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A Multivariate Model of the Term Structure*

TERENCE C. LANGETIEG

I. Introduction

THE TERM STRUCTURE OF interest rates has a very important role in economic theory. On the macroeconomic level the term structure serves as a key transmission link between the monetary and real sectors. On the microeconomic level the term structure not only serves as a tool for bond pricing, but the term structure also has an important role in the valuation of most financial claims.

The concept of an equilibrium term structure under uncertainty was introduced by Merton [41] in 1973. Since then, Brennan and Schwartz [4, 5], Vasicek [51], Cox, Ingersoll, and Ross [11, 12], Richard [46], Long [33], and Dothan [16] have made important contributions to the theory of the term structure and equilibrium bond pricing. At this time, "closed form" solutions for the equilibrium term structure have been derived in several special, but important, cases where the term structure is generated from univariate or bivariate stochastic processes such as the elastic random walk, geometric random walk, and square root process. However, it is arguable that the term structure of interest rates is embedded in a large macroeconomic system. Hence, it is arguable that there are generally many economic factors which are related to the term structure. The primary objective of this paper is to develop a multivariate model of the term structure that can accommodate an arbitrary number of economic relationships. The model is of a general form and can be linked with many alternative macroeconomic systems.

The major assumptions of the model to be developed are that the set of stochastic factors related to the term structure follow a joint elastic random walk, that the instantaneous riskless rate can be represented as a linear combination of the same factors, and that the market prices of risk (corresponding to the different factors) are, at most, time dependent. The resulting model of the term structure can be expressed in a form that is consistent with traditional notions of the term structure. Even with multiple factors, the term structure can always be expressed in a form characterized by expectations of future short-term rates plus a liquidity premium. We will restrict our attention to the pricing of default-free, pure-discount bonds. While this special case is of interest in its own right, a model for pricing the default-free, pure-discount bond also plays an important role in pricing default-free coupon bonds, risky pure-discount bonds, and, as Cox, Ingersoll, and Ross show, the valuation of most financial claims [12].

Assistant Professor of Finance, Graduate School of Business, University of Southern California, Los Angeles.

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In Section II we provide a brief review of prior research related to the term structure and bond pricing. Section III focuses on the characteristics of the stochastic process of the short-term riskless rate and other factors related to the term structure. Some of the more important properties of the joint elastic random walk are also presented. Section IV presents the form of the multivariate solution for the default-free, pure-discount bond when the underlying stochastic factors follow a general Ito process. Section V specializes the multivariate solution for the case of a joint elastic random walk. Examination of the bond pricing equation provides an interesting perspective on the nature of bond risk. In the multivariate case, risk is best measured by a vector rather than a single number. However, the risk vector has essentially the same interpretation as the duration concept in more traditional models. We also present explicit forms for the bond's expected return and variance over instantaneous and finite holding periods, and we examine the functional form of forward rates applicable for the multivariate case. Finally, we apply the model in the valuation of call options and pure-discount, risky corporate debt.

II. Theories of the Term Structure

The type of bond to be analyzed in this study is a default-free, pure-discount bond. By definition, the price at time t of a bond paying \$1 at time T, denoted as $P(t, T)$ is related to the prevailing yield (or discount rate) as follows:

$$P(t, T) = \exp[-R(t, T) \cdot (T - t)] \tag{1}$$

where $R(t, T)$ denotes the yield at time t of a bond maturing at time T and "exp" denotes the exponential function.

Theories of the term structure, or bond pricing, involve the manner in which $R(t, T)$, or $P(t, T)$, is determined. These theories can be divided into five major classes: (1) the pure expectations theory, (2) the liquidity preference theory, (3) the market segmentation or "preferred habitat" theory, (4) mean-variance theory, and (5) arbitrage pricing theory.

The first theory, the pure expectations theory, is usually interpreted to imply that the rate of return on a bond maturing at time T should be equal to the (geometric) average of the expected short-term rates from time t to time T. Denoting the expected value at time t of the short-term rate at time v as $E_t(r(v))$, the pure expectations hypothesis can be expressed as:

$$R(t, T) = \frac{1}{T - t} \int_t^T E_t(r(v))\, dv \tag{2}$$

Meiselman's [40] empirical work supports the pure expectations theory, but his results are questioned in studies by Grant [21], Buse [7], Malkiel and Kane [35], and others.

The second theory, the liquidity preference theory, asserts that long-term bonds will have a yield in excess of the geometric average of future short-term rates. In its original form, Hicks [25] casually observes that the longer the life of the bond, the greater the risk with respect to the potential for capital loss due to changes in interest rates. Assuming that all investors' preference for liquidity leads to an adoption of an instantaneous investment horizon and risk aversion, bonds with a longer term and greater instantaneous risk should have a larger term (or liquidity) premium. In a world where interest rates can experience only a single, instantaneous shift, Carr, Halpern, and McCallum [10], McEnally [39], Hopewell and Kaufman [27], and Fisher and Weil [18] correctly argue that interest rate risk can be measured by duration. However, from the perspective of continuous time modeling, Ingersoll, Skelton, and Weil [28] show that duration is a valid measure of a bond's interest rate risk *only* when the short-term rate follows random walk (i.e., the term structure experiences only shape-preserving shifts). While duration is valid in a random walk world, the random walk model has other theoretical inadequacies (see Section III).

Modigliani and Sutch [43] further refine the concept of a liquidity premium with their observation that an investor's perception of a bond's interest rate risk is relative to the investor's investment horizon, or "preferred habitat." Hence, a ten-year bond is "safer" than a one-year bond for the "hedging" investor with a ten-year investment horizon. If *all* investors were "hedgers," it is possible that aggregate supply of bonds at various maturities might not coincide with aggregate demand. Since markets must clear, the imbalance could lead to different yields on bonds with different maturities. The resulting term structure could be increasing, decreasing, humped, or of irregular shape. There are two cases where the preferred habitat theory degenerates to the pure expectation theory or the liquidity preference theory. First, an imbalance of supply and demand is not likely to persist in the long run, so the preferred habitat effect has no impact in the long run. Second, it is arguable that there is some horizon flexibility for both supply and demand. If there are a sufficient number of investors and issuers with flexible horizons, interest rate differentials due to the preferred habitat effect between maturities would be "arbitraged" away.

The liquidity preference theory and the preferred habitat theory have similar implications for the form of the term structure equation:

$$R(t, T) = \frac{1}{T-t}\left[\int_t^T E_t(r(v))\,dv + \int_t^T L(v, T)\,dv\right] \tag{3}$$

where $L(v, T)$ can be interpreted as the instantaneous term premium at time v of a bond maturing at time T.[1] The "premium" permits the model to capture

[1] Following Fama [17], $L(v, T)$ for can be interpreted in a continuous time model as $L(v, T) = E_t(H\tilde{P}R(v, T)) - E_t(r(v))$ where $H\tilde{P}R(v, T)$ is the (instantaneous) holding period return at time v for a bond maturing at T. Hence, the term structure in equation (3) can be re-written in terms of expected HPR's as:

$$R(t, T) = \frac{1}{T-t}E_t\left[\int_t^T H\tilde{P}R(v, T)\,dv\right] \tag{3'}$$

where

$$H\tilde{P}R(v, T) = \lim_{\Delta\to 0}\left[\frac{\ln[\tilde{P}(v+\Delta, T)] - \ln[\tilde{P}(v, T)]}{\Delta}\right] = \frac{\partial \ln[\tilde{P}(v, T)]}{\partial v} = \frac{-\partial(T-v)\tilde{R}(v, T)}{\partial v}$$

theoretical "deviations" from the pure expectations theory. The structure of term premiums could be monotonically increasing as would be implied by the liquidity theory, or of irregular shape as implied by the preferred habitat theory.

Empirical studies by Kessel [30], Cagan [9], and McCullough [37] and others support the hypothesis that a liquidity premium exists, and the next question we address is exactly what factors cause the liquidity premium to vary over time. Kessel and Cagan assert that the liquidity premium is positively related to the level of interest rates. However, Nelson [45] claims that the relationship is negative, while McCulloch claims that there is no relationship and that the liquidity premium structure does not vary over time. Roll [48] theoretically argues that the liquidity premium is a function of investors' wealth, confidence in expectations of future interest rates, degree of risk aversion, and maturity preference or preferred habitat. Fama [17] empirically finds the liquidity premium to be a positive function of the degree of uncertainty in expectations of future inflation rates which can be measured by the degree of fluctuation in recent short-term rates. At the time of this writing it is still empirically unclear what stochastic factors, if any, affect the structure of liquidity premiums.

A fourth theory of the term structure is related to the mean-variance theory of Markowitz [36] and to the capital asset pricing model. The popularity of mean-variance theory for managing portfolios of common stock led to an interest in determining the expected return and variance related to bond investment. Malkiel [34] shows that in a one period world a bond's return should be measured by the holding period return (HPR). Watson [52], Hempel and Yamitz [24], and Crane [13] use simulation to compare expected holding period returns and variances from portfolios of widely varying composition. Generally these studies show that the optimal bond portfolio varies according to risk preference and varies according to the future expected course of interest rates. However, these simulation-based models also share questionable assumptions concerning the dynamics of interest rate movements and the a priori selection of portfolio types used in the simulation. As in any simulation, their results reflect the assumptions underlying the simulation. An alternative method of estimating the HPR is to examine historical data. However, results from empirical analysis can only be suggestive of an underlying theory of the term structure and the dynamic process that drives the term structure.

A theoretical approach to determining a bond's expected HPR and variance

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Fama also shows that the term structure equation (or bond prices) can be expressed in a third alternative form in terms of forward rates:

$$R(t, T) = \frac{1}{T-t}\left[\int_t^T F(t, s)\, ds\right] \tag{3''}$$

where

$$F(t, s) = \lim_{\Delta \to 0}\left[\frac{\ln[P(t, s-\Delta) - P(t, s)]}{\Delta}\right] = \frac{-\partial \ln[P(t, s)]}{\partial s} = \frac{\partial (s-t)R(t, s)}{\partial s}$$

While we will calculate the (discrete) holding period return and forward rates, we will emphasize the more traditional equation (3) is the development of the term structure model. Of course, formulations based on (3′) or (3″) are equally valid.

