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Biased Estimators and Unstable Betas

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STUDIES [1, 2, 3, 5, 8, 10] ESTIMATING AND ANALYZING the stability of security betas have theoretical and practical implications. The measurement of pure, or systematic, risk of securities is important in portfolio selection and in applications of the Capital Asset Pricing Model (CAPM). Ex post measurements are the usual base for estimates of systematic risk, so conditions that produce biased estimates and/or unstable betas are noteworthy.

This paper shows that simultaneous violations of two ordinary-least-squares assumptions can lead to estimates that are both biased and unstable, even when the true betas are stable. Specifically, below we demonstrate that a combination of autocorrelated residuals and security returns that lead measured market returns yield biased estimates of the risk measure of the CAPM. The results of Friend and Blume [9] and Black, Jensen and Scholes [1] are consistent with this estimation problem. Several empirical studies suggest that significant autocorrelation is common for most levels of differencing [4, 6, 7, 14] and Francis [8] found that with monthly differencing intervals, for shorter periods (two to three years of data), returns for some securities lead measured market returns.

The estimation problem is likely for regression estimates that are based on a few monthly observations, since the results by Francis [8] suggest the problem may not occur for estimates based on monthly data for long periods.[1] However, autocorrelation and security returns that lead market returns may combine to produce the unstable betas that Blume [2], Brenner and Smidt [3], Fabozzi and Francis [5], and Levy [10] have noted, because these studies used monthly data and short time periods to estimate betas. Below a model that explains the instability problem is developed and tests show that the relationships hypothesized in that model are statistically significant.

I. The Model

Efficient least squares estimation requires that the disturbance terms be intertemporally independent ($E(\epsilon_{j,s}, \epsilon_{j,t}) = 0$ for $s \neq t$; where $\epsilon_{j,t}$ is the disturbance term for the theoretical regression model wherein the returns of security j are a

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[1] The existence of autocorrelation in stock returns and residuals is well documented by Cootner [4], Fama [6], Young [14], and others; Francis [8] notes some tendency for some securities to significantly lead or lag the market. He calculated coefficients over two year periods and considered monthly differencing intervals for stock returns. Over longer periods the tendency toward leading and lagging largely disappeared, except during periods of unusual price behavior.

linear function of the market returns in the t^{th} time period. Also, the intertemporal correlation between the disturbance term and the independent variable should be zero ($E(\epsilon_{j,s}, R_{m,t}) = 0$ for all s and t, where $R_{m,t}$ is the independent variable, (the market returns for period t)). When either of these assumptions is violated singly, the least-squares estimators are unbiased but inefficient. It is not generally recognized, however, that simultaneous violation of these assumptions can lead to biased estimators (see Rao and Miller [11]). Francis [8] gives empirical estimates which show measured security returns and measured market returns frequently have significant, non-zero, residual autocorrelations and/or significant, non-zero, intertemporal residual-market correlations.

In an efficient capital market, neither of these problems should occur, provided that returns are measured correctly. However, in an otherwise efficient capital market, these phenomena could arise from measurement errors. The development below focuses on the consequences of applying Ordinary-Least-Squares (OLS) procedures to ex post data that includes such measurement errors.[2] The source of these measurement errors is open to conjecture but it is likely that measurement errors for market returns arise with untraded securities, discontinuous and nonsynchronous trading processes, reporting frictions, or other imperfections in the trading and reporting mechanisms, rather than from market inefficiencies, per se.

I.A. The Model with Measurement Errors

The returns for a security depend linearly in the riskless rate of return and market returns. If market and security returns are adjusted for the riskless rate, and for their respective expected values, the returns for security j are given by

$$r_{jt} = \beta_j r_{mt} + \epsilon_{jt} \tag{1}$$

where r_{jt} and r_{mt} are measured, adjusted, returns for period t and ϵ_{jt} is the disturbance term for security j for period t. To incorporate the problems noted above assume that:

1. The market returns are measured with error, so that if r^*_{mt} is the true market return, we measure:

$$r_{mt} = r^*_{mt} + \eta_t \tag{2}$$

where η_t is the measurement error, taken to be independent and identically distributed over time, with an expected value of zero, and further η_t is taken to be independent of r^*_{ms} for all s and t.

