

Estimation of an adaptive stock market model with heterogeneous agents

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Abstract

Standard asset pricing models based on rational expectations and homogeneity have problems explaining the complex and volatile nature of financial markets. Recently, boundedly rational and heterogeneous agent models have been developed and simulated returns are found to exhibit various stylized facts, such as volatility clustering and fat tails. Here, we are interested in how well the proposed models can explain all the properties seen in real data, not just one or a few at a time. Hence, we do a proper estimation of some simple versions of such a model by the use of efficient method of moments and maximum likelihood and compare the results to real data and more traditional econometric models. We discover two main findings. First, the similarities with observed data found in earlier simulations rely crucially on a somewhat unrealistic modeling of the noise term. Second, when the stochastic is more properly introduced the models are still able to generate some stylized facts, but the fit is generally quite poor. © 2007 Elsevier B.V. All rights reserved.

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1. Introduction

Many of the well established models in economics and finance rely on two important cornerstones: homogeneous investors and market efficiency. There is little doubt that people differ in preferences, knowledge, and beliefs, but the homogeneity assumption may still serve as a useful approximation, since if the heterogeneity averages out and can be captured by a single representative agent, the analysis can be tremendously simplified. The other major assumption, the efficient market hypothesis (EMH), is closely related to the rational expectation hypothesis, first introduced by Muth (1961). There are strong and weak versions of the EMH, but generally it states that since all agents are rational and capable of processing information immediately and accurately, prices should reflect all available information. In such a market everyone agrees upon the fundamental price and fluctuations around this equilibrium arise only because of unexpected and random changes in fundamentals.

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Empirical investigations of financial series show signs of volatility clustering, excess kurtosis, high persistence and asymmetry in volatility, small autocorrelation in returns, but substantial correlation in absolute or squared returns, see [Pagan \(1996\)](#) for a comprehensive study of characteristic features in financial market data. These stylized facts are difficult to explain with just efficient market fluctuations and call for alternative explanations.

It is often argued, see e.g. [Fama \(1970\)](#), that an efficient market has no predictable patterns (conditioned on public information) since these would disappear as rational traders exploit them. Interestingly, this seemingly innocent statement reveals some unappealing features of the EMH. Firstly, the EMH is a very conservative theory, as its philosophy is that there is no point to try and find profitable methods, because if such methods existed they would already be in use. This is in sharp contrast to how the real world works, not just the financial world, where a never-ending search for improvements and superiority is driving the evolution forward. If, for instance, software entrepreneurs around the world reasoned that existing products could not be improved, because if it was possible then Microsoft with its huge resources would already have developed them, not many computer enterprises would have been started. Of course, this is not what we observe, and if such a static description of the software industry would have been true, Microsoft would not have become a dominant player in the first place. Secondly, the EMH is preserved because some traders, like the entrepreneurs just mentioned, apparently do not believe in it and are therefore constantly looking for business opportunities. The EMH, however, is silent on why some rational traders disagree with their, also rational, colleagues. Thirdly, these unfaithful traders' actions instantly influence prices, thereby erasing any predictable pattern, but it is not obvious how this mechanism should work in a world of different opinions.

Real markets are likely to be characterized by a constant interaction among its participants. Suppose a subset of traders test a strategy based on the belief that if the price hits a threshold, it will continue up. By chance, they are right and their success is observed by the other agents. More agents want to try this strategy, which raises the price further, reinforcing and motivating the strategy in a self-fulfilling manner. It is not perfectly rational to rely on such a strategy, especially not if the fundamental price is known, but in an uncertain world it can be hard to tell if a price rise is due to changes in fundamentals, or a speculative overvaluation. Sooner or later the price drops, by chance or because a view that the asset is overvalued is spread among the traders, inducing people to sell. The price drops further and may trigger another trading strategy, and so on. On average, forecastable structures may disappear as agents exploit them, as indicated by the low (linear) autocorrelations in asset returns found in empirical studies, but may be present during certain periods of time.¹ The above indicates that heterogeneity, uncertainty, adaptation, and expectation feedback are plausible components of real markets, and arise naturally if just the extreme assumptions and vague mechanisms behind the EMH are slightly lightened.

Bounded rationality, see e.g. [Sargent \(1993\)](#), as opposed to perfect rationality, is often used to describe how agents, with limited information about fundamentals, develop expectation models of what variables move prices and dividends. Agents are not irrational, but given their limited information they adapt to what they believe is optimal, and given their information set they act rationally. The endogenous uncertainty of the state of the world prevents the agents from forming and solving life-time optimization problems in favor of more simple reasoning and rules of thumb, see e.g. [Shefrin \(2000\)](#).

Recently, the concept of bounded rationality and evolutionary adaptive agents have been modeled in e.g. [Brock and Hommes \(1997a, 1998\)](#), [Chiarella and He \(2002a,b\)](#), [Gauersdorfer and Hommes \(2000\)](#), [Hommes \(2001\)](#), and in a more computationally oriented multi-agent framework in [Arthur et al. \(1997\)](#), [LeBaron et al. \(1999\)](#), [LeBaron \(2001a\)](#), [Lux \(1995\)](#), and [Lux and Marchesi \(2000\)](#).² See also [Hommes \(2006\)](#) and [LeBaron \(2006\)](#) for recent surveys. Common to all these heterogeneous agent models is the existence of different trader types: fundamentalists, who believe the asset price is determined solely by economic fundamentals, and technical traders who try to predict future prices by searching for patterns in historical prices. A general and most interesting finding is that these models qualitatively explain a number of the stylized facts mentioned above and that, in contrast to classic financial

¹ Many empirical investigations support the fact that technical trading strategies yield significant profits during certain time intervals. The classic paper is [Brock et al. \(1992\)](#). See also [Jegadeesh and Titman \(2001\)](#), and [Chiarella and He \(2002a,b\)](#) for more recent references on this topic.

² The agents in [Arthur et al. \(1997\)](#), [LeBaron et al. \(1999\)](#), and [LeBaron \(2001a,b\)](#) not only choose the best forecasting rule, but also have the ability to further develop and update it, i.e. the agents can learn. See also [Evans and Honkapohja \(2001\)](#) for a recent treatment of algorithmic learning.

theory, the technical traders are not driven out of the market. Both types of agents continue to coexist, as they do in real markets.³

The Adaptive Belief System (henceforth ABS) of Brock and Hommes (1997a, 1998) models the financial markets as an evolutionary interaction of competing agents, each with a specific trading strategy. The agents are boundedly rational since they choose the strategy that has worked best in the past according to some fitness function such as realized profits, accumulated wealth, or the utility of these quantities. The model includes many desirable key features such as adaptation and expectation feedback. It is complex, yet analytically tractable, and has inspired further extensions, for example in Gaunersdorfer and Hommes (2000), Chiarella and He (2002b), and De Grauwe and Grimaldi (2003, 2006).

Do the above models also quantitatively explain financial market movements? Rough calibrations in Brock and Hommes (1997a), Chen et al. (2001), Gaunersdorfer and Hommes (2000), Gaunersdorfer (2001), and LeBaron (2001b) indicate that some of the statistical properties of the simulated returns resemble those of the real data, but some do not. A close fit to some moments of the data is at the expense of a worse fit to others. It is important to stress that the heterogeneous expectation models are not without dynamic noise. The non-linear models are fed with an exogenous stochastic process, but the noise process is “nice”, which in this case means that it is normally distributed. Instead it is the internal dynamics of the models that should amplify and distort the randomness into the complicated and realistic price fluctuations we observe. This is in sharp contrast to the statistical models used in empirical finance of which the ARCH-class models, and the Markov switching models by far are the most popular, see Bollerslev et al. (1994) and Hamilton (1994) for numerous applications. Both models have proved to be quite successful in modeling financial data, but they do not offer any explanation of why volatility tends to cluster, or why there are switches between different magnitudes in volatility. It would be most satisfactory to be able to explain these phenomena with a structural model, where deviations from the fundamental value are triggered by randomness, and amplified by realistically modeled agents. Besides, if the existing statistical models are approximations of such underlying dynamics, the estimation of the structural model directly would most likely lead to econometric improvements.

In this paper we perform a proper estimation of an adaptive heterogeneous agent model to see if the preliminary simulation results stand to face reality. Unfortunately, when we correct the earlier simulations in Gaunersdorfer and Hommes (2000) and Gaunersdorfer (2001) with a more realistic description of the model stochastics, many of the similarities to real data are lost. We focus on two specifications of the extended version of the model in Brock and Hommes (1998), described in De Grauwe and Grimaldi (2003, 2006). One of the specifications contains a simple modeling of the noise term and is easily estimated by maximum likelihood. The other one has a more complex noise structure and to estimate it, we instead rely on the efficient method of moment (EMM) approach of Gallant and Tauchen (1996), also denoted indirect inference by Gouriéroux et al. (1993).

A general finding is that none of the two model specifications investigated here provide an adequate fit to the data, but the estimation techniques can be used to estimate other heterogeneous agent models, at least if they contain a reasonable number of parameters. It is our hope that a proper objective estimation also will become a helpful tool for the further theoretical developments in this field, by pointing out what the models may fail to capture, or the overall impact of certain parameters, or what other features are of importance and should be focused upon.

This paper is one of the first to estimate heterogeneous agent models on financial data, but there have been a few other recent attempts. Baak (1999) and Chavas (2000) estimate a heterogeneous agent model on hog and beef market data, and find evidence for heterogeneity of expectations and the existence of non-rational agents present in the market. Vigfusson (1997) reformulates the exchange rate model with fundamentalists and chartists developed by Frankel and Froot (1990) as a Markov switching model, and estimate it using daily Canada–US exchange rates. Winker and Gilli (2001) and Gilli and Winker (2003) estimate the exchange rate model of Kirman (1991) with fundamentalists and chartists using the daily DM–US\$ exchange rate, where the fraction of fundamentalists and chartists are driven by an exogenous herding process. Gilli and Winker find significant switching between the different investor types. Westerhoff and Reitz (2003) also estimate a model with fundamentalists and chartists, in which the share of the fundamentalists changes endogenously, depending upon the distance from the fundamental exchange rate. Recently,

³ There is a close connection to earlier work in behavioral finance on noise-trader models, see De Long et al. (1990). The distinction of two trader types is also made there, but the traders are not adaptive. The less rational noise-traders do systematic mistakes, and continue to do so without adjusting to the outcome of their strategy. They are truly irrational and not boundedly rational.