HPR is provided by Roll [47]. Roll integrates the dynamics of interest rate movements with the capital asset pricing model [32, 50]. Assuming that the capital asset pricing model is valid for bonds, Roll shows the relation between the bond's liquidity premium and the equilibrium expected rate of return. While Roll introduces an equilibrium setting for bond pricing, he does not provide an explicit bond valuation model.

A final theory of the term structure is provided by the application of continuous time modelling that is similar to the methodology used in option pricing. Starting from the assumption of no riskless arbitrage opportunities, Merton [42] and others [11, 12, 16, 46, 51] solve second order partial differential equations for the bond price. While these equations lead to closed-form solutions for the simpler cases, more complex cases generally require solution via numerical analysis [4, 5] . This arbitrage pricing approach provides a more general approach than other theories of the term structure. First, an explicit functional form of bond prices (or the term structure) is not assumed; rather it is derived from three fundamental assumptions:

1. The assumption that bond prices (and the term structure) are functionally related to certain stochastic factors. Of course, the functional form is not assumed.
2. The assumption that the underlying factors follow a particular stochastic process.
3. The assumption that markets are sufficiently perfect to permit a "no arbitrage" equilibrium to obtain.

The equilibrium functional form of bond prices includes many theories of the term structure as special cases. Richard [46] shows that the traditional form of the pure expectations hypothesis is valid when all the underlying factors are non-stochastic. Vasicek's [51] univariate elastic random walk model leads to a term structure as depicted in equation (3), although the notion of liquidity preference or market segmentation is not assumed. Building on Richard's [46] multivariate model, the objective of this paper is to extend Vasicek's model to the case of a multivariate elastic random walk. In this case the resulting term structure equation is still consistent with the traditional term structure as represented in equation (3).

III. The Dynamics of the Spot Rate

To complete the description of the term structure equation we must specify the generating process for the underlying stochastic factors. In this paper, the multivariate elastic random walk will be employed. To gain a better understanding of the multivariate case, we first examine the univariate elastic random walk for the short-term rate, noted as r. Such a process is Markovian and can be represented by a first order linear stochastic differential equation:[2]

$$dr\ (t) = \alpha(r,\, t)\ dt + \sigma(t)\ dz\ (t) \tag{4}$$

[2] See Arnold [1], Gilman and Skorohod [20], or Hoel, Sidney, and Port [26] for a discussion of stochastic differential equations and Browian motion. The stochastic term "$dz\ (s)$" in equation (4) has the following properties: 1) $E_t(dz(s)) = 0$ for $s \geq t$, 2) $\text{cov}_t(dz(s), dz(s^*))$ is zero if $s^* \neq s$ and ds if $s^* = s$, 3) $\int_t^v dz(s)$ is normally distributed with a zero mean and variance $(v - t)$.

where

$$\alpha(r, t) = a(t) + b(t)r(t)$$

The coefficients $a(t)$, $b(t)$, and $\sigma(t)$ may be taken as constants or as functions of time; hence they are deterministic. The coefficient $\alpha(r, t)$ is referred to as the instantaneous "drift" and is stochastic over time, since it depends on $r(t)$. The coefficient $\sigma(t)$ represents the instantaneous "diffusion" of the process. It is assumed that $\sigma(t)$ is independent of the level of $r(t)$; hence $\sigma(t)$ is deterministic. The term "$dz\ (t)$" represents a Wiener process or "Brownian motion." The term "$\alpha(r, t)\ dt$" represents the anticipated component of the change in the short-term rate, while "$\sigma(t)\ dz\ (t)$" represents the random or unanticipated component.

The first order linear process in equation (4) has also been termed an "elastic random walk" [51], an "Ornstein-Uhlenbeck" process [1], and a normal backwardation process [29]. The "elastic random walk" is probably the most descriptive name. Assuming that $b < 0$, future short-term rates will asymptotically tend toward their long term "normal" level of $-\frac{a}{b}$. However, when $b = 0$ the elastic random walk degenerates to the well-known random walk which has no normal level, and when $b > 0$ the process becomes explosive. At this time the true generating process of the spot rate has not been identified. However, Dobson, Sutch, and Vanderford [15], Brick and Thompson [6] have produced empirical support for the elastic random walk and random walk respectively.[3] Hence, there is some strength in the argument that the class of first order linear processes as depicted in equation (4) adequately represents the generating process for the spot rate.

While the elastic random walk and random walk are consistent with the data, both models have certain objectionable theoretical implications. The random walk model implies that spot rates will drift off to infinite positive and negative values with probability one. As Merton [41] shows, the possibility of an infinite negative yield implies that extremely long-term pure discount bonds tend to infinite positive values. As long as $b < 0$, the elastic random walk does not have this problem, since the short-term rate is always drifting toward its "normal" level. However, the elastic random walk does permit the transient occurrence of negative short-term rates. Since negative nominal interest rates will never occur in reality, the elastic random walk is an inadequate model when the short-term rate is close to zero. However, the elastic random walk will be adequate if the current short-term rate is well above zero, implying that there is only a small probability of reaching a negative level.

One way to prevent the occurrence of negative interest rates is to introduce a "reflecting barrier" at the zero level. However, this also adds considerable complication to the analysis of bond prices. A second approach is to assume that the short-term rate's diffusion coefficient is proportional to r^{α} when $\alpha > 0$, accompanied by a positive drift coefficient when the spot rate reaches the zero level.

[3] Dobson, Sutch and Vanderford [15] reject as internally inconsistent Malkiels' [35] weighted-regressive model, De Leenow's [14] weighted-extrapolative model, Muth's [44] adaptive expectations model, and Bierwag and Grove's [2] Polar-convex model. Of course, a different statistical test on different data may prove these processes to have merit.

Cox, Ingersoll, and Rose [12] analyze the cases where $\alpha = \frac{1}{2}$ and $1\frac{1}{2}$, and Dothan [16] analyzes the case where $\alpha = 1$. While these processes have a natural reflecting barrier at the zero level, there is an absence of empirical work related to such processes. Furthermore, these processes appear to be more difficult to work with, especially in a multivariate form when the underlying factors are stochastically dependent. While the elastic random walk will be employed here, it must be recognized that the elastic random walk is adequate only when the short-term rate is are sufficiently above the zero level to warrant the assumption that the probability of reaching negative level is negligible over a reasonable finite time interval.

The Multivariate Elastic Random Walk

A more general characterization of the short-term rate allows dependence on multiple economic factors. For instance, we might form a theoretical model where the short-term rate directly depends on the inflation rate and the real interest rate and indirectly depends on other economic factors that are stochastically related to the inflation rate and real rate. In this paper, it will be assumed that the short-term rate can be represented by a linear combination of n stochastic factors:

$$r = w_0 + \sum_{i=1}^{n} w_i x_i = w_0 + w'x \tag{5}$$

where x is a vector of stochastic factors that make up an underlying macroeconomic system and w is a vector of weights that are, at most, time dependent.[4] It will also be assumed that the stochastic process of the macroeconomic system can be expressed in reduced form as a linear process:

$$dx = (a + Bx)\,dt + \sigma dz \tag{6}$$

where

$dx' = [dx_1\ dx_2\ \cdots\ dx_n]$
$a' = [a_1\ a_2\ \cdots\ a_n]$
$B = n \times n$ matrix with elements (B_{ij})
$\sigma dz' = [\sigma_1 dz_1\ \sigma_2 dz_2\ \cdots\ \sigma_n dz_n]$
$dz_i =$ standard Wiener process with a zero mean and a variance equal to dt
a, B, and σ are, at most, time dependent

We might also refer to this process as a joint elastic random walk. Note that the random component is independent of the level of the process. Several useful results will be employed in this paper. Arnold [1] shows that the solution to (6) has the form:

[4] For example, Richard [46] shows that the nominal instantaneous riskless rate can be expressed as $r(t) = \Pi(t) + r^*(t) - \sigma_I^2$ where $\Pi(t)$ is the anticipated rate of inflation, $r^*(t)$ is the real riskless rate, and σ_I^2 is the instantaneous variance rate of the price level I which follows a stochastic process of the form $\frac{dI}{I} = \Pi(t)\,dt + \sigma_I\,dz_I$. Clearly this relation is consistent with equation 5. While $r(t)$ directly depends on $\Pi(t)$, $r^*(t)$, and σ_I^2, the stochastic process of $r(t)$ may also depend on other macroeconomic factors that influence the stochastic processes of $\Pi(t)$, $r^*(t)$, and σ_I^2.

$$x(v) = \psi(v-t)x(t) + \int_t^v \psi(v-s)a\,ds + \int_t^v \psi(v-s)\sigma dz\,(s) \qquad v \geq t \quad (7)$$

where $\psi(v-t)$ is the fundamental $(n \times n)$ solution matrix to: (7a)

$\dot{\psi}(v-t) = B\psi(v-t)$ subject to $\psi(t-t) = I$
$I = (n \times n)$ identity matrix
$t =$ current point in time

In the case where B is constant:

$$\psi(v-t) = \exp(B(v-t)) = I + \textstyle\sum_{n=1}^{\infty} B^n(v-t)^n/n! \quad (8)$$

If B is diagonal, $\psi(v-t)$ will be a diagonal matrix with diagonal elements $\exp(B_{ii}(v-t))$ and zero's elsewhere. In the general case (B not diagonal) we must find the eigenvalues and eigenvectors of B to put $\psi(v-t)$ into a tractable form (see footnote 21).