2. The lagged disturbances of individual securities can be correlated with the current measurement-error term (but not with the current true market return, r^*_{mt}), so that

$$E(\eta_t, \epsilon_{j,t-1}) = E(r_{mt}, \epsilon_{j,t-1}) \neq 0 \tag{3}$$

[2] Roll [12] argues that measurement of a "true" market index is an impossible task. He further argues that even correctly-measured stock market indices are inadequate proxies for tests of the CAPM. This paper concentrates on problems that arise with ex post estimates of betas, which are commonly used as the basis for estimates of systematic risk for securities.

3. The disturbances are serially correlated so that

$$\epsilon_{jt} = \rho_j \epsilon_{jt-1} + \nu_t \tag{4}$$

4. Finally, also assume that the following covariances are zero:

$$E(r_{mt}, r_{m,t+1}) = 0 \tag{5a}$$

$$E(r_{m,t-1}, \epsilon_{j,t}) = 0 \tag{5b}$$

$$E(r^*_{mt}, \nu_{jt}) = 0 \tag{5c}$$

$$E(n_t, \nu_{jt}) = 0 \tag{5d}$$

and assume that all of the above disturbances are homoskedastic. The next subsection demonstrates that, with these assumptions, OLS estimates of β are biased. Under these conditions, the bias component is likely to be unstable, because the relationships given by (3) and (4) are likely to be unstable. The instability in the estimated betas can occur even when the true beta is stable.

I.B. Estimates of β

When OLS methods are applied to estimate betas we have (below the limits on the summations and the security subscripts are dropped if they are obvious):

$$\hat{\beta}_j = [\textstyle\sum_{t=1}^{T} r_{jt} r_{mt}]/[\sum_{t=1}^{T} r^2_{mt}] \tag{6a}$$

$$\hat{\beta} = [\textstyle\sum ((\beta r_{mt} + \epsilon_t) r_{mt})]/[\sum r^2_{mt}] \tag{6b}$$

$$\hat{\beta} = [\textstyle\sum (\beta r^2_{mt} + r_{mt}\epsilon_t)]/[\sum r^2_{mt}] \tag{6c}$$

$$\hat{\beta} = \beta + [\textstyle\sum (\rho r_{mt}\epsilon_{t-1} + r_{mt}\nu_t)]/[\sum r^2_{mt}] \tag{6d}$$

$$\hat{\beta} = \beta + [\rho[\textstyle\sum r_{mt}\epsilon_{t-1}]/[\sum r^2_{mt}] + [\sum (r^*_{mt}\nu_t + \eta_t\nu_t)]/[\sum r^2_{mt}]] \tag{6e}$$

and if the regression coefficient for ϵ_{t-1} and r_{mt} is designated by λ, when the expected values are taken of $\hat{\beta}$ and of the right side of 6(e) we have:

$$E(\hat{\beta}) = \beta + \rho\lambda \tag{7}$$

since the numerator of the last term in 6(e) has an expected value of zero,

Now consider changes in estimated betas:

$$\Delta\hat{\beta} = \hat{\beta}_T - \hat{\beta}_{T-1} \tag{8a}$$

$$\Delta\hat{\beta} = (\beta_T + \rho_T\lambda_T) - (\beta_{T-1} + \rho_{T-1}\lambda_{T-1}) \tag{8b}$$

$$\Delta\hat{\beta} = \Delta\beta + \Delta(\rho\lambda) \tag{8c}$$

If β is stable, $\Delta\beta$ is zero and:

$$\Delta\hat{\beta} = \Delta(\rho\lambda), \tag{9}$$

so that changes in estimated betas would be due to changes in the product of the autocorrelation coefficient and the lagged-residual/contemporaneous-market-return coefficient. The next section develops estimates for ρ and λ and considers the relation of the estimates to the theoretical coefficients above.