Alfarano et al. (2005) estimate a simple agent-based model with fundamentalists and noise traders using daily gold price returns, exchange rates and the German DAX stock market index. More closely related to this paper are estimations by Manzan (2003) and Boswijk et al. (2005) of a modified version of the ABS of Brock and Hommes on yearly S&P 500 data. These authors find significant switching between fundamentalists and trend-followers and, in particular, explain the rapid rise in stock prices in the late nineties as being strongly amplified by trend-following trading rules.

The paper is organized as follows. Section 2 presents the model under scrutiny, the ABS, while Section 3 describes how stochastic is introduced in the model and how to estimate it. The empirical results of the estimation, and a comparison with real data as well as other statistical models, are given in Section 4. Finally, Section 5 presents some concluding remarks and suggestions for further research. A more detailed description of the EMM is found in the Appendix.

2. The adaptive heterogeneous agent model

In this section we present a somewhat generalized version of the ABS of Brock and Hommes, which De Grauwe and Grimaldi (2003, 2006) use in a related exchange rate framework. The model consists of three parts: (i) utility maximizing agents select optimal portfolios based on (ii) different forecasts rules or beliefs about the next period price, and (iii) evaluate the different rules and adopt in the coming period the one with best performance or highest fitness.

2.1. The utility maximization

Suppose there are two securities in the economy: one risky asset, a stock, that pays an uncertain dividend, and one infinitely supplied risk-free asset that pays the constant rate r . Let p_t be the ex-dividend stock price, and y_t its stochastic dividend. Following the framework of Brock and Hommes (1998), the wealth of investor h evolves according to

$$W_{h,t+1} = (1 + r)W_{h,t} + (p_{t+1} + y_{t+1} - (1 + r)p_t)z_{h,t} \quad (1)$$

where $z_{h,t}$ is the number of shares at time t . Let $E_{h,t}$ and $V_{h,t}$ denote the h investor's expectation and variance operators, conditioned on the information set $F_t = \{p_{t-1}, y_{t-1}, p_{t-2}, \dots\}$ of past prices and dividends. Assuming myopic investors with a mean-variance utility function and a common risk-aversion parameter $a > 0$, each investor solves

$$\max_{z_{h,t}} E_{h,t}[W_{h,t+1}] - \frac{a}{2} V_{h,t}[W_{h,t+1}] \quad (2)$$

for his optimal amount of shares, which yields

$$z_{h,t} = \frac{E_{h,t}[R_{t+1}]}{aV_{h,t}[R_{t+1}]}, \quad (3)$$

where $R_{t+1} = p_{t+1} + y_{t+1} - (1 + r)p_t$ is the excess profit. Thus, the investors are characterized by the same utility function and differ only in how they form their beliefs about the conditional expectation and variance.

Suppose $n_{h,t}$ is the fraction of investors with the same beliefs at time t . Summing over the demands from all groups of investors gives us the aggregated demand. With a fixed total number of shares (per capita) in the market, Z , we have

$$\sum_{h=1}^H n_{h,t} z_{h,t} = Z. \quad (4)$$

Using Eq. (3) in Eq. (4), the market clearing equilibrium price p_t is determined by

$$(1 + r)p_t = \left(\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2} E_{h,t}[p_{t+1} + y_{t+1}] - a_{h,t} Z \right) \frac{1}{\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2}}, \quad (5)$$

where the summation is over H groups, and $\sigma_{h,t}^2$ is a short-hand notation for $V_{h,t}[R_{t+1}]$. Assuming net zero supply of the risky asset ($Z=0$), and an IID dividend process with constant mean $E_t[y_{t+1}] = \bar{y}$, the market clearing price equation becomes:⁴

$$(1+r)p_t = \left(\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2} (E_{h,t}[p_{t+1}] + \bar{y}) \right) \frac{1}{\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2}}. \quad (6)$$

It is important to note that the fundamental rational expectations (RE) price is nested within the above model. With homogeneous expectations, the arbitrage market equilibrium (6) reduces to

$$(1+r)p_t = E_t[p_{t+1}] + \bar{y}. \quad (7)$$

Using the law of iterated expectations and assuming transversality, the RE price is given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \quad (8)$$

In a second specification we also allow the dividends to follow a random walk. In that case the market clearing price is determined by

$$(1+r)p_t = \left(\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2} (E_{h,t}[p_{t+1}] + y_t) \right) \frac{1}{\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2}}, \quad (9)$$

which implies that the RE price also follows a random walk⁵

$$p_t^* = \sum_{k=1}^{\infty} \frac{y_t}{(1+r)^k} = \frac{y_t}{r}. \quad (10)$$

It is assumed that the investors do not observe y_t before the market clearing price is set, that is, dividend is paid at the same time they have agreed on the price. In a perfectly rational world, all investors agree upon the fundamental price of the risky asset. Asset prices change because of unexpected changes in dividends only. In a heterogenous world, on the other hand, where prices are determined by Eqs. (6), (9), or more generally by Eq. (5), asset dynamics can show a much more complex behavior.

2.2. The forecast rules

So far we have said nothing about how the agents form their beliefs, that is, their conditional expectations about the future price. Let us assume the existence of two types of traders with the following prediction rules (for reasonably large sample sizes T):

$$E_{h,t}[p_{t+1}] = \begin{cases} p_{t-1}^* + v(p_{t-1} - p_{t-1}^*), & h = f \\ p_{t-1} + \lambda \alpha_{m,h} \sum_{i=0}^T (1 - \alpha_{m,h})^i (p_{t-1-i} - p_{t-2-i}), & h = \text{mo, co} \end{cases} \quad (11)$$

⁴ In order to render analytical tractability, Brock and Hommes (1997a,b, 1998) also made the additional assumption that beliefs about conditional variance are equal and constant for all types and times, that is, $\sigma_{h,t}^2 = \sigma^2 \forall h, t$.

⁵ The dividend processes considered here do not give rise to the exponentially growing prices we observe in real stock markets. As such, the models are silent about explaining the long-term trend in prices but could still be used to analyze other statistical properties of real financial data, apart from a constant drift. Boswijk et al. (2005) assumes a constant growth rate in dividends, thereby obtaining long-run trending prices. A similar approach could be taken here by assuming a drift in the random walk dividend process. The non-stationary case has also been discussed in Brock and Hommes (1997b) and more recently in Hommes (2002). See also the related model in Chiarella and He (2002a), where the underlying CRRA utility function leads to growing price (and wealth) processes.

where $0 \leq v \leq 1$, $|\lambda| < 1$ and $0 \leq \alpha_{m,h} \leq 1$. The first investor is called a *fundamentalist* ($h=f$) and believes that tomorrow's price will mean-revert towards the fundamental price by a factor v . When $v=1$, these traders are advocates of the efficient market hypothesis (EMH) and that prices follow a random walk. The second category is *chartists* or *technical traders*. They extrapolate into the future a geometrically declining moving average of past price changes, where the forgetting factor $\alpha_{m,h}$ determines the effective time window, and λ gives the degree of extrapolation.⁶ The chartists are further categorized as *momentum traders* or *trend chasers* ($h=mo$) if $\lambda > 0$, or *contrarians* ($h=co$) if $\lambda < 0$, and we see that $\lambda=0$ also corresponds to EMH-believers.⁷ Usually, only deviations from the latest observed price are investigated, in which case $\alpha_{m,h} \uparrow 1$ and Eq. (11) is given by $E_{h,t}[p_{t+1}] = p_{t-1} + \lambda(p_{t-1} - p_{t-2})$, but other extrapolation rules and lags are analyzed in Chiarella and He (2003). Note that the timing of the information set is of importance. In the Walrasian market equilibrium used here, the market clearing price depends on expectations of p_{t+1} . When forming these expectations the agents have not yet observed p_t , and therefore use the most recent information from $t-1$. Furthermore, in the special case of a simple IID dividend process with constant mean, p_{t-1}^* is replaced by p^* .

The linear updating rule of the chartists is stable if $|\lambda| < 1$ but may lead to too large deviations from the fundamental price. We therefore follow De Grauwe and Grimaldi (2003) and introduce a stabilizing force that becomes active if the price deviates too much from its fundamental value, by assuming that the risk aversion of the fundamentalists declines as the misalignment increases:

$$a_{h,t} = \begin{cases} \frac{a}{1 + \phi|p_{t-1} - p_{t-1}^*|}, & h = f \\ a, & h = mo, co, \end{cases} \quad (12)$$

where $\phi \geq 0$ measures the sensitivity to the misalignment $|p_{t-1} - p_{t-1}^*|$. If $\phi=0$ we are back to the case where all agents share the same risk aversion. The economic interpretation is that when the fundamentalists become more and more confident of the existence of a mispricing, they increase their share of the market. In reality, however, there may be many “limits to arbitrage” that prevent well-informed fundamental investors (if they exist) from exploiting the mispricing since they cannot know if it will persist for long and, perhaps, be pushed even further away from its fair value by other market participants. Nevertheless, the specification in Eq. (12) nests the constant risk aversion case, and for this reason it may be interesting to investigate the significance of ϕ in the estimations.

In this more general framework, the agents also care about the time-varying risk of their portfolio since $\sigma_{h,t}^2$ enters the market price equilibrium. Here, we follow Gaunersdorfer (2000) and De Grauwe and Grimaldi (2003) and define the investor's belief of the conditional variance of the excess profits as the geometrically declining weighted average (with forgetting factor $\alpha_{v,h}$) of the squared (one period ahead) forecast error made by the chartists and fundamentalists, respectively:⁸

$$\sigma_{h,t}^2 = \alpha_{v,h} \sum_{i=0}^T (1 - \alpha_{v,h})^i (E_{h,t-2-i}[p_{t-1-i}] - p_{t-1-i})^2, \quad h = f, mo, co, \quad (13)$$

where $0 \leq \alpha_{v,h} \leq 1$. Eq. (13) is interesting since it introduces a feed-back from the belief of the second moment to the market clearing price. Moreover, since each investor type can give different weight to past squared observations by the different forgetting factors, the impact of the past price variability can be different depending on which investor currently dominates the market. Essentially, the heteroscedasticity in the data could then potentially be explained by the mixture of these three beliefs of the conditional variance.⁹

⁶ Note that to ensure stability of the chartist prediction rule for all T , $|\lambda| < 1$. Eq. (11) can also be expressed in recursive form: $E_{h,t}[p_{t+1}] = p_{t-1} + \lambda m_{h,t}$ with $m_{h,t} = (1 - \alpha_{m,h})m_{h,t-1} + \alpha_{m,h}(p_{t-1} - p_{t-2})$.