We will also be interested in the expected values and the covariance matrix of $\tilde{x}(v)$ and $\tilde{x}(v^*)$ where v and v^* represent future points in time. Using a derivation similar to Arnold we find:

$$E_t(\tilde{x}(v)) = \psi(v-t)x(t) + \int_t^v \psi(v-s)a\,ds \qquad v \geqq t \quad (9)$$

$$\text{cov}_t(\tilde{x}(v), \tilde{x}(v^*)) = \int_t^{\min(v,v^*)} \psi(v-s) \textstyle\sum \psi(v^*-s)'\,ds \qquad v, v^* \geqq t \quad (10)$$

where

$\sum = \text{cov}_t(\sigma dz, \sigma dz) = E_t(\sigma dz \cdot \sigma dz')$
$= (n \times n)$ covariance matrix with elements $(\sigma_{ij}) = (\rho_{ij}\sigma_i\sigma_j)$
and $\rho_{ij} = E_t(dz_i \cdot dz_j)$
$\psi(v^*-s)' =$ transpose of $\psi(v^*-s)$

IV. A General Model for Bond Prices

In this section a general model for pricing default-free bonds will be developed. The model is general in that it allows the bond price to depend on an arbitrary number of stochastic factors following an arbitrary Ito process. Following a derivation similar to that of Richard [46] and Ross [49], the form of the solution for multivariate case can always be obtained. However, an explicit solution can be obtained only for special distributions. One mathematically tractable case occurs when the underlying stochastic factors follow a joint elastic random walk. This case will be analyzed in Section V.

Suppose we want to value a pure-discount, default-free bond paying $1 at time T. Let $P_T = P(t, T, x)$ denote the price of this bond at time $t \leq T$. We assume that bond P_T depends on n stochastic factors x. Let these factors follow a joint Ito process of the form:[5]

[5] An alternative, but equivalent, model is based on a joint Ito process of the form:

$$dx_i = \alpha_1(t, x_1, \cdots, x_n)\,dt + \textstyle\sum_{j=1}^{n} \sigma_{ij}(t, x_1, \cdots, x_n)\,dz_j \qquad i = 1, \cdots, n$$

$$dx_i = \alpha_i(t, x)\, dt + \sigma_i(t, x)\, dz_i \qquad i = 1, \cdots, n \tag{11}$$

where dz_i is a standard Wiener process. We also assume that the riskless rate can be expressed as a function of the same set (or subset) of stochastic factors, $r = r(t, x)$.

The bond pricing model we are developing is very general, since the functional form of r, α, and σ is arbitrary, and almost any reasonable continuous specification is possible. Of course, to develop an *explicit* bond pricing model, the functional forms of r, α, and σ must be theoretically or empirically specified.[6]

Assume that $P(t, T, x)$ is continuous in t and x with continuous partial derivatives with respect to t and x and with continuous second partial derivatives with respect to x. Then we can invoke Ito's formula to show that $dP_T = dP\,(t, T, x)$ will have the form:[7]

$$dP_T = \alpha_{P_T}\, dt + \sum_{i=1}^{n} \beta^i_{P_T}\, dz_i \tag{12}$$

where

$$\alpha_{P_T} = \frac{\partial P_T}{\partial t} + \sum_{i=1}^{n} \frac{\partial P_T}{\partial x_i} \alpha_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P_T}{\partial x_i \partial x_j} \sigma_i \sigma_j \rho_{ij}$$

$$\beta^i_{P_T} = \frac{\partial P_T}{\partial x_i} \sigma_i$$

Next, we introduce an equilibrium argument similar to that of Ross [49] which leads to a partial differential equation for the bond price. Consider a portfolio $\bar{P}$ of $(n + 2)$ pure-discount bonds, P_j, with different maturities $T_j > 0$.

$$\bar{P} = \sum_{j=1}^{n+2} \gamma_j P_j = \gamma' P$$

where

$$\gamma_j = \text{number of bond with maturity } T_j \text{ in portfolio}$$

$$P' = [P_1\, P_2 \cdots P_{n+2}] \qquad \gamma' = [\gamma_1\, \gamma_2 \cdots \gamma_{n+2}] \tag{13}$$

By Ito's formula, the stochastic differential equation for $\bar{P}$ is:

$$d\bar{P} = \sum_{j=1}^{n+2} \gamma_j \alpha_{P_j}\, dt + \sum_{i=1}^{n} \left[\sum_{j=1}^{n+2} \gamma_j \beta^i_{P_j}\right] dz_i$$

$$d\bar{P} = \gamma' \alpha + \sum_{i=1}^{n} \gamma' \beta^i\, dz_i$$

where

$$\alpha = [\alpha_{P_1}\, \alpha_{P_2} \cdots \alpha_{P_{n+2}}]$$

$$\beta^i = \beta^i_{P_1}\, \beta^i_{P_2} \cdots \beta^i_{P_{n+2}}] \tag{14}$$

Assuming unrestricted short sales, we can always choose the weights γ such that the portfolio has no risk and requires no investment.

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where the dz_j represent standard Weiner processes. Without lack of generality we can consider the dz to be independent processes in this model.

[6] See Cox, Ingersoll, and Ross [12] for one of the more elegant theories concerning the functional relation of the instantaneous spot rate to underlying economic stochastic factors.

[7] See Arnold [1, p. 90] for a discussion of Ito's formula.

$$\sum_{j=1}^{n+2} \gamma_j P_j = \gamma' P = 0 \qquad \text{(no investment)}$$

$$\sum_{j=1}^{n+2} \gamma_j \beta_{P_j}^i = \gamma' \beta^i = 0 \qquad i = 1, \cdots, n \quad \text{(no risk)} \tag{15}$$

Note that the portfolio is riskless in nominal terms and is also riskless in real terms.[8] If markets are sufficiently perfect (i.e., no commissions, taxes, or other costs in portfolio revision), then, as Ross [49] argues, the riskless, zero-investment portfolio should earn a zero rate of return in equilibrium.

$$\sum_{j=1}^{n+2} \gamma_j \alpha_{P_j} = \gamma' \alpha \overset{e}{=} 0 \tag{16}$$

Ross notes that the vector γ is orthogonal to the vector P and the n vectors β^i. Since γ is a $(n + 2)$ vector it can be orthogonal to at most $(n + 1)$ independent vectors in $(n + 2)$ space. Since γ is also orthogonal to α, we conclude that α can be expressed as a linear combination of P and β^i, $i = , \cdots, n$.

$$\alpha = \phi_0 P + \phi_1 \beta^1, \cdots + \phi_m \beta^n$$

$$\text{where} \quad \phi_i,\ i = 0, \cdots, n \quad \text{are scalars}^9 \tag{17}$$

If an underlying stochastic factor x_i is *tradable*, then $\phi_i = (\alpha_i - \phi_0 x_i)/\sigma_i$. When x_i is not tradable, then ϕ_i must be empirically estimated or theoretically specified.[10] Assuming there is a riskless instantaneous interest rate, denoted as r, then $\phi_0 = r$.[11] Multiplying each equation i in (17) by $1/P_i$, we obtain the equilibrium condition for the instantaneous expected rate of return of bond i:

$$\left(\frac{\alpha_{P_i}}{P_i}\right) \overset{e}{=} r + \phi_1\left(\frac{\beta_{P_i}^1}{P_i}\right) + \cdots + \phi_n\left(\frac{\beta_{P_i}^n}{P_i}\right) \tag{18}$$

[8] We are focusing on a "nominal" equilibrium rather than a "real" equilibrium. A portfolio which is riskless in nominal terms is also riskless in real terms, due to the assumption of zero investment. For example, if we form the arbitrage argument in real terms rather than norminal, it can be shown that in $A(T)$, resulting from the general bond valuation in equation (20), $\phi\sigma$ must be replaced by $(\phi\sigma + \sigma_{.I})$ where $\sigma_{.I}$ is a vector with components $\left(\text{cov}\left[dx_i, \frac{dI}{I}\right]\right) = (\sigma_i \sigma_I \rho_{iI})$, where $\rho_{iI} = E(dz_i \cdot dz_I)$, and where $\frac{dI}{I} = \Pi(t)\, dt + \sigma_I\, dz_I$ represents the stochastic process of the price level index I. In a general model $\Pi(t)$, σ_I, and ρ_{iI} may be stochastic functions of x and time.

With a slight change in interpretation, the model developed in this paper for a nominal equilibrium can be adapted for a real equilibrium. To preserve the *normality* of $A(T)$, we also assume that: (1) σ_I and ρ_{iI} are at most time dependent functions; and (2) if nominal bond prices depend on the stochastic price index, the dependence is on $\ln(I)$ which follows a linear process of the same form as x. With these assumptions, the nominal bond in real equilibrium is analyzed by simply replacing $\phi\sigma$ by $(\phi\sigma + \sigma_{.I})$ throughout the text of this paper where ϕ is the market price of risk in real terms.

[9] The market prices of risk σ_i, $i = 0, \ldots, n$ may depend on time and all of the underlying stochastic factors x. In general we would expect ϕ to depend on factors such as the level of aggregate wealth and the level of investor risk aversion, for example see [12], [33], and [48].

[10] If *all* the stochastic factors affecting bond prices are tradable, then we could make the assumption of risk neutrality to simplify the solution technique. However, as Richard [46] observes, factors affecting the spot rate [e.g., inflation rate, real interest rate, etc.] are not tradable.

[11] If there is no riskless rate, then Ross [49] notes that ϕ_0 represents the expected return on a zero-beta asset (i.e., an asset with $\beta^i = 0$ for $i = 1, \cdots, n$).