II. Estimates of the Bias Component

Estimates are needed to test (9) above. The dependent variable, $\Delta\hat{\beta}$, can be readily produced but estimates of $\Delta(\rho\lambda)$ cannot be produced directly. The observed residuals, denoted e_t, and the observed market returns, cannot be used to estimate $\lambda = E(r_{mt}, \epsilon_t^*)/E(r_{mt}^2)$ because $\sum r_{mt}e_t = 0$, by construction. However, ρ and λ can be estimated separately and straight-forward estimates are:

$$\hat{\rho} = [\sum e_t e_{t-1}]/[\sum e_t^2] \tag{10}$$

and

$$\hat{\lambda} = [\sum r_{m,t} e_{t-1}]/[\sum r_{mt}^2] \tag{11}$$

When these simple estimates are examined below, $\hat{\rho}$ is shown to be a biased estimator for ρ but $\hat{\lambda}$ is an unbiased estimator for λ.

First consider $\hat{\lambda}$:

$$\hat{\lambda} = [\sum r_{m,t} e_{t-1}]/[\sum r_{mt}^2] \tag{12a}$$

$$= [\sum (r_{mt})(r_{j,t-1} - \hat{\beta} r_{m,t-1})]/[\sum r_{mt}^2] \tag{12b}$$

$$= [\sum (r_{mt})(\beta r_{m,t-1} + \epsilon_{t-1} - \hat{\beta} r_{mt-1})]/[\sum r_{mt}^2] \tag{12c}$$

$$= [\sum r_{mt}\epsilon_{t-1}]/[\sum r_{mt}^2] + (\beta - \hat{\beta})[\sum r_{mt} r_{mt-1}]/[\sum r_{mt}^2] \tag{12d}$$

and when expected values are taken

$$E(\hat{\lambda}) = \lambda, \tag{13}$$

since the numerator for the last term in (12d) has expected value zero under assumption (5a). Thus $\hat{\lambda}$ is an unbiased estimate for λ.

Now consider $\hat{\rho}$:[3]

$$\hat{\rho} = [\sum e_t e_{t-1}]/[\sum e_{t-1}^2] \tag{14a}$$

$$\hat{\rho} = [\sum \epsilon_t \epsilon_{t-1}]/[\sum e_{t1}^2] - \rho\lambda[\sum r_{mt}\epsilon_{t-1}]/[\sum e_{t-1}^2] \tag{14b}$$

and with adjustments to the denominators then taking expected values we have:

$$E(\hat{\rho}) = (E(\epsilon^2)/E(e^2))\cdot\rho - (\rho\lambda E(r_{mt}^2)/E(e_{t-1}^2)) \cdot E(r_{mt}\epsilon_{t-1}) \tag{15a}$$

[3] The steps required to move from [14a] to [14b] depend on an expansion and regrouping of the $e_t e_{t-1}$ term.

$$\begin{aligned} e_{jt}e_{j,t-1} &= (r_{jt} - \hat{\beta} r_{mt})(r_{j,t-1} - \hat{\beta} r_{m,t-1}) \\ &= (\beta r_{mt} + \epsilon_t - \hat{\beta} r_{mt})(\beta r_{m,t-1} + \epsilon_{t-1} - \hat{\beta} r_{m,t-1}) \\ &= ((\beta - \hat{\beta}) r_{mt} + \epsilon_t)((\beta - \hat{\beta}) r_{m,t-1} + \epsilon_{t-1}) \\ &= ((\beta - \hat{\beta})^2 r_{mt} r_{m,t-1} + (\beta - \hat{\beta})(r_{mt}\epsilon_{t-1} + r_{m,t-1}\epsilon_t) + \epsilon_t, \epsilon_{t-1}) \end{aligned}$$

and since $\beta - \hat{\beta} = -\rho\lambda$, when expected values are taken we have (given assumptions (5a) and (5b)):

$$E(\epsilon_t, \epsilon_{t-1}) = -\rho\lambda E(r_{mt}\epsilon_{t-1}) + E(\epsilon_t \epsilon_{t-1}).$$

$$E(\hat{\rho}) = \rho[(\sigma_\epsilon^2/\sigma_e^2) - (\lambda^2\sigma_{rm}^2/\sigma_e^2)] \tag{15b}$$

so if $\hat{\rho}$ is adjusted by the reciprocal of the term in brackets in (15b) we have

$$E(\hat{\rho})\cdot((\sigma_\epsilon^2/\sigma_e^2) - \lambda^2(\sigma_{rm}^2/\sigma_e^2)) = E(\rho^*) = \rho \tag{16}$$