⁷ The choice of traders is partly motivated by empirical studies who discover profitability for momentum strategies over short time intervals, while contrarian strategies generates profits over longer time intervals. See Chiarella and He (2002b) for further references.

⁸ In the implementation, the recursive form may be more convenient: $\sigma_{h,t}^2 = (1 - \alpha_{v,h})\sigma_{h,t-1}^2 + \alpha_{v,h}(E_{h,t-2}[p_{t-1}] - p_{t-1})^2$.

⁹ From a time series perspective this is not a new way of modelling heteroscedasticity. In the HARCH models of Müller et al. (1997) and Dacorogna et al. (1998) squared returns of different frequencies are important determinants of market volatility. Also in the regime switching models, the volatility of an exogenous noise process differs in the different regimes. The main difference here, and a possible shortcoming, is that $\sigma_{h,t}^2$ affects the price directly but it does not affect future beliefs, $\sigma_{h,t+i,i}^2 \geq 1$, as (H)ARCH-type models do.

2.3. The evolutionary fitness measure

One important thing remains in order to complete the model and that is to specify how the fractions $n_{h,t}$ evolve over time. Let us assume the existence of an evolutionary fitness function or performance measure, $U_{h,t}$. Based on the performance measure, agents make a decision of which group to join and whose belief they should rely on. The probability that an agent chooses strategy h is formed on the basis of discrete choice or ‘Gibbs’ probabilities (see [Manski and McFadden, 1981](#); [Anderson et al., 1993](#), for a discussion and economic applications of discrete choice models):

$$n_{h,t} = \frac{\exp(\beta(U_{h,t-1} - C_h))}{\sum_{h=1}^H \exp(\beta(U_{h,t-1} - C_h))}, \quad (14)$$

where $C_h \geq 0$ measure the cost of strategy h , and $\beta \geq 0$ is the intensity of choice measuring how fast agents switch between different prediction strategies. Usually $C_f > 0$ for the fundamentalists to represent an information cost associated with revealing the fundamental price (p_{t-1}^* or p^*), in the spirit of [Routledge \(1999\)](#). If $\beta = 0$, the traders are indifferent to deviations in fitness and all fractions will be constant and equal to $1/H$. The other extreme case, $\beta = \infty$, corresponds to the case where all traders immediately switch to the most successful trading strategy last period. In the intermediate case with a finite and positive β , agents make their predictions according to their fitness, but choose less optimal strategies with a certain probability. The market displays herd behavior, but with an inertia and a skepticism about the optimal strategy.

With mean-variance investors a natural performance measure, adopted in [De Grauwe and Grimaldi \(2003, 2006\)](#), is the utility from past profits of a one unit investment or, for short, past risk-adjusted profits.¹⁰ The risk-adjusted profit for strategy h at time t is given by

$$\pi_{h,t} = R_t s(z_{h,t-1}) - \frac{a_{h,t}}{2} \sigma_{h,t}^2, \quad (15)$$

where $s(x)$ is the signum function, that is, $s(x) = 1$, if $x > 0$, $s(x) = 0$ if $x = 0$, and $s(x) = -1$ if $x < 0$. Thus from Eq. (3), when agents’ forecast of the sign of the excess profit $E_{h,t-1}[R_t]$ is correct their risk-adjusted profits increase. Also note that although the risk aversion of the fundamentalists may differ from the chartists, this does not affect the sign of $z_{h,t-1}$ from Eq. (3). A suitable performance measure can then be defined as

$$U_{h,t} = \pi_{h,t} + \eta U_{h,t-1}, \quad (16)$$

where $0 \leq \eta \leq 1$ is a memory parameter. If $\eta = 0$, only the performance in the last period is of interest, while with a positive η , the weights given to past utilities of profits decrease exponentially. The main building blocks of the ABS are thus the price Eq. (5) in conjunction with the forecasts rules of the price and the conditional variance, the evolutionary dynamics (14), and the fitness function (16).

3. The model stochastics

The dynamic properties of a similar two-agent ABS with a fundamentalist and a trend-follower, both with a common and constant perception of risk and a constant dividend process, have been thoroughly investigated in [Gauersdorfer and Hommes \(2000\)](#), and [Gauersdorfer et al. \(2000\)](#). For different parameter configurations, the system is shown to undergo different bifurcations and to exhibit complicated non-linear, and even chaotic, price behavior. Still, the purely deterministic model is too simple to capture the dynamics of real stock markets. Until now we have been deliberately vague about how the stochastic is introduced in the model. Adding IID noise to a constant dividend enters the price Eq. (6) only from R_t in Eq. (15) via Eqs. (16) and (14), and appears insufficient to render realistic price dynamics if its standard deviation is small. In fact, [Gauersdorfer \(2001\)](#) and [Gauersdorfer and Hommes \(2000\)](#) just use a constant

¹⁰ It should be noted that the approach in Brock and Hommes is somewhat different as they use the utility from past realized profits, that is $\pi_{h,t} = R_t z_{h,t-1} - a\sigma^2 z_{h,t-1}^2/2$, where $z_{h,t-1} = E_{h,t-1}[R_t]/(a\sigma^2)$ is the demand from Eq. (3) with $\sigma_{h,t}^2 = \sigma^2$, a constant. The profitability of a rule will then also depend on if the demand is temporarily high or low. Both approaches were investigated here with small differences in the results.

dividend and instead add a dynamic Gaussian noise term ε_t , representing a model approximation error, to the market clearing Eq. (6):

$$p_t = \frac{1}{1+r} \left(\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2} (E_{h,t}[p_{t+1}] + \bar{y}) \right) \frac{1}{\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2}} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \rho). \quad (17)$$

This results in a noisy chaotic system from which simple returns, $r_t = p_t/p_{t-1} - 1$, are calculated. In a number of simulations in the above references, this model generates return series that exhibits low autocorrelation, volatility clustering, and excess kurtosis, the trademarks of real financial markets. The problem with Eq. (17) is that the generated prices move around a constant $p^* = \bar{y}/r$, either as a pure random walk if $\nu = 1$ and $\lambda = 0$ in the different investors' forecast rules or, more generally, in a persistent but mean-reverting way if $\nu \lesssim 1$. Either way, the price will deviate considerably for shorter or longer periods from p^* and by adding noise of the same variance to all prices, those prices that are relatively low will vary as much in absolute terms as prices that are far above p^* . To be more specific, if noise with a standard deviation of one is added to a price of 10, the noise-to-price ratio is 10%, while if the price drops to 2 the relative magnitude now becomes 50%. It is not a very realistic way to model noise that is not normalized to the price level, particularly since prices also can be negative, and return series calculated from such prices will show signs of the above stylized facts by construction only. In fact, if the non-linearities of the system are turned off and prices are just generated from a random walk, $p_t = p_{t-1} + \varepsilon_t$, the resulting return series, $r_t = \varepsilon_t/p_{t-1}$, will actually show signs of all the stylized facts mentioned above, such as large autocorrelation in squared or absolute returns, and fat tails!¹¹ To avoid these purely artificial results, a more proper and common way is to model the stochastic part as multiplicative noise:

$$p_t = \frac{1}{1+r} \left(\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2} (E_{h,t}[p_{t+1}] + \bar{y}) \right) \frac{1}{\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2}} + p_{t-1} \varepsilon_t \equiv \hat{p}_t + p_{t-1} \varepsilon_t, \quad \varepsilon_t \sim N(0, \rho). \quad (18)$$

In this case, pure IID returns are nested within the model as should be a minimum requirement for a return specification. As we will see, the ability of the model to generate real dynamics is greatly affected: The non-normal behavior of the simulations reported in the mentioned references disappear to a large extent if the stochastic is modeled in the proposed way.

Our main concern is to estimate the model. Assuming a suitable error distribution for ε_t , in our case we choose the Gaussian since any non-normal behavior preferably should stem from the intrinsic non-linear dynamics, the likelihood function for estimating the model with a constant dividend from the observed prices (p_t) is given by:

$$\ln L = -\frac{1}{2} \sum_{t=1}^T \left(\ln(2\pi) + \ln(\rho^2) + \left(\frac{p_t - \hat{p}_t}{p_{t-1}} \right)^2 \frac{1}{\rho^2} \right). \quad (19)$$

In the second specification where the dividend is assumed to follow a random walk, the stochastic now enters the pricing mechanism not only through $n_{h,t}$ as before, but also in the market clearing Eq. (9), and more significantly through p_t^* in Eq. (11) since variations in y_t directly affects the fundamental price via Eq. (10). Somewhat simplified, one may say that in the former case of a constant dividend, prices moved persistently around a constant p^* , while prices in the present specification instead move around a random walk fundamental price, p_t^* . It should be clear that by a random walk dividend we mean one with a multiplicative noise part:

$$y_t = y_{t-1} + y_{t-1} \varepsilon_{\text{rw},t}, \quad \varepsilon_{\text{rw},t} \sim N(0, \rho_{\text{rw}}) \quad (20)$$

¹¹ Other persistent processes instead of the random walk produces qualitatively the same result, e.g. $p_t = p^*(1-a) + ap_{t-1} + \varepsilon_t$, with $a \lesssim 1$, say, 0.95. The parameter choice of the simulated series in the above mentioned references all roughly imply a price persistence of this magnitude.

so that the fundamental price process (10) does not suffer from the shortcomings previously mentioned.¹² This way of introducing stochastics gives much richer dynamics and ideally it should be the only way randomness enters in the model, but nothing prevents from adding a model approximation error as well, yielding

$$p_t = \frac{1}{1+r} \left(\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2} (E_{h,t}[p_{t+1}] + y_t) \right) \frac{1}{\sum_{h=1}^H \frac{n_{h,t}}{\sigma_{h,t}^2}} + p_{t-1} \varepsilon_t, \quad \varepsilon_t \sim N(0, \rho), \quad (21)$$

where $\varepsilon_{rw,t}$ and ε_t are independent and normally distributed.