An equilibrium bond pricing equation for any pure discount bond $P(t, T, x)$ paying with certainty \$1 at time T is obtained by substituting for α_P and β_P^i, $i = 1, \cdots, n$ from 12 into 18 and rearranging to get:

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P}{\partial x_i \partial x_j} \sigma_i \sigma_j \rho_{ij} + \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} (\alpha_i - \phi_i \sigma_i) + \frac{\partial P}{\partial t} - rP = 0$$

$$\text{subject to} \quad P(T, T) = 1 \tag{19}$$

Following Richard [46], the solution to 19 has the form:[12]

$$P(t, T, x) = E_t[\exp[A(T)]]$$

$$A(T) = -\int_t^T r(v)\,dv - \int_t^T \frac{1}{2} \phi\sigma' \textstyle\sum^{-1} \phi\sigma\,dv - \int_t^T \phi\sigma' \textstyle\sum^{-1} \sigma dz\,(v) \tag{20}$$

where

$\phi\sigma' = [\phi_1\sigma_1\ \phi_2\sigma_2 \cdots \phi_n\sigma_n]$
$\sigma dz' = [\sigma_1 dz_1\ \sigma_2 dz_2 \cdots \sigma_n dz_n]$
$\sum^{-1}$ = inverse of the $(n \times n)$ covariance matrix $\sum$, see equation (10), where $\sum$ is assumed to be full rank

The bond pricing solution in (20) can be successfully employed only in special situations. If we do not know the probability density of the exponent, $A(T)$, then the "expectation" in (20) cannot be evaluated. However, in this case, numerical analysis could be applied directly to equation (19). Even when we do know the probability density of $A(T)$, evaluation of the "expectation" in (20) will generally require use of numerical methods. Equation (20) has been shown to have a closed-form solution for three alternative distributions. First, Cox, Ingersoll, and Ross [12] solve (20) for the univariate square root process (i.e., $\sigma = c\sqrt{x}$ where c is deterministic). Dothan [16] solves (20) for the univariate geometric random walk (i.e., $\sigma = cx$). Finally, Vasicek [51] solves (20) for the univariate elastic random walk (i.e., $\sigma = c$). Cox, Ingersoll and Ross [12] and Richards [46] also solve for the two factor multivariate square root process where the two stochastic factors are independent. In the next section we will analyze (20) for the case of the n-factor multivariate elastic random walk, without assuming independence of the stochastic factors.

V. The Multivariate Elastic Random Walk Model

In Section II it was argued that the short-term interest rate might depend directly on economic factors such as the inflation rate and the real interest rate, and indirectly on many other economic factors. Hence, we assert that an adequate model for pricing default-free bonds will generally require multivariate modelling. To develop a tractable multivariate model, we introduce three new assumptions.

[12] Richard [46] derives the form of the solution for the case where only two stochastic factors are present. However, his derivation is done in matrix form so his result is directly extendable to the n factor case.

First, we assume that the price of a pure-discount, default-free bond depends on an arbitrary number of stochastic factors which follow a joint elastic random walk (as in equation 6). As noted in Section II, there is empirical evidence that the stochastic process of the short-term rate can be represented (or approximated) by an elastic random walk (as in equation 4). Of course, the elastic random walk assumption requires empirical verification for all of the underlying stochastic factors. The second new assumption is that the short-term rate can be represented by a linear combination of the same stochastic factors (as in equation 5). Finally, we assume the market price of risk of each stochastic factor (e.g., ϕ_i for $i = 1, \cdots, n$ as defined in equation 17) is non-stochastic. This implies that a deterministic model can adequately describe changes in the market price of risk over time. Of course, each of these three assumptions requires rigorous empirical verification. However, even if we are unable to produce strong empirical support for these assumptions, it is still possible that the resulting model might serve as an adequate approximation for bond pricing. The primary result of these three assumptions is a bond pricing model that is mathematically tractable and a model of the term structure that is consistent with traditional intuition.

Using the first two assumptions we can express the (instantaneous) short-term rate, $r(v)$, as follows:

$$\begin{aligned} r(v) &= w_0 + w'x(v) \\ &= w_0 + w'\left[\psi(v-t)x(t) + \int_t^v \psi(v-s)a\,ds + \int_t^v \psi(v-s)\sigma dz\,(s)\right] \end{aligned} \tag{21}$$

Equation (21) follows directly from equations (5) and (7). Next, replace $r(v)$ in the general bond price solution in equation (20) by $r(v)$ in (21).

$$P(t, T) = E_t e^{A(T)} \tag{22}$$

where

$$A(T) = -\int_t^T \left[w_0 + w'\psi(v-t)x(t) + w'\int_t^v \psi(v-s)a\,ds + w' \cdot \int_t^v \psi(v-s)\sigma dz\,(s)\right] dv - \int_t^T \frac{1}{2}\phi\sigma' \textstyle\sum^{-1} \phi\sigma\,dv - \int_t^T \phi\sigma' \textstyle\sum^{-1} \sigma dz\,(v)$$

By inspection, we see that the exponent $A(T)$ is normally distributed when the market prices of risk, ϕ, are non-stochastic.[13] Since $A(T)$ is normal, we can readily evaluate the expectation in equation (20).[14]

$$P(t, T) = \exp[E_t(A(T)) + \tfrac{1}{2}V_t(A(T))] \tag{22a}$$

[13] Normality of $A(T)$ also depends on the assumption of an elastic random walk and the assumption that the spot rate can be represented as a linear combination of the underlying stochastic factors.

[14] When $A(T)$ is not normally distributed, the expectation in (22) cannot be evaluated as in (22a). However, there are alternative means of evaluating (22) for non-normal distributions.

where

$$E_t(A(T))$$

$$= -\int_t^T \left[w_0 + w'\psi(v-t)x(t) + w' \int_t^v \psi(v-s)a\, ds + \frac{1}{2}\phi\sigma' \textstyle\sum^{-1} \phi\sigma \right] dv$$

$$V_t(A(T))$$

$$= \int_t^T \int_t^T \left[\int_t^{\min(v,v^*)} w'\psi(v-s) \textstyle\sum \psi(v^*-s)'w\, ds \right] dv\, dv^*$$

$$+ \int_t^T \phi\sigma' \textstyle\sum^{-1} \phi\sigma\, dv$$

$$+ 2\int_t^T \left[\int_t^v w'\psi(v-s)\phi\sigma\, ds \right] dv$$

To develop an explicit form for $P(t, T)$ requires only the evaluation of the integrals in (22a). However, it is far more instructive to consider an alternative method for evaluating $P(t, T)$.

Bond Risk

Through inspection of the differential equation for $P(t, T)$ in equation (12) it is clear that bond risk is intimately related to the gradient vector $\frac{\partial P(t, T)}{\partial x'} = \left[\frac{\partial P(t, T)}{\partial x_1} \cdots \frac{\partial P(t, T)}{\partial x_n} \right]$. We can calculate $\frac{\partial P(t, T)}{\partial x'}$ directly from equation (22a) and use the fact that $V_t(A(T))$ does not depend on $x(t)$:

$$\begin{aligned}\frac{\partial P(t, T)}{\partial x'} &= \frac{\partial \exp[E_t(A(T)) + \tfrac{1}{2}V_t(A(T))]}{\partial x'} \\ &= P(t, T)\frac{\partial E_t(A(T))}{\partial x'} \\ &= P(t, T)\left[-\int_t^T w'\psi(v-t)\, dv \right] \end{aligned} \tag{23}$$

It is convenient to normalize the gradient by expressing it in terms of "risk" per dollar of investment. Let $\nabla = \nabla(t, T)$ denote this *normalized gradient.*

$$\begin{aligned}\nabla(t, T)' &= \frac{1}{P(t, T)}\frac{\partial P(t, T)}{\partial x'} \\ &= -\int_t^T w'\psi(v-t)\, dv \end{aligned} \tag{24}$$

Since w and $\psi(v-t)$ are at most functions of time, so is the normalized gradient. When w and B are both constant over time (24) integrates to:

$$\nabla(t, T)' = -w'\Gamma(t, T) \tag{25}$$

where

$\Gamma(t, T) = \int_t^T \psi(v-t)\,dv = B^{-1}(\exp(B(T-t)) - I)$
B^{-1} = inverse of B and is assumed to exist

We will see that the normalized gradient vector $\nabla(t, T)$ and the matrix $\Gamma(t, T)$ play a central role in bond pricing. We will refer to $\nabla(t, T)$ as the *risk vector* at time t of a default-free bond promising a single payment at time T.[15] By inspection of (24) or (25), it is clear that bond risk is closely associated with $\psi(\cdot)$, which represents a fundamental solution matrix for the joint process of the underlying stochastic factors (see equation 7). In equation (25) the bond risk vector is expressed as a linear combination of the columns of $\Gamma(t, T)$ which in turn is a function of the fundamental matrix.

To give the risk vector $\nabla(t, T)$ a more traditional interpretation, consider the special case where $B = 0$ (i.e., random walk model) and where the bond depends on only the stochastic short-term rate (i.e., $n = 1$ and $w = 1$). In this case, $\nabla(t, T)$ reduces to $-(T-t)$, where $(T-t)$ is recognized to the *duration* at time t of a pure discount bond maturing at time T. If the short-term rate was to experience a small, instantaneous change of Δr, we would expect an instantaneous return on the bond to be $-(T-t)\Delta r$. Hence, duration is a valid measure of instantaneous bond risk in a one factor, random walk world. In the general and more realistic case where $B \neq 0$, the bond risk vector ∇ has an interpretation similar to that of the duration concept.[16] A small, instantaneous change of Δx in the underlying stochastic factors will produce an instantaneous bond return of $\nabla'\Delta x$.

It is interesting to observe the relation of the risk vector and bond maturity. When $B = 0$ (random walk case) and w is constant, $\Gamma(t, I) = I$, and so the risk vector is $-w'I(T-t)$. When w_i is positive (negative), the bond price is negatively (positively) related to changes in factor i. The absolute value of bond risk associated with factor i monotonically increases without bound as T increases. When $B \neq 0$, we must specify the form of B before the relation between risk and maturity can be determined. Consider the special case where B is diagonal with non-zero diagonal components B_i and zeros elsewhere. In this case the bond risk associated with factor i is $-w_i\left(\dfrac{\exp(B_i(T-t)) - 1}{B_i}\right)$. When w_i is positive (negative) the bond price is negatively (positively) correlated with factor i. When $B_i > 0$, the absolute value of the risk measure also increases without bound as maturity increases. When we make the reasonable assumption that $B_i < 0$, the absolute value asymptotically increases to the finite limit $|w_i/B_i|$. The more general case where B is nondiagonal is discussed in footnote 21. Finally, it should

[15] Cox, Ingersoll and Ross [11] call the risk vector "basis risk."

[16] The interpretation of the risk vector $\nabla(t, T)$ corresponding to the multivariate elastic random walk is similar to the interpretation of the duration concept in that both are defined by the expression $(1/P(t, T))(\partial P(t, T)/\partial x)$. Langetieg and Babcock [31] show that many applications involving duration can be extended to the elastic random walk model.

also be noted that while the absolute value of the risk measure is an increasing function of term, this property is not preserved for coupon bonds [12], [18], [23].