Thus ρ^* would be an unbiased estimator of ρ. There is a problem when one tries to produce an estimate for ρ^*. Unbiased estimators for σ_{rm}^2, σ_e^2, and λ are readily available but σ_ϵ^2 has no observable unbiased statistic. If σ_e^2 is used to estimate σ_ϵ^2 it overstates the variance since $\sigma_\epsilon^2 = \sigma_e^2 - \beta^2\sigma_\eta^2$ (see Theil [13], p. 608) and, since η is also unobservable, the bias cannot be adjusted directly.

One suboptimal alternative is to accept an imperfect estimate that could be estimated with observable values. If σ_e^2 is substituted for σ_ϵ^2 in (16), an estimate for ρ (designated as ρ' below), is produced:

$$\rho' = \rho^*/(1 - \lambda^2(\sigma_{rm}^2/\sigma_e^2)) \tag{17a}$$

$$E(\rho') = E(\hat{\rho})\cdot[(\sigma_\epsilon^2/\sigma_e^2) - \lambda^2(\sigma_{rm}^2/\sigma_e^2)]/[1 - \lambda^2(\sigma_{rm}^2/\sigma_e^2)] \tag{17b}$$

The advantage of ρ' over ρ^* is that all parameters in ρ' can be directly estimated and the disadvantage of ρ' relative to ρ^* is that it is not an unbiased estimator for ρ. However, ρ' is an improvement over $\hat{\rho}$ since, for $\rho > 0$, $E(\hat{\rho}) < E(\rho') < \rho$; and for $\rho < 0$, $E(\hat{\rho}) > E(\rho') > \rho$.[4] This suboptimal, but tractable, estimator, ρ'; is estimated below with statistics substituted for the parameters in (17a) above.[5] Estimates of ρ' are used below to test the relationship between $\Delta\hat{\beta}$ and $\Delta(\rho\lambda)$.

III. Empirical Tests

Using the notation developed above and assuming that the true beta is stable, the instability in estimated betas between two periods, 1 and 2, would be:

$$\Delta\hat{\beta} = \Delta\rho\lambda \tag{18}$$

To test this proposition, beta coefficients were estimated for all available securities using the Quarterly Compustat tapes for four overlapping two year periods from 1967–1971. These two year periods are labeled one through four and they correspond to the periods 67–68, 68–69, 69–70, and 70–71, respectively. Monthly excess returns[6] were regressed on the excess return of the monthly Fisher Arithmetic Index, an equally-weighted-average of the returns on all stocks listed on the New York Stock Exchange (about 1,800 firms). The thirty-day treasury

[4] These inequalities presume $(\sigma_\epsilon^2/\sigma_e^2) > (\lambda^2(\sigma_{rm}^2/\sigma_e^2))$, which is consistent with empirical estimates. Table I of this paper shows that the ranges for $|\hat{\lambda}|$ are generally less than one and Fama (6; p. 103 and Tables 4.3, 4.4) reports estimates of error variances that are two to fifteen times the estimate of market variances. This inequality also requires that $\sigma_\epsilon^2 = \sigma_e^2 - \beta^2\sigma_\eta^2 \geq \beta^2\sigma_\eta^2$ so that $\sigma_\epsilon^2 \geq \frac{1}{2}\sigma_e^2$, but this is a weak requirement that almost all securities would meet.

[5] It must be noted that when statistics are used to estimate ρ' the estimate for ρ' is less efficient than $\hat{\rho}$.

[6] Excess returns include both dividends and capital gains less the rate on 30-day treasury bills. Using quarterly Compustat data the dividend can be either arbitrarily lumped into the third month of the quarter or spread evenly over all three months. Both methods were tried and only trivial differences in results were noted. The results below are based on the first method.