In reality the level of observed (usually quarterly or annually) dividends might be low, but their volatility might not be, and the influence of y_t on prices via p_t^* in the forecasting rules could very well turn out to be important.¹³ Furthermore and most interestingly, in recent work Yoon (2003) finds that a process of the form $p_t = (1 + \alpha_t)p_{t-1} + p_{t-1}\varepsilon_t$ where α_t is a stochastic process, despite its simplicity it displays many features seen in real financial prices. Such *stochastic unit root processes*, also analyzed in Granger and Swanson (1997), bear some resemblance to the price process in Eq. (21) and the present framework could therefore be seen as an attempt towards a more theoretical justification for these promising, but ad hoc, processes.

Unfortunately, the error terms are now more embedded in the non-linear structure, which means that a computable likelihood function is no longer easy to derive. We therefore have to rely on simulation based econometric methods, where some moments of the simulated data are matched to those of the real data. This idea follows Hansen's (1982) generalized method of moments and is described in the simulated method of moment procedure of Duffie and Singleton (1993). In order to avoid selecting the moments on an ad hoc basis, Gallant and Tauchen (1996), and Gouriéroux et al. (1993) systemize which moments to match in what has become known as the efficient method of moments (EMM) or indirect inference. This estimation procedure involves simulating data from the structural ABS with noise added as in Eqs. (20) and (21), compute simulated returns, $r_t = p_t/p_{t-1} - 1$, evaluate a score vector of an estimated auxiliary model with these simulated returns, and then use this vector as moment conditions to be minimized in order to obtain the unknown parameters of the ABS. Further details of the EMM estimation are given in the Appendix.

4. Empirical results

The complete ABS with noise is described by the price Eqs. (18) or (21) with the fundamental price processes (8) or (10), the forecast rules of the conditional price and variance, the evolutionary dynamics (14), and the fitness function (16). Our ambition is to estimate the two ABS specifications in order to more objectively investigate if any of the versions of the model is capable of describing real market data. First, we estimate an auxiliary GARCH model to be used in the EMM estimation of a two-agent ABS with the more complex stochastic structure in Eq. (21). Here, we are forced to economize on the number of parameters to estimate. Second, we do a maximum likelihood estimation of a similar ABS but with the additive noise distribution in Eq. (18) only, in order to see how much, if anything, is lost with this simpler approach. At this stage, we estimate a more flexible three-agent ABS to investigate the gain of this added complexity. With a pure additive noise term, the number of parameters is no longer a crucial constraint.

As previously mentioned, the choice of dividend processes used here imply that prices do not grow exponentially over the long run in sharp contrast to real prices, but by using a growing non-stationary dividend process, rapidly increasing prices can be obtained. In order to have a close connection to the theoretical results and earlier simulations, we hold on to the original formulation of the dividend processes without drift. This prevents us from analyzing the mean of the return series, but other statistical properties of the data can be investigated. In our empirical work we therefore use detrended daily data of the S&P 500 index from January 1980 to December 2000, that is we calculate a return series from the index, normalize it to zero mean, and re-compute index observations from the demeaned return series assuming an

¹² De Grauwe and Grimaldi (2003, 2006) add IID noise to their fundamental price (more specifically exchange rate) process regardless of its level, which implies that their simulated prices also experience the problem of moving as much when the price is low compared to when it is high. On the other hand, their prices are not constrained to be positive, and their returns are computed as price differences. This means that the noise is correctly modeled if their price is interpreted as the log of the price, but it is not obvious why, in their utility based exchange rate model, agents should agree upon the log of the price, and not the price itself.

¹³ Although possible, no restrictions on the size of the noise of the dividend process are imposed to try to match the variability of any observed dividend series. In other words, p_{rw} may take any value in order to fit the data. One could think of y_t in more general terms as an unobserved fundamental process consisting of dividends and measurement errors.

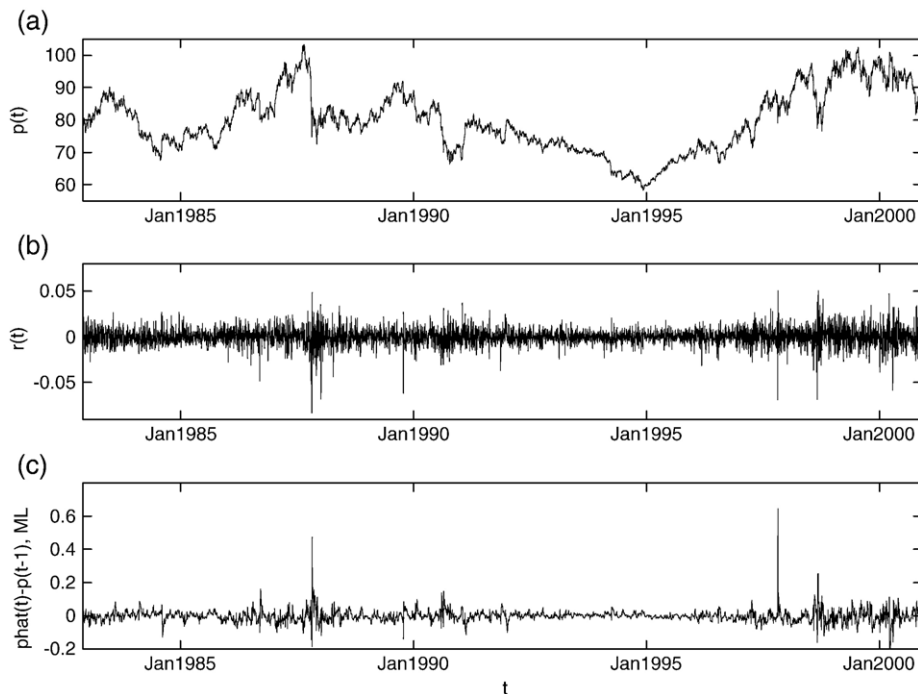


Fig. 1. (a) Daily S&P 500 detrended prices December 1982–December 2000. The observations belonging to the October 1987 crash and the two consecutive days are excluded. (b) Corresponding S&P 500 returns and (c) differences between the predicted price $\hat{p}_t$ from the model with a constant dividend in Eq. (18), and p_{t-1} .

arbitrary starting value of 100. This way of detrending the data assumes that prices, in addition to the time-variation in expected returns induced by the model, have a constant growth. If this is not the case, an alternative detrending method would, for instance, be to use a rolling window of suitable size to demean the return series which, in turn, could have a different impact on the results. The present method could in any case be seen as a reasonable first approximation. Fig. 1a and b show the detrended index series and the return series.¹⁴ It is evident from Fig. 1a that the variability of the price series depends on the index level, which would not have been the case for a price series generated from Eq. (17).

In order to capture as much of the (non-normal) characteristics of the data as possible it is important to have long time series. Still, a few events like the 1987 crash have a large impact on the statistical properties of the data set and the resulting estimation, see e.g. Engle and Lee (1994). Since, to our knowledge, no structural or statistical model has successfully captured these extreme observations jointly with the rest of the data, we choose to eliminate the observations belonging to the crash and the two consecutive days.¹⁵ Another reason is to facilitate the comparisons with the simulations in Brock et al. (2005), Gaunersdorfer and Hommes (2000) and Hommes (2001), who also exclude these data points from their sample. The volatility clustering and the extreme events are seen in Fig. 1b. Even after the censoring, it is quite a challenge to find a model that fits this data set.

4.1. ML estimation of the auxiliary model

Natural choices of auxiliary models are the ARCH-class models, initially proposed by Engle (1982). They can potentially capture many stylized facts of financial data, such as volatility clustering and asymmetry, and excess kurtosis. There have been many extensions of the original ARCH model, and several of them have proved to be quite successful in modeling many financial variables, not just equity indices, see e.g. the survey in Bollerslev et al. (1994). Important for our concern, they also have a well-defined likelihood function.

¹⁴ For reasons that will become clear later, the likelihood of the GARCH estimates is based on $t > 500$. The first two years of data (500 observations) are also excluded in Fig. 1. As can be inferred from Fig. 1, the detrended price declined from 100 to around 80 between January 1980 and January 1982.

¹⁵ The excluded returns are −20.4%, 5.3%, and 9.1%, resulting in a sample length of 4549 return observations.

Table 1
GARCH estimation with normally distributed errors

	β_1	α_0	α_2	α_3	α_L	$\omega \times 10^5$	γ_1	γ_2	γ_3
Estimate	0.049	0.011	0.591	0.169	0.064	0.164	0.799	−0.779	0.885
t-value	3.20	2.48	2.73	0.85	4.77	4.39	52.6	−34.0	37.2

The specification we use is the AR(o)-GARCH(p, q) process of Bollerslev (1986), extended with a leverage term to account for the fact that large downward moves tend to have larger effects on future volatility than upward moves of comparable size, see Pagan and Schwert (1990) for an early discussion how this asymmetry could be modelled. Our choice is similar in spirit to the one used in Engle and Lee (1994), who model the S&P 500 index on a daily frequency from 1971 to 1990:

$$\begin{aligned}
 r_t &= \beta_1 r_{t-1} + \varepsilon_t, \\
 \varepsilon_t &= \sigma_t u_t, \\
 g_t &= (\alpha_0 + \alpha_L D_t) \varepsilon_t^2 \\
 \sigma_t^2 &= \omega + \sum_{j=1}^p \gamma_j \sigma_{t-j}^2 + \sum_{j=1}^q \alpha_j g_{t-j},
 \end{aligned} \tag{22}$$

where $D_t = 1$ if $\varepsilon_t < 0$, zero otherwise, and $\alpha_1 \equiv 1$.¹⁶ A positive α_L thus reflects the leverage effect of the lagged squared innovations.