The Diffusion Process

Having determined the bond's risk vector, we can calculate the bond's equilibrium expected (instantaneous) return from 18:

$$
\begin{aligned}
\frac{\alpha_P}{P(t,T)} &\overset{e}{=} \left[r(t) + \left(\frac{\beta_P^1}{P(t,T)}\right)\phi_1 + \cdots + \left(\frac{\beta_P^n}{P(t,T)}\right)\phi_n \right] dt \\
&= \left[r(t) + \left(\frac{\frac{\partial P(t,T)}{\partial x_1(t)}}{P(t,T)}\right)\sigma_1\phi_1 + \cdots + \left(\frac{\frac{\partial P(t,T)}{\partial x_n(t)}}{P(t,T)}\right)\sigma_n\phi_n \right] dt \\
&= [r(t) + \nabla(t,T)'\sigma\phi]\, dt
\end{aligned}
\tag{26}
$$

where

$$
\nabla(t,T)' = \frac{1}{P(t,T)}\left[\frac{\partial P(t,T)}{\partial x_1(t)} \cdots \frac{\partial P(t,T)}{\partial x_n(t)}\right]
$$

We can think of $\nabla'\sigma\phi$ as an instantaneous term or risk premium. In the special case of constant coefficients and diagonal B with non-zero components, the term premium is simply: $-\sum_{i=1}^{n} w_i\left(\frac{\exp(B_i(T-t))-1}{B_i}\right)\sigma_i\phi_i$. According to the liquidity preference theory, we would expect that the term premium increases with bond maturity. This would be the case if w_i is positive and ϕ_i is negative for each factor x_i. However, if factor x_i is important for hedging against adverse changes in the investor's investment or consumption opportunity set, then ϕ_i could be positive and the term premium could be negative. Of course, a negative term premium is inconsistent with the liquidity preference theory. A rigorous argument concerning the sign of the term premium or the relation of the term premium to bond maturity cannot be made until w, B, and ϕ are specified.

Having determined the functional form of the bond's expected return, we can now fully describe the bond's diffusion process. Substitute (26) into (12) to get:

$$
\frac{dP(t,T)}{P(t,T)} = [r(t) + \nabla(t,T)'\sigma\phi]\, dt + \nabla(t,T)'\sigma dz \tag{27}
$$

By applying Ito's formula we find the solution to (27) is:[17]

$$
\begin{aligned}
\tilde{P}(s,T) = P(t,T)\exp\Bigg[\int_t^s r(v)\, dv \\
+ \int_t^s \left[\nabla(v,T)'\sigma\phi - \frac{1}{2}\nabla(v,T)' \textstyle\sum \nabla(v,T)\right] dv + \int_t^s \nabla(v,T)'\sigma dz\,(v)\Bigg]
\end{aligned}
\tag{28}
$$

[17] To solve (27), first find the diffusion process for $\ln(P(s))$ by using Ito's formula. The diffusion process for $\ln(P(s))$ can be directly integrated. Once $\ln(P(s))$ is determined, we recover $P(s)$ by the relation $P(s) = \exp[\ln(P(s))]$.

From equation (21) we can separate $r(v)$ into a deterministic part and a stochastic part. Substituting for $r(v)$ and simplifying produces a particularly useful form of equation (28):[18]

$$\tilde{P}(s, T) = P(t, T)\exp\left[\int_t^s E_t(r(v))\,dv + \int_t^s L(v, T)\,dv + \nabla(s, T)'[\tilde{x}(s) - E_t(x(s))]\right] \qquad t \leqq s \leqq T \quad (29)$$

where

$$E_t(r(v)) = w_0 + w'E_t(x(v)) = w_0 + w'\left[\psi(v - t)x(t) + \int_t^v \psi(v - u)a\,du\right]$$

$$L(v, t) = \nabla(v, T)'\sigma\phi - \tfrac{1}{2}\nabla(v, T)' \textstyle\sum \nabla(v, T)$$

Equation (29) is particularly useful for several reasons. First, it can be used to determine the value of the bond at time t by setting $s = T$ and $P(T, T) = 1$. Second, we will use (29) to directly calculate holding period risk and return. Note that $\tilde{P}(s, T)$ depends on the uncertain value of $\tilde{x}(s)$. Since $\tilde{x}(s)$ has a normal distribution, $\tilde{P}(s, T)$ is lognormally distributed and the expected value and variance of $\tilde{P}(s, T)$ are easily calculated. Finally, (29) has a form that is simple and intuitive. The change in the bond's value from time t to time s can be decomposed into three sources: (1) the expected spot rate from t to s, $\int_t^s E_t(r(v))\,dv$, (2) the term premiums from t to s, $\int_t^s L(v, T)\,dv$, and (3) uncertainty due to unanticipated changes in the underlying stochastic factors, $\nabla(s, T)'[\tilde{x}(s) - E_t(x(s))]$.

Bond Prices and the Term Structure Equation

The current bond price is derived by setting $s = T$ in (29) and solving for $P(t, T)$. Note that when $s = T$, the "uncertain term" drops out since $\nabla(T, T) = 0$.

$$P(t, T) = \exp\left[-\int_t^T E_t(r(v))\,dv - \int_t^T L(v, T)\,dv\right] \qquad (30)$$

[18] To derive equation (29) from equation (28) note that the stochastic part of (29) can be reformulated as:

$$\begin{aligned}\int_t^s [\tilde{r}(v) - E_t(\tilde{r}(v))]\,dv + \int_t^s \nabla(v, T)'\sigma dz\,(v) &= -\int_t^s \nabla(v, s)'\sigma dz\,(v) + \int_t^s \nabla(v, T)'\sigma dz\,(v) \\ &= \nabla(s, T)' \int_t^s \exp(B(s - v))\sigma dz\,(v) \\ &= \nabla(s, T)'\,[\tilde{x}(s) - E_t(\tilde{x}(s))]\end{aligned}$$

Equation (29) is valid only for the multivariate elastic random walk model, but (28) is valid for the general bond model in equation (20). However, the form of ∇, the risk vector, will be different for alternative stochastic processes.

where $L(v, T)$ and $E_t(r(v))$ are defined in (29). From equation (1), the term structure equation becomes:

$$R(t, T) = \frac{1}{T - t}\left[\int_t^T E_t(r(v))\,dv + \int_t^T L(v, T)\,dv\right] \tag{31}$$

The term structure is expressed in a form that is consistent with the traditional notion of a term structure determined by expectations of future short term rates *plus* a liquidity premium. As Vasicek [51] observes, the label "liquidity" premium is clearly a misnomer since investor liquidity plays no role in the derivation of (31).

As we can see from the definition of $L(v, T)$ in equation (29), $L(v, T)$ is deterministic since $\nabla(t, T)$ is deterministic. Therefore, the current level of the underlying stochastic factors impacts the bond price only through the expectations of future spot rates. Hence, we make the interesting observation that: *the bond price (and the term structure) will depend on stochastic factor x_i if and only if the spot rate (or the instantaneous drift of factors affecting the spot rate) depends on factor x_i.*[19]

We next examine the question of when the pure expectation hypothesis holds. This would require that $L(v, T) = 0$. From (29) $L(v, T)$ is composed of two parts: $\nabla'\sigma\phi$, which will disappear under risk neutrality, and $-\frac{1}{2}\nabla' \sum \nabla$, which will disappear when the spot rate is non-stochastic. Hence, we conclude, as Richard [46], that the pure expectations hypothesis will hold only in a world of certainty. Even when investors are risk neutral there will still be a non-zero term premium in an uncertain world.

The Special Case of Constant Coefficients and Distinct Eigenvectors

While the bond pricing equation in (30) is in an intuitive form, the explicit calculation of bond prices requires further integration. To illustrate the integration of $-\int_t^T E_t(r(v))\,dv$, consider the special case where the coefficients w_0, w, a and B are constant over time and B^{-1} exists.[20]

$$-\int_t^T E_t(r(v))\,dv = -(T - t)(w_0 - w'B^{-1}a) + \nabla(t, T)'[x(t) + B^{-1}a] \tag{32}$$

[19] To prove this claim, we require the assumption of a joint elastic random walk, the assumption that the short-term rate is a linear combination of certain stochastic factors, and the assumption that the market prices of risk (i.e., ϕ_i, $i = 1, \cdots, n$) are nonstochastic. Hence this claim is dependent on this set of assumptions. It is clear that bond prices will depend on all stochastic factors related to the expected short-term rate through the term $Z = w'\psi(v - t)x(t) = w' \exp(B(v - t))x(t)$ (see equation 29 and 30). When B is diagonal, Z will depend on $x_i(t)$ if and only if $w_i \neq 0$. When B is nondiagonal, Z could depend on x_i even if $w_i = 0$. This will occur if x_i effects the "drift" of another factor x_j where $w_j \neq 0$.

[20] We will generally assume constant coefficients throughout this paper. In the case where the coefficients change deterministically, integration is more difficult and may require numerical methods. However, the case of time dependent coefficients offers little additional insight into the nature of bond prices, so we focus on the constant coefficient case. Matrix B^{-1} will exist providing B is of full rank. Matrix B may be less than full rank if one of the stochastic factors follows a random walk rather than an elastic random walk. Integration in this case is still possible but at the expense of the simpler matrix notation used throughout this paper.

Loosely speaking, we can interpret $-\int_t^T E_t(r(v))\,dv$ as the contribution of the "pure expectations theory" to the bond's price (or the term structure). We see that this integral is a function of time left to maturity $(T-t)$, the coefficients w_0, w, B, and a, the risk vector of $\nabla(t, T)$, and the current level of the process $x(t)$.

