon rate, ρ, and the interest rate, the three variable rate consols have a range in risk from 46% to 57% of that of the straight consol. The callable consol has 58% as much risk. Obviously these variable rate contracts have not succeeded in reducing risk much below that achievable through conventional contracts. In particular Figure 2 reemphasizes the similarity of the callable consol and the variable rate loan with a guaranteed maximum coupon. This finding indicates that variable rate mortgages, which typically contain a guaranteed maximum, may be little different from standard mortgages which include the right to prepay. If this is true they are not the panacea for the mortgage market they are claimed to be.

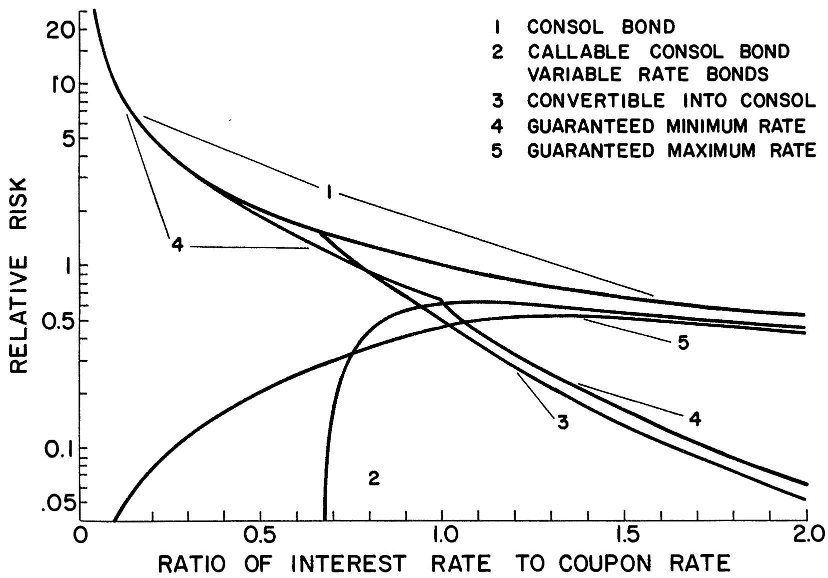

Figure 2.

[9] In the case of the guaranteed maximum variable rate loan an issuance fee, like points in a mortgage, could increase the value to the creditor to par.

[10] $-\rho F_r/F$ is a dimensionless quantity which is a function solely of r/ρ for each bond. $\phi \equiv -F_r/F$ does not have this property so the former is plotted for scaling purposes.

V. Variable Rate Loans Subject to Default Risk

Although some other countries, notably Israel, have issued variable rate bonds, the U.S. has not yet chosen to do so. Therefore all variable rate securities available here are subject to some degree of default risk. Clearly no payment structure could be constructed which would make the time path of the price of a risky variable rate bond completely deterministic. Nevertheless, in this section we prove that interest payment schedules which eliminate that portion of the risk due to fluctuations in interest rates can be constructed. The schedules have the desirable properties that they are independent of the stochastic properties of the state variables and, under reasonable conditions, are linear in the spot rate. In general they depend only upon the characteristics of the issuing firm.

Unfortunately, these schedules are rather complex and the simpler formulas actually used to determine interest payments on risky variable rate loans are not proper ones. A typical contract would call for the interest paid to be related to some prevailing index of interest rates. Most often interest is calculated by multiplying the outstanding principal by this index. As we saw in Section III the index should be based upon short term rates—a practice not uniformly adopted. Furthermore, this simple scheme is generally inappropriate even if a short term rate is used.

One case in which it is correct is for a risky consol bond written by a firm which pays no dividends, has no other debt, and has a variance proportional to the spot rate $\sigma^2(Y, V) = \sigma^2 r$. If this bond pays interest of rB, the valuation equation (4) may be written as

$$\hat{L}[F] + V\sigma'_{YV}F_{YV} + r[\tfrac{1}{2}\sigma^2V^2F_{VV} + (V - B)F_V + B - F] = 0. \tag{27}$$

Clearly there exists a solution to (27) subject to $F(0, \cdot) = 0$ and $F(\infty, \cdot) = B$ which depends upon V alone.[11] Therefore, the bond's value is unaffected by changes in the state variables, in particular those state variables which determine the interest rate.

If any one of these conditions is not true, then the interest formula is no longer a simple one. The general rule for figuring the scheduled payment is to determine the interest that the corporation would be charged on a short term loan with other terms the same. To illustrate this problem consider the same variable rate consol examined above, but now assume that the firm's variance is a constant independent of the state variables.