We allow the disturbances u_t to be generated by a standard Normal distribution, but a specification with the normalized Student- t distribution of Bollerslev (1987) was also tested. The Student- t distribution is capable of handling fatter tails of the distribution, but the estimation does not rely on the powerful QML results of Bollerslev and Wooldridge (1992): the QML theory applies if the assumed normality assumption is false, but not if an assumed Student- t distribution is misspecified. The Gaussian results may therefore be more robust which, in turn, could explain the worse fit of the EMM estimation of the structural model when using a GARCH-model with Student- t distributed errors as an auxiliary model. In the following, only estimates of the Gaussian case is presented.¹⁷

The GARCH estimation of the nine parameters are presented in Table 1. We use three lags in the conditional variance process, i.e. $p=q=3$, which is more than usually used in modeling asset returns. Almost all estimates are significant, but standard information criteria (Hannan–Quinn and Schwarz, but not Akaike) show signs of over-parameterization. However, there are two reasons for choosing this specification: first and foremost, the order condition of identification is that there are at least as many parameters in the auxiliary model as there are in the structural model. Second, the information criteria favor additional parameters in the conditional variance process compared to the conditional mean, and this specification maximizes the Hannan–Quinn and Schwarz information criteria for this particular number of parameters. The relevant persistence measure is $\sum \gamma_j + (1 + \sum \alpha_j)(\alpha_0 + \alpha_L/2)$, meaning that volatility forecasts decay with the power of 0.981. The significance of α_L supports the leverage effect in the data.

Table 2 takes a closer look at the observed data and the standardized residuals of the estimations. The Ljung–Box test indicates that the model successfully removes the dependences in the second moment, but the issue of non-normality remains. We see that the standardized residuals retain almost all skewness and a large part of the excess kurtosis of the original data. By comparing the sample moments by the moments predicted by the assumed distributions, it is clear that the model overestimates the skewness and underestimate the kurtosis. The implication is that more flexible distributions may be needed, like the exponential generalized beta family in Wang et al. (2001), despite the previously mentioned QML results.¹⁸

¹⁶ For numerical reasons, the threshold D_t is implemented with the differentiable function $d(x) = \exp(-K^*x)/(1 + \exp(-K^*x))$, $K=10,000$.

¹⁷ Interestingly, Engle and Lee (1994) also find that the use of an auxiliary Gaussian GARCH model, instead of one with Student- t distributed errors, in the EMM estimation of a stochastic volatility model, results in a model that is able to fit the excess kurtosis in the S&P 500 data better.

¹⁸ A further expansion into semi-nonparametric (SNP) conditional densities introduced in Gallant and Nychka (1987) is also possible, and applied, amongst others, to interest rate modeling in Andersen and Lund (1997).

Table 2

Moments and diagnostics of the standardized residuals and the S&P 500 return data

Data	SD	Skewness	Kurtosis	$Q(10)$	$Q^2(10)$
Std. res.	1.00	−0.43	6.36	12.8	2.95
<i>p</i> -value				0.24	0.98
S&P 500	0.0095	−0.46	8.65	40.2	697.3
<i>p</i> -value				0.00	0.00

Note: $Q(10)$ and $Q^2(10)$ are the Ljung–Box statistics for the return and squared return data with 10 lags, respectively. The *p*-values are of a χ^2_{10} distribution.

4.2. EMM and ML estimations of the structural model

To obtain our EMM estimates, the GARCH scores are evaluated with simulated data from the structural model as to minimize Eq. (24), see the Appendix. We choose the sample size to five times the length of the original data, $N = 5 \times T$.¹⁹ This is the effective sample size, that is after the first 2000 simulated values are discarded in order to let any initial effects to wear off.²⁰ It should be mentioned that we encountered some numerical difficulties. Different starting values resulted in different terminal parameter estimates, suggesting that the minimizing function is very flat, or has several local minima. The value of the objective function, and the statistical properties of the different solutions were quite similar though. A global optimization technique, such as simulated annealing (see Press et al., 1992; Goffe et al., 1994 for implementations and further references) would be useful, but is for the moment hardly not feasible due to computational limitations.

Identification requires that the parameters in the structural model to be less than in the auxiliary model and since EMM estimation is sensitive to overfitting of the auxiliary model, we restrict the number of parameters to estimate with EMM to eight and fix the rest to some plausible figures. The selected value of r corresponds to a yearly risk-free rate of 3.7%, which roughly equals the average of three month US Libor interest rates over the period, and the starting value of y_t is chosen so that the starting value of p_0^* equals the starting value of the detrended price process, 100, that is $y_0 = 3.7$.

When we use the simpler noise structure in Eq. (18) we can allow ourselves to estimate a larger number of parameters. In this case, we fix r to 3.7% as before and $\bar{y} \approx 3$ to give a fundamental price close to 80, which coincides with the average of the detrended price series in Fig. 1a. Apart from this, we estimate all 14 parameters of the three-agent model.

The estimates and corresponding standard errors of our two ABSs are displayed in Table 3. Starting with the ML estimates, we see that most of the parameters are highly significant. The mean-reverting parameter of the fundamentalists, v , is very small and significant and so is the extrapolation parameter, λ_{mo} , of the trend chasers. The third agent is, on the other hand, significantly identified as a contrarian ($\lambda_{co} < 0$). Interestingly, since ϕ is small and insignificant all agents share the same risk aversion, and there seems to be no need for a stabilizing force driving the chartists out of the market via Eq. (12). The success of contrarian strategies is in the literature documented for longer time horizons, while the momentum strategies seem to be more profitable on shorter horizons, see e.g. Chiarella and He (2002b). Here, the conclusions are somewhat mixed since the contrarians use a longer time-window (α_m -parameters closer to 0) than the trend chasers when they form their beliefs about the future price, while the opposite is true for the use of past price observations in the updates of their beliefs of the conditional variance (α_v). Furthermore, the information cost of the fundamentalists, C_F , is clearly identified and some weight is also attributed to past utilities, although the investors are restricted in this respect since η is identical for all trader types.

The significantly estimated parameters indicate that the three-agent model in Eq. (18) captures additional features in the data compared to a random walk, which the model nests. However, in spite of the significant estimates, the standard deviation of the noise term is not far from the standard deviation of the raw data in Table 2, which suggests a quite modest model fit. In fact, the model residuals calculated from Eq. (18), $(p_t - \hat{p}_{t-1})/p_{t-1}$, are virtually indistinguishable

¹⁹ It would be desirable to have an even larger N in order to reduce the Monte Carlo error. Unfortunately, the considerable computational time prevent us from this at the present but the chosen N is, for instance, substantially larger than what Engle and Lee (1994) use in their EMM estimation of a stochastic volatility model on S&P 500 data.

²⁰ We further discard the first 500 draws of the score vector because of possible initial transients. The same is also done when estimating W_T in (25).

Table 3

ML and EMM estimates of the ABS

Model	ν	λ_{mo}	λ_{co}	ϕ	a	$\rho (\times 10^3)$	$\rho_{rw} (\times 10^3)$	$\alpha_{m,mo}$	$\alpha_{m,co}$	$\alpha_{v,f}$	$\alpha_{v,mo}$	$\alpha_{v,co}$	β	C_f	η
ML	0.06	0.04	-0.40	0.01	1.00	9.47	—	0.55	0.10	0.01	0.27	0.50	1.91	0.19	0.04
S.E	0.01	0.00	0.00	0.00	0.01	0.00	—	0.02	0.01	0.01	0.03	0.05	0.02	0.02	0.00
EMM	0.55	0.11	—	0.07	1	5.25	6.36	0.84	—	0.42	0.42	—	1.99	0	0
S.E	0.75	0.98	—	0.25	—	2.58	0.83	2.13	—	0.22	0.22	—	20.44	—	—

Note: ν is the mean-reverting parameter in Eq. (11), λ_{mo} and λ_{co} are the extrapolation parameters in Eq. (11), ϕ measures the fundamentalist's time-variation in the risk aversion a in Eq. (12), ρ and ρ_{rw} are the standard deviations of the exogenous noise processes in Eqs. (20) and (21), the α -parameters are the forgetting factors of the different investors' belief of the conditional mean, Eq. (11), and variances, Eq. (13), β measures the intensity of choice and C_f is the information cost of the fundamentalists in Eq. (14), and η is the memory parameter in the performance measure Eq. (16). Parameters with no entries for standard errors are fixed in the estimations. In the EMM estimation $\alpha_{v,f}$ and $\alpha_{v,mo}$ are restricted to be identical.

from the return series in Fig. 1b. Furthermore, Fig. 1c plots the difference between the predicted price $\hat{p}_t$ and p_{t-1} , where the latter would be the predicted price if all non-linearities are turned off and prices just follow a random walk. Again, the differences are very small which suggests that the model in Eq. (18) does not add much to the overall price dynamics.

Fig. 2 shows a four-year subsample of deviations of the observed price from the constant fundamental price, together with the two chartists' market weights. Both have an average share of around 50% and the weight of the fundamentalists (not shown) is effectively zero throughout the whole sample. The dynamic of the chartists is quite hectic with fast switches between the two groups. No chartist group is dominating the market for long and during times of turbulence, the switching seem to increase even further. The fast switching is not a consequence of a three-agent model. Also a two-agent model (not shown) with one fundamentalist and a chartist behave in the same way. A fast switching between agent groups seem to be necessary to try to fit the observed data.