To integrate the term premium, $-\int_t^T L(v, T)\,dv$, we again focus on the case of constant coefficients. We also focus on the case where the eigenvalues of matrix B are distinct.[21] In this case, the integral is evaluated as:[22, 23]

[21] The eigenvectors of an $(n \times n)$ matrix B are defined by the relationship:

$$Bc_i = \Lambda_i c_i \qquad i = 1, \cdots, n$$

where Λ_i denotes the eigenvalue associated with eigenvector c_i. We treat the case where the eigenvectors are linearly independent (or distinct). In this case, the matrix B can be replaced by $C\Lambda C^{-1}$ where C is an $(n \times n)$ matrix with columns equal to the eigenvectors of B and Λ is a $(n \times n)$ diagonal matrix with diagonal elements equal to the corresponding eigenvalues. We can also express $\psi(T-t) = \exp[B(T-t)]$ as $C\exp[\Lambda(T-t)]C^{-1}$. Hence the bond risk vector $-w'\Gamma(t, T)$, or $-w'B^{-1}[\exp(B(T-t)) - I]$, can be expressed as $-w'C(\Lambda^{-1}[\exp(\Lambda(T-t)) - I])C^{-1}$. When all the eigenvalues are ***negative***, the bond risk vector asymptotically approaches the finite limit $w'C\Lambda^{-1}C^{-1} = wB^{-1}$ as T increases. When one Λ_i is positive, then the absolute value of one or more of the components of the risk vector will tend to infinity as T increases. This will lead to an undesirable behavior of long-term yields and bond prices. When the eigenvalues are not distinct, then the use of Jordan canonical forms will be necessary, but this case will not be investigated here.

[22] The result in equation (33) can be obtained by using the general result in equation (39b) and setting $s = T$. A full derivation of the (33) or (39b) is too lengthy to present here but will be made available to the interested reader. Note that the second integral in (39b) can also be written as:

$$-\frac{1}{2}\int_t^s \nabla(v, T)' \textstyle\sum \nabla(v, T)\,dv = -\frac{1}{2}\,wB^{-1}\left[\int_t^s [\psi(T-v) - I] \textstyle\sum [\psi(T-v)' - I]\,dv\right](B^{-1})'w$$

$$= \frac{1}{2}\,w'B^{-1}[\psi(T-s)\Omega\psi(T-s)' - \psi(T-t)\Omega\psi(T-t)'](B^{-1})'w$$

$$-\frac{1}{2}\,w'B^{-1}B^{-1}[\psi(T-s) - \psi(T-t)] \textstyle\sum (B^{-1})'w$$

$$-\frac{1}{2}\,w'B^{-1} \textstyle\sum [\psi(T-s)' - \psi(T-t)'](B^{-1})'(B^{-1})'w$$

$$-\frac{1}{2}\,w'B^{-1} \textstyle\sum (B^{-1})'w\cdot(s-t)$$

[23] Ω is defined by the following three transformation to the original covariance matrix $\sum$, which is defined in equation (10). Note that when the eigenvalues are negative and $\sum$ is positive definite, then Ω will be negative definite due to the second transformation.

Transformation 1: $\sum^* = C^{-1} \sum (C^{-1})'$

where

$C = (n \times n)$ matrix with columns equal to the eigenvectors of B, see equation (6), which by assumption are distinct (see footnote 21).

Transformation 2: $\sum^{**} = (n \times n)$ matrix with components $(\sigma_{ij}^{**}) = \dfrac{(\sigma_{ij}^*)}{\Lambda_i + \Lambda_j}$

where

$$(\sigma_{ij}^*) = \text{component } (i, j) \text{ of matrix } \textstyle\sum^*$$

$$\Lambda_i = i\text{th eigenvalue of } B$$

Transformation 3: $\Omega = C \sum^{**} C'$

$$-\int_t^T L(v, T)\, dv = w'B^{-1}[\Gamma(t, T) - I\cdot(T-t)]\,\sigma\phi$$

$$+ \tfrac{1}{2} w'\Gamma(t, T)\Omega\Gamma(t, T)'w$$

$$- \tfrac{1}{2} w'[B^{-1}\Gamma(t, T)\Omega + \Omega\Gamma(t, T)'(B^{-1})']w$$

$$+ \tfrac{1}{2}\, w'B^{-1} \textstyle\sum (B^{-1})'w\cdot(T-t) \tag{33}$$

where

$I = (n \times n)$ identity matrix
$\Gamma(t, T) = B^{-1}[\exp(B(T-t)) - I]$, see equation (25)
$\Gamma' =$ transpose of Γ
$\Omega =$ a transformed covariance matrix[21]

A final special case is the *fully separable* elastic random walk. In this case both B and $\sum$ are diagonal. When B and $\sum$ are diagonal then each stochastic factor (x_i) follows a fully independent process. The term structure equation in (31) decomposes into n term structures corresponding to n stochastic factors:

$$R(t, T) = w_0 + \textstyle\sum_{i=1}^{n} R_i(t, T) \tag{34}$$

where

$$R_i(t, T) = w_i\left[\frac{\phi_i\sigma_i - a_i}{B_i} - \frac{1}{2}w_i\frac{\sigma_i^2}{B_i^2}\right]$$

$$+ \frac{1}{(T-t)}w_i\left[\frac{\exp(B_i(T-t)) - 1}{B_i}\right]$$

$$\cdot\left[x_i(t) - \left(\frac{\phi_i\sigma_i - a_i}{B_i} - \frac{1}{2}w_i\frac{\sigma_i^2}{B_i^2}\right)\right]$$

$$-\frac{1}{4}\frac{1}{(T-t)}w_i^2\frac{\sigma_i^2}{B_i}\left(\frac{\exp(B_i(T-t)) - 1}{B_i}\right)^2$$

$B_i =$ diagonal component i of diagonal matrix B

$\sigma_i^2 =$ diagonal component i if diagonal matrix $\sum$

Setting $n = 1$, $w_0 = 0$, and $w_1 = 1$, it is easily verified that $R_1(t, T)$ reduces to Vasicek's [51] term structure equation for the univariate elastic random walk. Vasicek shows that the relation between the bond's term T and yield $R_1(t, T)$ may be monotonically increasing, monotonically decreasing, or humped, depending on the level of stochastic factor x_1. In the multivariate case, we might find some $R_i(t, T)$ increasing in T while others might be decreasing or humped in T. *Hence, in the multivariate case, $R(t, T)$ can be monotonically increasing, decreasing or subject to multiple humps*! The coefficients must be specified before a rigorous analysis of the shape of the term structure can be done.

Whatever the shape of the term structure, in the reasonable case where each B_i is negative, the asymptotic limit of the term structure as the term approached infinity is:

$$R(\infty) = w_0 + \sum_{i=1}^{n} w_i \left[\frac{\phi_i \sigma_i - a_i}{B_i} - \frac{1}{2} w_i \frac{\sigma_i^2}{B_i^2} \right] \qquad \text{(separable case)} \tag{35}$$

Returning to the more general case, where B is nondiagonal (and the eigenvalues of B are distinct and negative) the asymptotic finite limit of the term structure is:

$$R(\infty) = w_0 - w'B^{-1}a + w'B^{-1}\sigma\phi - \tfrac{1}{2}w'B^{-1}\textstyle\sum (B^{-1})'w \tag{35'}$$

Holding Period Returns

The investor with a finite investment horizon (or holding period) is interested in the characteristics of the bond's holding period return. The holding period return from time t to time s of a bond paying \$1 at time $T \geqq s$ is defined as:

$$H\tilde{P}R(t, s, T) + 1 = \frac{\tilde{P}(s, T)}{P(t, T)} \qquad t < s \leqq T \tag{36}$$

To calculate the expected HPR and variance of the HPR we again apply equation (29). Since $\tilde{P}(s, T)$ is lognormally distributed we find:

$$E_t(HPR + 1) = \exp\left[\int_t^s E_t(r(v))\, dv + \int_t^s L(v, T)\, dv + \tfrac{1}{2} V_t \left[\nabla(s, T)'\tilde{x}(s)\right] \right] \tag{37}$$

$$V_t(HPR + 1) = [E_t(HPR) + 1]^2 \cdot [\exp(V_t\, [\nabla(s, T)'\tilde{x}(s)]) - 1] \tag{38}$$

From (37) we see that the expected return from time t to time s is determined by: 1) current expectations of the spot rates from t to s, 2) term premiums from t to s, and 3) the variance of $\tilde{x}(s)$ weighted by the bond's risk vector at time s, $\nabla(s, T)$. To evaluate these three terms we again assume constant coefficients and focus on the special case where B has distinct eigenvalves and B^{-1} exists. We find:

$$\int_t^s E_t(r(v))\, dv = (s - t)[w_0 - w'B^{-1}a] + w'B^{-1}[\exp(B(s - t)) - I][x(t) + B^{-1}a] \tag{39a}$$

$$\int_t^s L(v, T)\, dv = \int_t^s \nabla(v, T)'\sigma\phi\, dv - \frac{1}{2}\int_t^s \nabla(v, T)'\textstyle\sum\nabla(v, T)\, dv \tag{39b}$$

where

$$\int_t^s \nabla(v, T)'\sigma\phi\, dv = w'B^{-1}[\Gamma(s, T) - \Gamma(t, T) + I\cdot(s - t)]\sigma\phi$$

$$-\frac{1}{2}\int_t^s \nabla(v, T)'\textstyle\sum\nabla(v, T)\, dv = \tfrac{1}{2}\, w'[\Gamma(s, T)\Omega\Gamma(s, T)' - \Gamma(t, T)\Omega\Gamma(t, T)']w$$

$$- \tfrac{1}{2}\, w'B^{-1}[\Gamma(s, T) - \Gamma(t, T)]\Omega w$$

$$- \tfrac{1}{2}\, w'\Omega[\Gamma(s, T)' - \Gamma(t, T)'](B^{-1})'w$$

$$- \tfrac{1}{2}\, w'B^{-1}\textstyle\sum(B^{-1})'w \cdot (s - t)$$

$$\begin{aligned} \tfrac{1}{2}\, V_t[\nabla(s, T)'\tilde{x}(s)] &= \tfrac{1}{2}\nabla(s, T)'cov_t(\tilde{x}(s), \tilde{x}(s))\nabla(s, T) \\ &= \tfrac{1}{2}\, \nabla(s, T)'[\exp(B(s - t))\Omega\exp(B(s - t))' \\ &\quad - \Omega]\nabla(s, T) \end{aligned} \tag{39c}$$

where

$$\nabla(s, T)' = -w'\Gamma(s, T)$$

$$\Gamma(s, T) = B^{-1}(\exp(B(T - s)) - I)$$

Note that (39a) is the same for all bonds with maturity $T \geq s$. Note that the *HPR* is related to both the risk vectors at time t, $\nabla(t, T)$, and time s, $\nabla(s, T)$. As s approaches T, the variance term in 39c goes to zero and the holding period return becomes with certainty $\dfrac{1}{P(t, T)} - 1$. Holding s constant, as T approaches infinity, $\Gamma(t, T)$ and $\Gamma(s, T)$ tend to $-B^{-1}$ (when the eigenvalues of B are negative, see footnote 21), and (39b) and (39c) tend to the finite limits:

$$\int_t^s L(v, \infty)\, dv = [w'B^{-1}\sigma\phi - \frac{1}{2}\, w'B^{-1}\textstyle\sum(B^{-1})'w] \cdot (s - t) \tag{39b'}$$

$$\tfrac{1}{2}\, V_t[\nabla(t, \infty)'\tilde{x}(s)] = \tfrac{1}{2}\, w'B^{-1}[\, \exp(B(s - t))\Omega\exp(B(s - t))' - \Omega](B^{-1})'w \tag{39c'}$$