We wish a consol value which is independent of the state variables, thus the valuation equation can be simplified to

$$\tfrac{1}{2}\sigma^2V^2F'' + [rV - \delta(r, V)]F' - rF + \delta(r, V) = 0. \tag{28}$$

Without any loss of generality we can write $\delta(r, V) = k(V) + K(r, V)$, and (28)

[11] The terms in brackets is the consol debt equation with a constant unit interest rate examined by Merton [7] and others. The solution given in (32) with $\rho = 1$ is valid for this variable rate loan. If the bond were not a consol then the term F_t would have to be included, and the interest rate would affect its price since F_t is not multiplied by the otherwise common factor r.

is therefore valid, for all r, for any function $k(V)$ admitting a solution to

$$\tfrac{1}{2}\sigma^2V^2F'' - k(V)F' + k(V) = 0 \tag{29}$$

with $F(0) = 0$ and $F(\infty) = B$. This solution will be the price of the variable rate bond. The corresponding function $K(r, V)$ is given by

$$K(r, V) = r\frac{F - VF'}{1 - F'}. \tag{30}$$

Consequently, the variety of possible interest immunizing payment schedules is virtually endless. The only general requirement which must be satisfied is that the payment be linear in the spot rate.

One natural choice for $k(V)$ would be that function which yields a bond whose value would be equal to that of a consol making constant payments C written by an identical firm in an economy in which the interest rate were a constant, ρ. This requires that $k(V) = C - \rho(F - VF')/(1 - F')$ and (29) becomes

$$\tfrac{1}{2}\sigma^2V^2F'' + (\rho V - C)F' - \rho F + C = 0. \tag{31}$$

Ingersoll [5] has given the solution to (32) as

$$F(V) = \frac{C}{\rho}\left[1 - P\left(\nu, \frac{2C}{\sigma^2V}\right)\right] + VP\left(\nu + 1, \frac{2C}{\sigma^2V}\right) \tag{32}$$

where $\nu \equiv 2\rho/\sigma^2$ and $P(\cdot)$ is the incomplete gamma function ratio.

The resulting payment schedule is

$$\delta(r, V) = C + (r - \rho)\,\frac{2C}{\sigma^2}\frac{\Gamma(\nu, 2C/\sigma^2V)}{\Gamma(\nu + 1, 2C/\sigma^2V)} \tag{33}$$

where $\Gamma(\cdot)$ is the incomplete complimentary gamma function. The fraction multiplying $r - \rho$ is positive so payments are increasing in r as would be expected. This fraction is increasing in V so payments are increasing (decreasing) in V when r is greater (less) than the comparison rate ρ. Payments vary from C when $V = 0$ to $(r/\rho)C$ for $V = \infty$.

In setting up this contract there is the latitude to choose any reference rate ρ desired. The level portion of the payment, C is then fixed at the value which sets the bond's initial price to par. Figure 3 plots the scheduled coupons as a function of firm value for various possible contracts. It is assumed that the firm's variance is 0.1 and that the intial leverage was 50%. It is clear that if the interest rate is close to the reference rate, payment schedules are not very sensitive to changes in firm value. For this reason it is probably desirable to set the reference rate close to the interest rate.

Unfortunately a subsequent change in the interest rate can dramatically affect this sensitivity. For example, in Figure 3 the interest rate is 12.5%. For a reference rate of 15% payments vary from 16.1% to 13.5% of par. For a reference rate of 10% payments vary from 11.5% to 14.1% of par. In each instance the range is about the same, 2.6 percentage points. If the interest rate were to drop to 7.5%, payments would range from 16.1% to 8.4% of par for the former contract and 11.5% to 8.9%