The EMM estimates of a two-agent system with fewer free parameters, but also a more complex disturbance structure in Eq. (21), show quite a different picture. Here, the risk aversion parameter is fixed but the fundamentalists

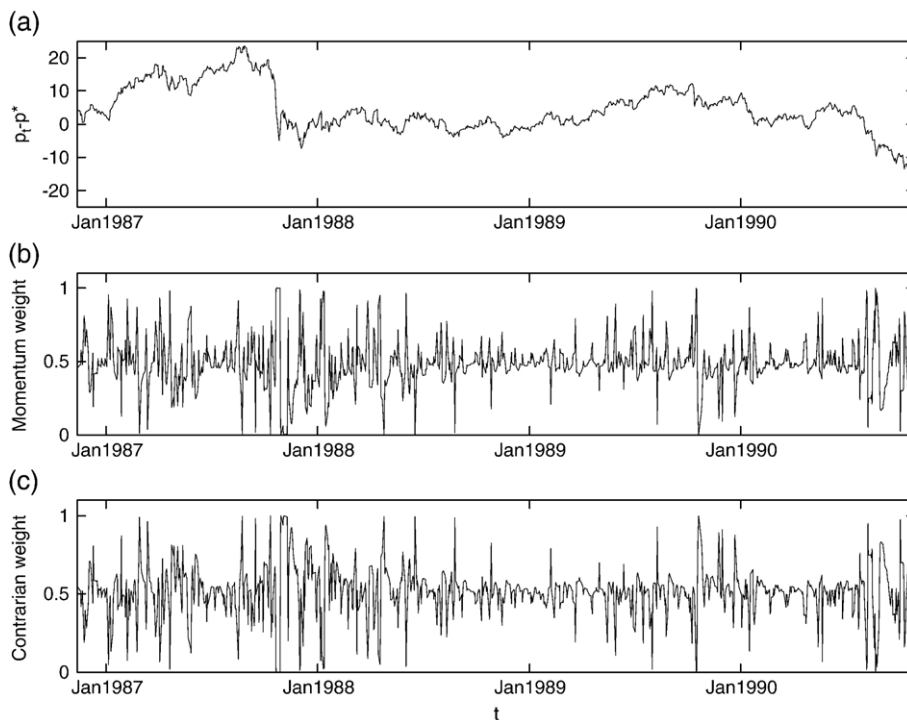


Fig. 2. (a) Four years (1000 observations) of daily differences between the observed price and the fundamental price ($p^* = 80$) from the ML estimation, and the corresponding weights of (b) the momentum traders, and (c) the contrarian traders.

are still allowed to differ by the influence of ϕ . In addition, both fundamentalists and trend chasers are assumed to share the same time-window in determining the time-varying risk in Eq. (13). The β -parameter that determines the speed of the switches between investor groups is of the same magnitude as in the earlier case, but it is no longer significant. In fact, no parameter is estimated to be significant at the 5% level, except for the variances of the two exogenous error terms! Although insignificant, the mean-reversion behavior of the fundamentalists is now much stronger, and the only group of technical traders consists in this case of momentum traders since the estimate of the extrapolation parameter is positive, 0.11.

Despite the disappointing findings from the EMM estimations, the top part of Fig. 3 shows a representative part of around four years (1000 daily prices) of the difference between the simulated prices in the EMM estimation, Eq. (21), and the fundamental prices $p_t^* = y_t/r$, from Eqs. (10) and (20). Periods of small price deviations are constantly interrupted by episodes where prices wander away from the fundamental price quite substantially and persistently. From the bottom part of the figure, we see that these movements coincide with the weight of the trend-followers: when the price deviations increase, the trend chasers tend to dominate the market and vice versa. In fact, these bubbles and crashes are typically triggered by exogenous noise and then further reinforced by the growing population of chartists until the price starts to move in the other direction, again triggered by a shock of such magnitude that the perception among the two investor groups of where the market is going is altered. This everlasting struggle among the investors determines the dynamics of the system and is the core of the model. When prices start to move away, say up, from the fundamental price, the fundamentalists expect a back-drop but are overwhelmed by the steadily increasing chartists group who profit from the price increase which, in turn, attracts more investors to become chartists in a self-fulfilling manner. When almost everyone has become a chartist, the upward pressure on prices slows down and the more pessimistic view of the fundamentalists becomes relatively important. A negative shock may then result in a price decline, which increases the profits of the fundamentalists, and so the journey back towards the fundamental price starts again. The market dynamics is here quite different compared to the former model specification with a only one noise process and a constant fundamental price, in which case no investor group dominated for long. It appears as if the

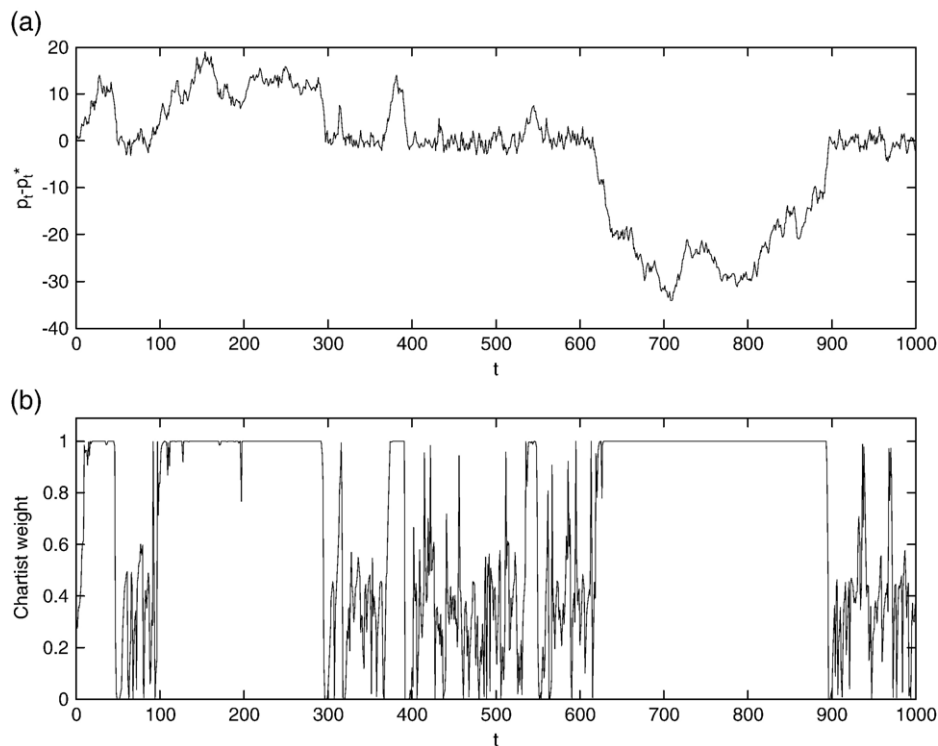


Fig. 3. (a) 1000 differences between the simulated and fundamental prices from the EMM estimation, and (b) the corresponding weights of the momentum traders.

Table 4

Moments and diagnostic statistics of the S&P 500 return series, the single simulated series in the EMM estimation, the 1000 series simulated from the EMM and ML estimates (EMM-MC and ML-MC), and the 1000 series from a Normal distribution (N-MC)

	SD ($\times 10^3$)	Skewness	Kurtosis	$Q(10)$	$Q^2(10)$	χ^2
S&P 500	9.50	−0.46	8.65	40.16	697.27	
<i>p</i> -value				0.00	0.00	
EMM	7.06	−0.014	3.22	24.73	182.45	29.27
<i>p</i> -value				0.01	0.00	0.00
EMM-MC	6.95	−0.017	3.16	14.36	32.59	
95% confidence interval	(6.4,7.3)	(−0.09,0.09)	(2.9,3.5)	(4.6,29.5)	(7.8,86.6)	
ML-MC	9.73	0.0034	3.19	36.07	34.99	
95% confidence interval	(9.3,10.7)	(−0.08,0.09)	(2.9,3.8)	(6.3,196.5)	(4.2,133.1)	
N-MC	9.50	0.0012	3.00	9.95	10.18	
95% confidence interval	(9.3,9.7)	(−0.07,0.07)	(2.9,3.2)	(3.4,20.8)	(3.4,21.6)	

Note: $Q(10)$ and $Q^2(10)$ are the Ljung–Box statistics for the return and squared return data with 10 lags, respectively. The test of overidentifying restrictions in the last column is distributed as χ^2_1 .

random fundamental price introduces an additional way of fitting the data and takes the pressure of the market participants to have to generate all movements in the observed prices themselves.

Such a boundedly rational exuberance is interesting per se, but it does not reveal if the resulting return series show any resemblances with true data. In addition, the fact that the parameters in the model with a simpler noise structure is estimated significantly, while they are not in the EMM estimations, does not in itself imply that the former model is superior. If the simpler model is misspecified, the standard errors cannot be trusted anyway. Since the model with the simpler noise term was non-robust (the significance of its parameters disappeared when the more complex noise structure was added), this may point towards a misspecification problem. Table 4 therefore shows a closer investigation of the behavior of the two model specifications. We see from the single simulated return series of length N from the EMM estimations that the model generates some non-normal behavior. The kurtosis is 3.2, which is clearly non-normal, but still far from the 8.6 of the S&P 500. The Ljung–Box portmanteau test for dependences in returns resembles that of the real data, while the same test statistic for dependences in squared returns, which often is regarded as an indication of ARCH-effects, although highly significant is lower than for the real data.

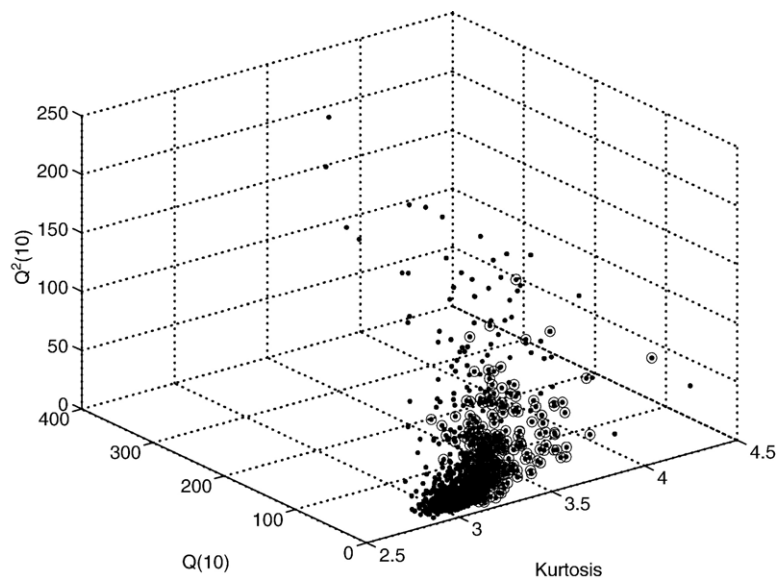


Fig. 4. A scatter plot of the kurtosis, and the Ljung–Box statistics with 10 lags in returns ($Q(10)$) and squared returns ($Q^2(10)$) of the 1000 simulated series in ML-MC. The simulations with a standard deviation between 0.0093 and 0.010 (and with a kurtosis above the mean of 3.19) are indicated with a circle.

Unfortunately, the small but promising features of the model do not come without a cost. The standard deviation of the EMM series is considerably less compared to the S&P 500. It is not obvious why the model in some sense trades the second moment for the fourth one, but a simulation of 1000 series of the same length as the original data (4549 return observations) with the EMM parameter estimates (denoted EMM-MC) confirms the smaller standard deviation and the excess kurtosis, although the kurtosis is not significantly different from 3 at the 5% level. Furthermore, we have nine parameters in the auxiliary GARCH model and eight parameters in the structural model, leaving one degree of freedom for the χ^2 test for an overidentifying restriction in Eq. (27). Not surprisingly, Table 4 provides striking evidence against the two-agent structural model with two exogenous noise processes in favor of the auxiliary GARCH model. The p -value is basically zero. Somewhat noteworthy, the quite large Ljung–Box test statistics of dependence in squared returns in the EMM estimation is no longer present in (any of) the 1000 simulated series.