Table I

Period	Number of cases	$\bar{\beta}$	$\bar{\hat{\rho}}'$ (s.e.)[a]	$\hat{\rho}_i'$ range low/high	$\bar{\hat{\rho}}$	$\hat{\rho}_i$ range low/high	$\bar{\lambda}$	$\hat{\lambda}_i$ range low/high
1	228	.937	−.094[b] (.013)	−.56/+.37	−.090[b] (.012)	−.56/+.34	.057[b] (.023)	−1.05/+1.27
2	251	.928	−.128[b] (.012)	−.60/+.37	−.123[b] (.012)	−.58/+.37	.000 (.017)	−.848/+.965
3	271	.880	−.135[b] (.013)	−.64/+.46	−.130[b] (.012)	−.62/+.42	.016 (.013)	−.770/+.730
4	276	.892	−.162[b] (.012)	−.78/+.32	−.155[b] (.011)	−.67/+.32	.030[b] (.013)	−.902/+.773

[a] The values in parentheses are standard errors of the means.
[b] Significantly non-zero at $\alpha \leq .01$.

Table II

Period	a	b	R^2	F	Sig. Level	Sample Size
A	.075	.693	.027	6.30	.000	225
B	.038	1.19	.059	15.42	.000	246

bill rate was used as a proxy for the riskless rate of return and only firms with fiscal years ending on December 31 were included in the sample.

The results of these estimates are reported in Table I. As the significance levels demonstrate, the autocorrelation and intertemporal correlations are significantly non-zero for this broadly-based sample. The wide ranges demonstrate that there are subsets of securities whose betas could be dramatically affected by concurrent autocorrelation and intertemporal market-residual correlations. Although the means and standard errors in Table I show that ρ and λ are significantly non-zero for most subperiods, the resultant instability in estimated betas is not directly presented. This problem is the primary concern of this paper and is considered next.

To test (18), the changes in the estimates of betas were regressed on the changes in the products of $\hat{\rho}_i'$ and $\hat{\lambda}_i$ for successive two-year periods (the estimates of ρ' are designated as $\hat{\rho}'$). The change between the 67–68 period and the 69–70 period is labeled period A. Similarly, the change between the 68–69 and 70–71 periods is labeled period B. The A-period regression can be symbolically represented as:

$$\hat{\beta}_3 - \hat{\beta}_1 = a + b((\hat{\rho}_3'\hat{\lambda}_1) - (\hat{\rho}_1'\hat{\lambda}_1)) \tag{19}$$

The results of those regressions are presented in Table II.

Although these r^2 values are low, there is a significant relationship ($\alpha < .001$) between the variables for both periods examined. The low r^2 are not surprising given the relatively small sample sizes for the $\hat{\beta}$, $\hat{\rho}'$, and $\hat{\lambda}$, (24 monthly returns), the imprecise estimates of ρ, and the likelihood of other intervening influences, including changes in the true betas. These results do confirm the hypothesized relation for the changes in estimated betas and changes in the autocorrelation and market/error-term correlation.

Thus, during the periods examined, some of the observed instability in estimated beta coefficients could be "explained" by a combination of autocorrelated errors and stock returns that tend to lead overall market activity. These results show that simple applications of OLS regression, in the presence of measurement errors, produces unstable and biased betas. Adjustments for these problems should improve the stability of estimates of betas. The suggested adjustment would be to simply subtract the product of the estimate of λ and ρ' from the estimated beta.[7]

IV. Summary

The results in this paper demonstrate that concurrent autocorrelated residuals and intertemporal correlations between market returns and residuals can lead to biased, unstable, OLS estimates of betas. The instability can be observed in estimates of betas even if true betas are constant over time.

The model in Section I demonstrates that these statistics present problems in real data analysis when OLS regression is brutally applied to ex-post data. The empirical results demonstrate that changes in estimated betas are significantly associated with changes in the product of the estimates of autocorrelations for residuals and the estimates for intertemporal market-residual covariances. These results suggest that more stable estimates of betas can be obtained by correcting for these econometric problems that could arise from measurement errors in an otherwise efficient market.

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[7] This adjustment is implied by equation (9), which also implies that $E(b) = 1.0$ in the above regressions. The results above are consistent with this hypothesized value.