When $s = t$, we focus on the bond's instantaneous return $\dfrac{dP(t, T)}{P(t, T)}$ with:

$$E_t\left[\frac{dP(t, T)}{P(t, T)}\right] = [r(t) + \nabla(t, T)'\sigma\phi]\, dt = [r(t) - w'\Gamma(t, T)\sigma\phi]\, dt \tag{40a}$$

$$\begin{aligned} V_t\left[\frac{dP(t, T)}{P(t, T)}\right] &= V_t[\nabla(t, T)'\sigma dz]\, dt \\ &= [\nabla(t, T)'\textstyle\sum\nabla(t, T)]\, dt = [w'\Gamma(t, T)\textstyle\sum\Gamma(t, T)'w]\, dt \end{aligned} \tag{40b}$$

As T approaches t, $\Gamma(t, T)$ goes to zero, and the bond's instantaneous return approaches the instantaneous riskless rate, $r(t)$, with certainty (i.e., $V_t(\cdot) = 0$). When the eigenvalues of B are negative, as T approaches infinity, $\Gamma(t, T)$ tends to $-B^{-1}$ and the expected value and variance tend to finite limits:

$$E_t\left[\frac{dP(t, \infty)}{P(t, \infty)}\right] = [r(t) + w'B^{-1}\sigma\phi]\, dt \tag{40a'}$$

$$V_t\left[\frac{dP(t, \infty)}{P(t, \infty)}\right] = [w'B^{-1}\textstyle\sum(B^{-1})'w]\, dt \tag{40b'}$$

Forward Rates

Forward rates are of interest since the forward rate represents the equilibrium yield on a futures contract (see Roll [47]). The (instantaneous) forward rate (defined in footnote 1) is found to be:

$$F(t, T) = -\frac{\partial \ln[P(t, T)]}{\partial T}$$

$$= -E_t(r(T)) - \int_t^T w'\psi(v - t)\sigma\phi\, dv$$

$$-\frac{1}{2}\int_t^T w'[\psi(T - v)\textstyle\sum\Gamma(v, T)' + \Gamma(v, T)\sum\psi(T - v)']w\, dv \quad (41)$$

In the case where coefficients are constant and B has distinct eigenvectors, (41) reduces to:[24]

$$F(t, T) = E_t(r(T)) + L(t, T)$$

where

$$L(t, T) = \nabla(t, T)'\sigma\phi - \tfrac{1}{2}\, \nabla(t, T)'\textstyle\sum\nabla(t, T) \quad (42)$$

Hence with constant coefficients the forward rate is equal to the current expectation of the spot rate at time T plus the current (instantaneous) liquidity premium on a bond maturing at time T.

Application to Pricing Options and Risky Bonds

The original Black-Scholes option pricing model [3] is based on a nonstochastic term structure. Merton [41] extends the Black-Scholes model to accommodate a stochastic term structure in the case where the price of a pure-discount, default-free bond follows a Ito process of the form:

$$dP(t, T) = \alpha(x, t)\, dt + P(t, T)\sigma_p(t)\, dz_p \quad (43)$$

While the drift coefficient is of a general specification, the diffusion coefficient *must* be specified such that it is equal to a deterministic function times the current bond price. The multivariate model based on the elastic random walk in equation (27) is completely consistent with the process in equation (43) and should be regarded as only one of many possible consistent specifications. Merton's valuation formula for the european call option, W, written on an optioned asset, A, with exercise price at time T of E, and with a term structure satisfying (43) is as follows:

[24] With constant coefficients it seems plausible that equation (41) will reduce to equation (42) whether or not B possesses distinct eigenvalues, but (41) will not reduce to (42) if coefficients are not constant over time.

$$W(t) = W(A(t), P(t, T), t; T, E, \sigma^2)$$

$$= A(t)\cdot N\left[\frac{\ln\dfrac{A(t)}{E\cdot P(t, T)} + \dfrac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right]$$

$$- E\cdot P(t, T)N\left[\frac{\ln\dfrac{A(t)}{E\cdot P(t, T)} - \dfrac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right] \quad t \leqq T \qquad (44)$$

where

$A(t)$ = value at time t of optional asset

$P(t, T)$ = value at time t of a pure discount default free bond paying \$1 at time T

E = exercise price

N = standard normal cumulative distribution

ln = natural logarithm

$(T-t)$ = time left until maturity

$$\sigma^2 = \frac{1}{(T-t)}\int_t^T [\sigma_A^2 - 2\sigma_{AP}^2(v) + \sigma_P^2(v)]\,dv$$

$$\sigma_A^2 = V\left[\frac{dA}{A}\right]\Big/ dt \text{ where } dA = \alpha_A(x, t)\,dt + \sigma_A A\,dz_A$$

$$\sigma_P^2(v) = V\left[\frac{dP(v, T)}{P(v, T)}\right]\Big/ dt, \text{ see equation (43)}$$

$$\sigma_{AP}(v) = \text{cov}\left[\frac{dA}{A}, \frac{dP}{P}\right]\Big/ dt$$

In (44) σ_p^2 and σ_{AP} are assumed to be modeled as deterministic functions of time while σ_A^2 is assumed constant. Since W does not depend on α_p or α_A, these terms can be specified as continuous functions in $x(t)$ and t where $x(t)$ represents the vector of underlying economic factors. Of course, $A(t)$ and $P(t, T)$ can be regarded as functions of $x(t)$ too.

Next $dP(t, T)$ from equation (27) will be introduced in Merton's call pricing model. First, we note that equation (27) can be re-written in a form consistent with equation (43) by defining

$$\alpha_p(x, t) \equiv P(t, T)[r(t) + \nabla(t, T)'\,\sigma\phi] \qquad (45a)$$

$$\text{and} \qquad \sigma_p(t)dz_p \equiv \nabla(t, T)'\sigma dz \qquad (45b)$$

Note that the random variables $\sigma_p(t)dz_p$ and $\nabla(t, T)'\sigma dz$ are stochastically equivalent.

It follows from (45b) that:

$$\sigma_p^2(t)\,dt = V_t\left[\frac{dP}{P}\right] = E_t[(\sigma_p(t)dz_p)^2] = E_t[(\nabla(t, T)'\sigma dz)^2]$$

$$= \nabla(t, T)'\textstyle\sum\nabla(t, T)\,dt$$

$$\sigma_{AP}(t)\,dt = E_t[\sigma_A dz_A \cdot \sigma_p dz_p] = [\sigma_A dz_A \cdot \nabla(t, T)'\sigma dz] = \nabla(t, T)'\rho\,dt$$

where

$$\rho' = [E(\sigma_A\sigma_1 dz_A dz_1)\; E(\sigma_A\sigma_2 dz_A dz_2)\cdots E(\sigma_A\sigma_n dz_A dz_n)] \tag{46}$$

Using (46), σ^2 in equation (44) becomes:

$$\sigma^2 = \frac{1}{(T-t)}\int_t^T [\sigma_A^2 - 2\nabla(v, T)'\rho + \nabla(v, T)'\textstyle\sum\nabla(v, T)]\,dv \tag{47}$$

Assuming constant coefficients (e.g., B, w, $\sum$, ρ and σ_A^2) and assuming B has distinct eigenvalues and B^{-1} exists, (47) integrates to:

$$\begin{aligned}\sigma^2 = \sigma_A^2 &- \frac{2}{T-t}\cdot[w'B^{-1}[\Gamma(t, T) - I\cdot(T-t)]\rho]\\ &+ \frac{1}{T-t}\cdot w'\Big[\Gamma(t, T)\Omega\Gamma(t, T)' - B^{-1}\Gamma(t, T)\Omega\\ &\qquad - \Omega\Gamma(t, T)'(B^{-1})' + B^{-1}\textstyle\sum(B^{-1})'\cdot(T-t)\Big]w\end{aligned} \tag{48}$$

Black and Scholes [3] and Merton [41] argue that the common stock of the firm can be viewed as a call option. In certain special cases equation (43) can be used to value common stock.[25] Let W represent the value of the common stock, A the value of the firm's assets, and F the contractual and certain *face value* at time T of the corporation's pure-discount (but risky) debt. In a no-tax world, the value of the firm's assets must equal the combined values of the firm's common stock and debt. Denoting the *value* of the firm's debt as D, it follows that:

$$\begin{aligned}D(t) &= D(A(t), P(t, T), t; T, F, \sigma^2) = A(t) - W(t)\\ &= A(t)N\left[-\frac{\ln\dfrac{A(t)}{F\cdot P(t, T)} + \dfrac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right]\\ &\quad + F\cdot P(t, T)N\left[\frac{\ln\dfrac{A(t)}{F\cdot P(t, T)} - \dfrac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right]\end{aligned} \tag{49}$$

[25] Equation (43) will provide a valid valuation of common stock when: (1) the firm terminates with certainty at time T so that bondholders cannot declare bankruptcy before T; (2) there are no dividends to stockholders; (3) all bonds are pure discount with a face value at time T of E; (4) the stock is a function of at most the value of the assets, the value of a default-free pure discount bond maturing at T, and time; (5) capital markets are sufficiently perfect to permit a continuous no arbitrage equilibrium to obtain; and (6) the stochastic process of both the value of firm's assets and the default-free bond can be represented by an Ito process with a diffusion coefficient that is equal to the level of the process times a deterministic function.