Table 4 also shows the results from 1000 simulations from the three-agent model with the simpler noise structure (ML-MC). This specification is capable of generating excess kurtosis (albeit insignificantly at the 5% level) and other non-normal properties without a lower standard deviation, as in the EMM case; on the contrary, the second moment distribution is substantially positively skewed. In fact, for both EMM-MC and ML-MC the distributions of the other statistics (apart from the skewness) all have a longer right tail than what returns simulated from a Normal distribution (N-MC) would indicate.

However, it does not follow from Table 4 if the simulated series that have, say, excess kurtosis also exhibit volatility clustering and ARCH-effects, or if the desirable features cancel out. Fig. 4 therefore shows a three-dimensional scatter plot of the kurtosis and the two Ljung–Box statistics for the ML-MC series. The majority of the simulated returns displays no signs of any stylized facts but it is also apparent from the figure that some series have both excess kurtosis

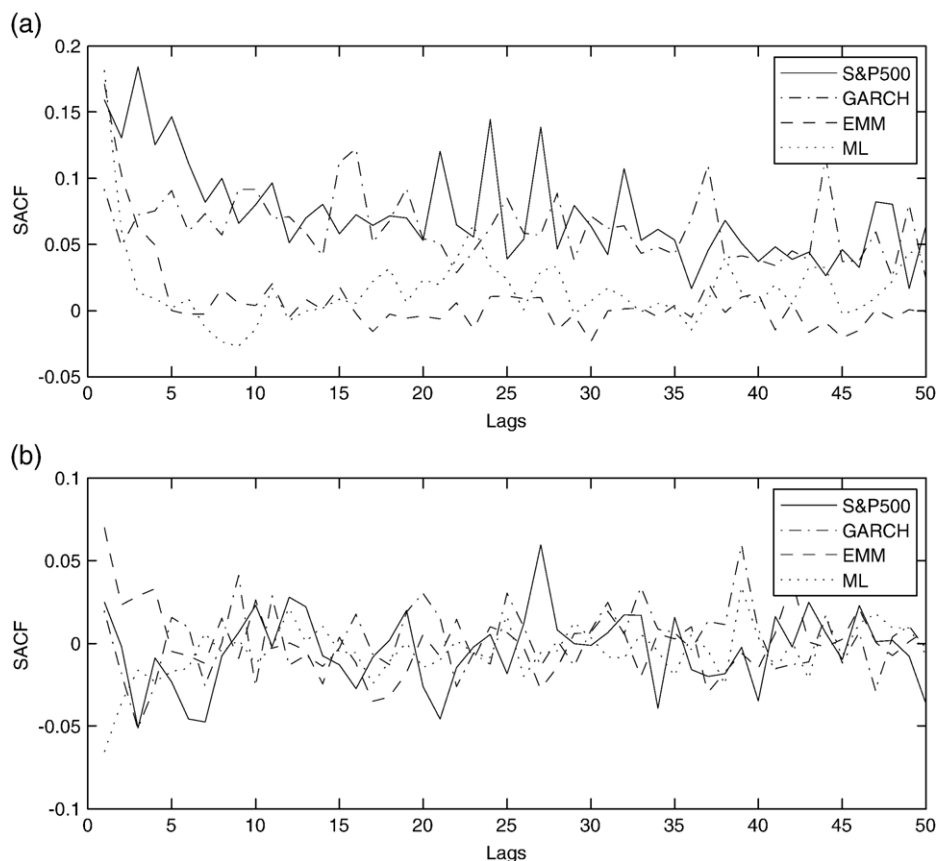


Fig. 5. (a) Sample autocorrelation functions (SACFs) of squared returns for the S&P 500, a simulated series from the auxiliary AR(1)-GARCH(3,3) model in (22), and two simulated series from the EMM-MC and ML-MC series in Table 4. (b) SACFs of returns for the same series as in (a).

and dependences in squared returns. Unfortunately, several series also have large Ljung–Box statistics in returns, and such autocorrelation pattern is not what we see in real markets. Many of the series that generates the non-normal behavior also have a standard deviation considerable larger than the mean of 0.0095. In Fig. 4 we therefore narrow the number of interesting series by indicating those simulations with a standard deviation between 0.0093 and 0.010 (and with a kurtosis above the mean of 3.19) with a circle. It is clearly seen that there are few, if any, series left that jointly exhibit excess kurtosis, a low autocorrelation in returns and volatility clustering in a way real returns series do. Although not displayed, a similar visual analysis of the EMM-MC series paints a similar picture.

Finally, in Fig. 5, we show the sample autocorrelation function of returns and squared returns for the S&P 500 data, a simulation of the auxiliary AR-GARCH model, and the series from EMM-MC and ML-MC that has the most promising combination of the stylized facts. For ML-MC, this would be the highest point with the circle in Fig. 4 ($SD=0.0099$, Kurtosis=3.5, $Q(10)=32$, $Q^2(10)=176$). We see that the autocorrelations of squared returns are quite different for the two structural models compared to the other series. It appears as if the Ljung–Box rejections in squared returns for the structural models stem from the first lags only, which is in stark contrast to the slow decay of the autocorrelation pattern in the squared S&P 500 and GARCH series. The sample autocorrelations in returns are quite small for all lags and for all series, but it looks like the pattern also differs somewhat for the structural models compared to the other two: the autocorrelation in the first lag is somewhat larger for the EMM series and appears also to be larger in magnitude, and negative, for the ML series.

To summarize, there is little doubt that the GARCH model more successfully captures many of the stylized facts of real market returns. It should be stressed that excess kurtosis and autocorrelation in the second moment still are possible to generate with the structural models, but only at the expense of a much worse fit to other moments, mainly an unrealistically large autocorrelation in returns. This is also the reason why these parameter configurations are rejected by the data.

5. Conclusions

In recent years, different structural models that try to explain the complex behavior of financial markets have been proposed. A class of models that have shown promising theoretical results are the ABSs, originating from Brock and Hommes (1997a), where heterogeneous agents equipped with different expectations determine the market price. A key feature is adaptation: a successful forecast rule will attract other investors, and vice versa. The quantitative aspects of the model are, however, not as carefully explored. The procedure has mostly been limited to comparing the size of different moments and the autocorrelation structure to some stock market indices, and by fitting GARCH models to simulated series, as in Chen et al. (2001) and Gaunersdorfer (2000). Here, we try to find out how well the proposed models can explain all the properties seen in real data, not just one or a few at a time.

Hence, in this paper we estimate two versions of an ABS by the use of maximum likelihood and the EMM technique of Gallant and Tauchen (1996) and Gouriéroux et al. (1993). We discover two main findings. First, the similarities with observed data found in earlier simulations based on Eq. (17) is to a large extent caused by the way noise is added to the model. If the noise is modeled as in Eq. (18) or (21) many of the documented stylized facts disappear. The noise term in Eq. (17) can be interpreted as a shock to the supply of shares or as a shock due to small fraction of random noise traders. This way of including noise in agent-based simulations models is not uncommon and may therefore be responsible for some of their successful results, but in large simulation models (often with many stochastic interacting variables) it is more complicated to find out exactly what drives the volatility clustering and other statistical non-normal properties. Second, when the stochastic is more properly introduced we find that the models are able to generate some stylized facts, but that the fit generally is quite poor. The results are in some sense disappointing since we cannot find an adequate fit to the observed data. On the other hand we should be encouraged, since the models under scrutiny are simple prototypes and still seem to have potential to explain some empirical facts. We did also discover local minima. It may therefore still be the case that there exists a global minimum that generates the desired real market behavior, but which we failed to find. In this respect, a global optimization algorithm would be most helpful, but unfortunately also very time consuming.

Furthermore, we estimate a heterogeneous agent model on 18 years of daily S&P 500 data, using both a simpler and another more complicated noise dynamics. Neither of the specifications seem to explain the observed data particularly well, which may be linked to the fact that the underlying model does not capture the clustered volatility well, compared e.g. to a GARCH model. Boswijk et al. (2005) have estimated a closely related 2-type heterogeneous agent model, with

a simpler noise term, on *yearly* S&P 500 data (130 observations) and obtained significant trend and mean reversion parameter estimates of the two trader types. They also found that the trend following agents were strongly dominating during the “dot com bubble” in the late nineties. Interestingly, yearly data show no signs of volatility clustering and much lower levels of kurtosis than present in the daily data. A possibility could thus be that this type of heterogeneous agent models does well in replicating the “bubble and crash” dynamics around a benchmark fundamental, at least if such dynamics do not imply excess kurtosis of any larger magnitudes, but is less successful in capturing the clustered volatility in the data. Of course, any success should ultimately be measured compared to other, possibly non-linear, statistical models that are designed to model deviations from a long-term fundamental level, such as (vector) error correction models (see e.g. Hamilton, 1994) and the smooth transition regression (STR) models of Teräsvirta (2004), which also could be used as auxiliary models in the EMM framework.

A potentially more serious shortcoming is that the model only involves a few trader types, while in reality there are many. It is straightforward to extend the model to a true multi-agent framework, such as Arthur et al. (1997) and Lux (1995), but at the loss of tractability and an increasing number of parameters. An elegant theory is developed in Brock et al. (2005), who introduces the Large Type Limit (LTL) system. The basic idea is that the many agents’ parameters are assumed to be drawn from some convenient distribution, typically a multivariate normal, thus reducing the degrees of freedom tremendously while still keeping the essence of a multi-agent model.

The estimation techniques used in this paper can be applied to the extended models as well. Hopefully, estimations similar to these will help future research to augment the models with features that match the observed data more closely. It is difficult to see, though, how this can be accomplished without a specification that more directly deals with a time-varying second moment. Alternatively, the possibility of using other processes, for instance a stochastic unit root process, see e.g. Yoon (2003), as a dividend process could also be a fruitful way of getting the models closer to reality.