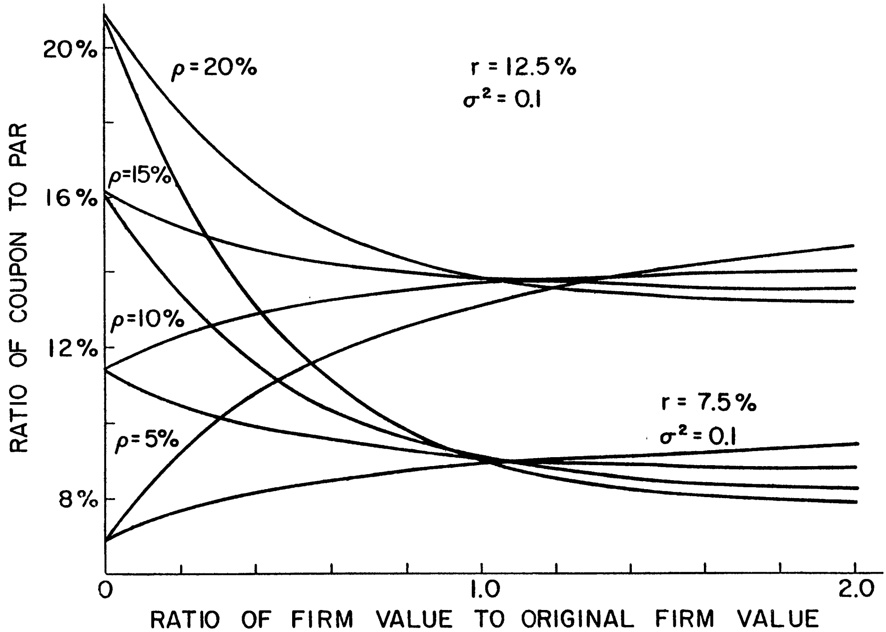

Figure 3.

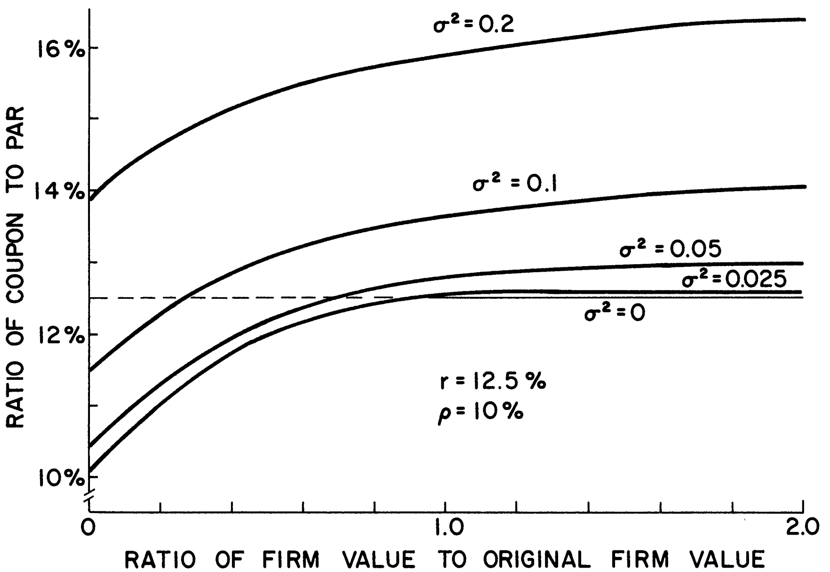

Figure 4.

of par for the latter. This would indicate that for payment stability over time, the reference rate should be somewhat below the spot rate if it is expected to fall and vice versa.

Figure 4 shows the firm value sensitivity of contract payments for firms of inherently different risk. Again an initial leverage of 50% was selected. Riskier firms, of course, have greater required payments for any level of V and r.[12] However, the value sensitivity of payments is virtually identical for any realistic level of risk.

For example, the difference between the required coupon rates on firms with risk levels of $\sigma^2 = 0.05$ and $\sigma^2 = 0.1$ only ranges from 84 basis points (at .6) to 106 basis points (at 2.0). Comparing $\sigma^2 = 0.2$ to $\sigma^2 = 0.1$ the differences ranges only from 224 basis points (at 0.8) to 238 basis points (at 2.0).

This implies that some standardization of contracts might be possible with little loss of immunizing potential. If the $\sigma^2 = 0.1$ contract were accepted as standard and a firm with half (twice) the risk were required to pay a coupon 95 basis points less (231 points more) then the maximum deviation from the optimal contract payment would be only 11 (7) basis points. Note that the riskless contract would not be a good choice for purposes of standardization because at low firm values the sensitivity of interest payments is substantially in excess of zero. Nevertheless, this is implicit in common usage when the floating rate is set at a constant premium above the treasury rate.

VI. Conclusion

We have presented here a number of findings about variable rate loan contracts. In particular we demonstrated that perfect interest rate immunization is achievable on default-free contracts through the simple expedient of paying continuous coupons at the currently prevailing spot rate of interest. Similarly the accumulation of interest at this rate permits perfect immunization with discrete coupon payments.

Loans which are risky in terms of default can also be immunized through variable coupon payments; however, the proper schedule is no longer simple to determine. We derived the coupon formula for a special case under commonly adopted assumptions. There we saw that coupon payments were increasing in the interest rate and the firm's business risk but could be either increasing or decreasing in its financial risk (leverage).

Certain contracts explicitly limit the degree to which the coupon can be varied. We examined contracts of this type and, at least for the special case considered, concluded that typical provisions of this type render variable rate contracts little better than standard contracts in terms of immunization.

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[12] The dashed portion of the $\sigma^2 = 0$ line is plotted for reference only. Under the payment scheme proposed, a riskless firm could not decrease in value.

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