While equation (49) can be used to value risky, pure discount corporate bonds with a certain maturity date, the typical corporate bond is often more complex. The life of many bonds, and their coupon streams, is not certain due to bankruptcy, callability, or convertibility. Furthermore, tax effects (e.g., ordinary income, preferential treatment of capital gains, tax deferral) must also be introduced.

Since the coefficients of the bond model are left in an unspecified form, application of the model for either bond pricing or option pricing requires further specification of the macroeconomic system of stochastic factors driving bond prices. In particular, application requires:

1) estimation of the coefficients of the macrosystem's stochastic process (e.g., the coefficients a, B, and $\sum$)
2) identification of economic factors related to the spot rate (e.g., the linear weighing vector w, see footnote 4)
3) estimation of the market prices of risk, ϕ.

This estimation stage also serves as a check on the reasonableness of the assumptions of the model. Once the coefficients are specified, the bond pricing model itself can be empirically tested. The pricing model also has many directly testable implications (e.g., lognormally distributed bond prices, deterministic liquidity premiums, dependence on only the economic factors related directly or indirectly to the spot rate).

IV. Summary

Our objective was to develop a model of the term structure that is rich enough to accommodate a large number of possible macroeconomic relationships. Use of the model requires that the spot rate can be adequately represented as a linear function (of an arbitrary number) of economic factors that follow a joint elastic random walk. Also assuming nonstochastic market prices of risk, an extremely simple, yet intuitive model of the term structure is obtained. As traditional theory suggests, the term structure (even in the multivariate case) is a simple composite of expected spot rates plus a term premium. Under the above assumptions, the term premium is a simple deterministic function of the bond's risk vector, where the risk vector has an interpretation similar to that of the traditional duration concept. The risk vector is shown to play an important role in the derivation of equilibrium bond prices and in the functional form of the equilibrium term structure, bond holding period returns, forward rates, option pricing, and the pricing of a special form of risky corporate debt.

REFERENCES

1. Arnold, Ludwig, *Stochastic Differential Equations*, John Wiley & Sons, 1974.
2. Bierwag, G. O. and M. A. Grove, "A Model of the Term Structure of Interest Rates," *Review of Economics and Statistics*, February 1967, pp. 50-62.
3. Black, Fisher and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, v. 81, 1973, pp. 637-654.
4. Brennan, Michael J. and Eduardo S. Schwartz, "Savings Bonds, Retractable Bonds, and Callable Bonds," *Journal of Financial Economics*, 1977, pp. 67-88.

5. Brennan, Michael J. and Eduardo S. Schwartz, "A Continuous Time Approach to the Pricing of Bonds," Working Paper No. 476, Faculty of Commerce and Business Administration, University of British Columbia, 1977.
6. Brick, John R. and Howard E. Thompson, "Time Series Analysis of Interest Rates: Some Additional Evidence," *Journal of Finance*, March 1978, pp. 93–104.
7. Buse, A., "Interest Rates, the Meiselman Model and Random Numbers," *Journal of Political Economy*, February 1967, pp. 49–62.
8. Buse, A., "Expectations, Prices, Coupons, and Yields," *Journal of Finance*, September 1970, pp. 807–818.
9. Cagan, Phillip, "A Study of Liquidity Premiums on Federal and Municipal Securities," in *Essays on Interest Rates* edited by Jack M. Guttentag and Phillip Cagan, Columbia University Press, 1969.
10. Carr, J. L., P. J. Halpern, and J. S. McCallum, "Correcting the Yield Curve: A Re-Interpretation of the Duration Problem," *Journal of Finance*, September 1974, pp. 1287–1294.
11. Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, "Duration and the Measurement of Basis Risk," *Journal of Business*, January 1979, pp. 51–62.
12. Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, "A Theory of the Term Structure of Interest Rates," Working Paper #19, Graduate School of Business, Stanford University.
13. Crane, Dwight B., "A Stochastic Programming Model for Commercial Bank Portfolio Management," *Journal of Financial and Quantitative Analysis*, January 1971, pp. 951–976.
14. De Leeuw, Frank, "A Model of Financial Behavior, in the *Brookings Quarterly Econometric Model of the United States Economy*, edited by J. S. Duesenberry et al., Rand McNally, 1965, pp. 465–530.
15. Dobson, Steven W., Richard C. Sutch and David E. Vanderford, "An Evaluation of Alternative Empirical Models of the Term Structure of Interest Rates," *Journal of Finance*, September 1976, pp. 1035–1065.
16. Dothan, L. V., "On the Term Structure of Interest Rates," *Journal of Financial Economics*, No. 6, 1978, pp. 59–69.
17. Fama, Eugene F., "Forward Rates as Predictors of Future Spot Rates," *Journal of Financial Economics*, 1976, pp. 361–377.
18. Fisher, L. and R. Weil, "Coping with Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business*, October 1971, pp. 408–432.
19. Fisher, Stanley, "The Demand for Index Bonds," *Journal of Political Economy*, June 1975, pp. 509–534.
20. Gihman, I. I. and A. V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, 1972.
21. Grant, J. A., "Meiselman on the Structure of Interest Rates: A British Test," *Economica*, February 1964, pp. 51–71.
22. Grant, J. A., "Reply," *Economica*, November 1964, pp. 419–422.
23. Haugen, Robert A. and Dean W. Wichern, "The Elasticity of Financial Assets," *Journal of Finance*, September, 1974, pp. 1229–1240.
24. Hempel, George H. and Jess B. Yamitz, "A Risk-Return Approach to the Selection of Optimal Government Bond Portfolios," Working Paper, Washington University, July 1974.
25. Hicks, John R., *Value and Capital*, Oxford Press, 1946.
26. Hoel, Paul G., Sidney C. Port and Charles J. Store, *Introduction to Stochastic Processes*, Houghton Mifflin, 1972.
27. Hopewell, M. H. and G. C. Kaufman, "Bond Price Volatility and Term to Maturity: A Generalized Respecification," *American Economic Review*, September 1973, pp. 749–753.
28. Ingersoll, Jonathan E., Jeffrey Skelton, and Roman L. Weil, "Duration Forty Years Later," *Journal of Financial and Quantitative Analysis*, November 1978, pp. 627–650.
29. Keynes, John Maynard, *A Treatise on Money*, Harcourt, Brace and Company, 1930.
30. Kessel, Reuben A., "The Cyclical Behavior of the Term Structure of Interest Rates," Occasional Paper 91, *National Bureau of Economic Research*, 1965.
31. Langetieg, Terence C. and Guilford C. Babcock, "Applications of Duration in the Selection of Bonds," Working Paper, Graduate School of Business, University of Southern California, 1978.
32. Lintner, John, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economic and Statistics*, February 1965, pp. 13–37.

33. Long, John B., Jr., "Stock Prices, Inflation, and the Term Structure of Interest Rates," *Journal of Finance Economics*, 1974.
34. Malkiel, Burton G., "Expectations, Bond Prices, and the Term Structure of Interest Rates," *Quarterly Journal of Economics*, May 1962, pp. 197–218.
35. Malliel, Burton G. and Edward J. Kane, "Expectations and Interest Rates: A Cross-Sectional Test of the Error-Learning Hypothesis," *Journal of Political Economy*, July/August 1969, pp. 453–470.
36. Markowitz, H., *Portfolio Selection: Efficient Diversification of Investment*, John Wiley, 1959.
37. McCulloch, J. Huston, "The Tax Adjusted Yield Curve," *Journal of Finance*, June 1975, pp. 811–830.
38. McCulloch, J. Huston, "An Estimate of the Liquidity Premium," *Journal of Political Economy*, 1975, pp. 95–118.
39. McEnally, Richard W., "Duration as a Practical Tool for Bond Management," *Journal of Portfolio Management*, Summer 1977, pp. 53–57.
40. Meiselman, David, *The Term Structure of Interest Rates*, Prentice-Hall, 1962.
41. Merton, Robert C., "The Rational Theory of Options Pricing," *Bell Journal of Economics and Management Science*, Volume 4, 1973, pp. 141–183.
42. Merton, Robert C., "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, v. 29, 1974, pp. 449–470.
43. Modigliani, Franco and Richard M. Sutch, "Innovations in Interest Rate Policy," *American Economic Review Papers and Proceedings*, May 1966, pp. 178–197.
44. Muth, J. F., "Optimal Properties of Exponentially Weighted Forecasts," *Journal of the American Statistical Association*, June 1960, pp. 299–306.
45. Nelson, Charles R., *The Term Structure of Interest Rates*, Basic Books, 1972.
46. Richard, Scott F., "An Arbitrage Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, Number 6, 1978, pp. 33–57.
47. Roll, Richard, "Investment Diversification and Bond Maturity," *Journal of Finance*, March 1971, pp. 51–66.
48. Roll, Richard, *The Behavior of Interest Rates: The Application of the Efficient Market Model to U.S. Treasury Bills*, Basic Books.
49. Ross, Stephen A., "Return, Risk, and Arbitrage," appearing in *Risk and Return in Finance*, edited by I. Friend and J. L. Bicksler, Ballinger Publishing, 1977.
50. Sharp, William R., "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance*, September 1974, pp. 425–442.
51. Vasicek, Oldrich, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, Number 5, 1977, pp. 177–188.
52. Watson, Gene, "Tests of Maturity Structure of Commercial Bank Government Security Portfolios: A Simulation Approach," *Journal of Bank Research*, Spring 1972, pp. 34–46.