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Appendix A

The key idea behind the efficient method of moment, which builds upon quasi maximum likelihood (QML) principles, is surprisingly simple: use the score of an auxiliary model (or score generator) evaluated under the structural model as the vector of moment conditions in order to calibrate the parameters of the structural model. The auxiliary model that generates the scores should approximate the actual distribution of the data closely, but it does not have to nest it. If it does, then one obtains ML efficiency. Furthermore, identification requires that the number of auxiliary model parameters is larger than those of the structural model.

To be more specific, suppose that the log likelihood function of the auxiliary model is $\frac{1}{N} \sum \ln f(r_t | X_t, \beta)$. This is not the true data generating process and the estimates, $\hat{\beta}$, may or may not be consistent. The data generating process is instead our structural model, parameterized by θ , and we assume that there is a value θ^0 such that the density of the observed data, r_t , is the same as the density of the simulated returns, $r_t(\theta^0)$. If we further assume the existence of a binding function, $\beta = b(\theta)$, we have that the unknown density of the structural model $p(r_t | X_t, \theta^0) = f(r_t(\theta^0) | X_t(\theta^0), \hat{\beta})$. The binding function defines β^0 , the quasi true vector, by $\beta^0 = b(\theta^0)$, from which it now follows that $\lim_{N \rightarrow \infty} \hat{\beta} = \beta^0$, the consistency result we need.

We can now simulate a time series of size N from the structural model, denoted $\{r_t(\theta), X_t(\theta)\}$, in order to generate the moment conditions:

$$m_N(\theta, \hat{\beta}) = \frac{1}{N} \sum_{t=1}^N \frac{\partial \ln f(r_t(\theta) | X_t(\theta), \hat{\beta})}{\beta}, \quad (23)$$

which converges to zero as $T \rightarrow \infty$ when $\theta = \theta^0$. This occurs because then $\hat{\beta}$ converges to β^0 which, in turn, is the QML estimate with first order condition given by Eq. (23). For other θ , Eq. (23) will not converge to zero. It is essential that N is large so that the Monte Carlo variance becomes negligible.

The EMM estimator of θ is defined by

$$\hat{\theta} = \arg \min_{\theta} m_N(\theta, \hat{\beta})' W_T^{-1} m_N(\theta, \hat{\beta}), \quad (24)$$

where W_T is a weighting matrix. Following GMM theory, the optimal choice of W_T is a consistent estimator of the asymptotic covariance matrix of the scores. If the auxiliary model is a reasonable approximation of the data, W_T is often estimated from the outer product gradient

$$W_T(\hat{\beta}) = \frac{1}{T} \sum_{t=1}^T \frac{\partial \ln f(r_t | X_t, \hat{\beta})}{\hat{\beta}} \frac{\partial \ln f(r_t | X_t, \hat{\beta})}{\hat{\beta}'}. \quad (25)$$

Most conveniently, W_T does not depend on the structural parameter vector θ .

The estimated asymptotic covariance matrix of $\hat{\theta}$ is

$$\text{Cov}(\hat{\theta}) = \frac{1}{T} \left(\frac{\partial m_N(\hat{\theta}, \hat{\beta})}{\partial \theta} W_T^{-1}(\hat{\beta}) \frac{\partial m_N(\hat{\theta}, \hat{\beta})}{\partial \theta'} \right)^{-1} = \frac{1}{T} (M_{\theta}' W_T^{-1}(\hat{\beta}) M_{\theta})^{-1}, \quad (26)$$

where the Jacobian M_{θ} in general must be computed numerically. A general specification test is also available. Under the null hypothesis that the structural model is true, T times the minimized value of the EMM criterion function is asymptotically distributed as χ^2 with degrees of freedom equal to the number of overidentifying restrictions:

$$T m_N(\hat{\theta}, \hat{\beta})' W_T^{-1} m_N(\hat{\theta}, \hat{\beta}) \rightarrow^d \chi_{\dim(\beta) - \dim(\theta)}^2. \quad (27)$$

References

- Alfarano, S., Lux, T., Wagener, F., 2005. Estimation of agent-based models: the case of an asymmetric herding model. *Computational Economics* 26, 19–49.
- Andersen, T., Lund, J., 1997. Estimating continuous-time stochastic volatility models of the short-term interest rate. *Journal of Econometrics* 343–377.
- Anderson, S., de Palma, A., Thisse, J., 1993. *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge.
- Arthur, W., Holland, J., LeBaron, B., Palmer, R., Taylor, P., 1997. Asset pricing under endogenous expectations in an artificial stock market. In: Arthur, W., Lane, D., Durlauf, S. (Eds.), *The Economy as an Evolving Complex System*, vol. II. Addison–Wesley, Reading.
- Baak, S., 1999. Test for bounded rationality with a linear dynamic model distorted by heterogeneous expectations. *Journal of Economic Dynamics and Control* 23, 1517–1543.
- Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 31, 307–327.
- Bollerslev, T., 1987. A conditional heteroskedastic time series model for speculative prices and rates of return. *Review of Economics and Statistics* 69, 542–547.
- Bollerslev, T., Wooldridge, J., 1992. Quasi-maximum likelihood estimation of dynamic models with time varying covariances. *Econometric Reviews* 11, 143–172.
- Bollerslev, T., Engle, R., Nelson, D., 1994. Arch models. In: Engle, R., McFadden, D. (Eds.), *Hand-book of Econometrics*, vol. 4. Elsevier, Amsterdam, pp. 2959–3038.
- Boswijk, H., Hommes, C., Manzan, S., 2005. Behavioral heterogeneity in stock prices. CeNDEF Working Paper, vol. 05-12. University of Amsterdam.
- Brock, W., Hommes, C., 1997a. A rational route to randomness. *Econometrica* 65, 1059–1095.
- Brock, W., Hommes, C., 1997b. Models of complexity in economics and finance. In: Hey, C., et al. (Ed.), *System Dynamics and Financial Models*. Wiley, pp. 3–41.
- Brock, W., Hommes, C., 1998. Heterogeneous beliefs and bifurcation routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control* 22, 1235–1274.
- Brock, W., Lakonishok, J., LeBaron, B., 1992. Simple technical trading rules and the stochastic properties of stock returns. *Journal of Finance* 47, 1731–1764.
- Brock, W., Hommes, C., Wagener, F., 2005. Evolutionary dynamics in financial markets with many trader types. *Journal of Mathematical Economics* 41, 7–42.

- Chavas, J., 2000. On information and market dynamics: the case of the U.S. beef market. *Journal of Economic Dynamics and Control* 24, 833–853.
- Chen, S.-H., Lux, T., Marchesi, M., 2001. Testing for non-linear structure in an artificial financial market. *Journal of Economic Behavior and Organization* 46, 327–342.
- Chiarella, C., He, X., 2002a. An adaptive model on asset pricing and wealth dynamics with heterogeneous trading strategies. *Quantitative Finance Research Group Research Paper*, vol. 84. University of Technology, Sydney.
- Chiarella, C., He, X., 2002b. Heterogeneous beliefs, risk and learning in a simple asset pricing model. *Computational Economics* 19, 95–132.
- Chiarella, C., He, X., 2003. Dynamics of beliefs and learning under aL-process — the heterogeneous case. *Journal of Economic Dynamics and Control* 27, 503–531.
- Dacorogna, M., Müller, U., Olsen, R., Pictet, O., 1998. Modelling short term volatility with GARCH and HARCH models. In: Dunis, C., Zhou, B. (Eds.), *Nonlinear Modelling of High Frequency Financial Time Series*. Wiley.
- De Grauwe, P., Grimaldi, M., 2003. Bubbling and crashing exchange rates. *CESifo Working Paper Series*, p. 1045.
- De Grauwe, P., Grimaldi, M., 2006. Exchange rate puzzles: a tale of switching attractors. *European Economic Review* 50, 1–33.
- De Long, J., Shleifer, A., Summers, L., Waldmann, R., 1990. Noise traders risk in financial markets. *Journal of Political Economy* 98, 703–738.
- Duffie, D., Singleton, K., 1993. Simulated moments estimations of Markov models of asset prices. *Econometrica* 61, 929–952.
- Engle, R., 1982. Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50, 987–1007.
- Engle, R., Lee, G., 1994. Estimating diffusion models of stochastic volatility. In: Rossi, P. (Ed.), *Modelling Stock Market Volatility: Bridging the Gap to Continuous Time*. Academic Press, San Diego.
- Evans, G., Honkapohja, S., 2001. *Learning and Expectations in Macroeconomics*. Princeton University Press.
- Fama, E., 1970. Efficient capital markets: a review of theory and empirical work. *Journal of Finance* 25, 383–417.
- Frankel, J., Froot, K., 1990. Chartists, fundamentalists, and trading in the foreign exchange market. *American Economic Review* 80, 181–186.
- Gallant, R., Nychka, D., 1987. Semi-nonparametric maximum likelihood estimation. *Econometrica* 55, 363–390.
- Gallant, R., Tauchen, G., 1996. Which moments to match? *Econometric Theory* 12, 657–681.
- Gaunersdorfer, A., 2000. Endogenous fluctuations in a simple asset pricing model with heterogeneous agents. *Journal of Economic Dynamics and Control* 24, 799–831.
- Gaunersdorfer, A., 2001. Adaptive beliefs and the volatility of asset prices. *Central European Journal of Operations Research* 5–30.
- Gaunersdorfer, A., Hommes, C., 2000. A nonlinear structural model for volatility clustering. *CeNDEF Working Paper*. University of Amsterdam. 00-02.
- Gaunersdorfer, A., Hommes, C., Wagener, F., 2000. Bifurcation routes to volatility clustering. *CeNDEF Working Paper*. University of Amsterdam. 00-04.
- Gilli, M., Winker, P., 2003. A global optimization heuristic for estimating agent-based models. *Computational Statistics & Data Analysis* 42, 299–312.
- Goffé, W., Ferrier, G., Rogers, J., 1994. Global optimization of statistical functions with simulated annealing. *Journal of Econometrics* 60, 65–99.
- Gourieroux, C., Monfort, A., Renault, E., 1993. Indirect inference. *Journal of Applied Econometrics* 8, 85–119.
- Granger, C., Swanson, N., 1997. An introduction to stochastic unit root processes. *Journal of Econometrics* 80, 35–61.
- Hamilton, J., 1994. *Time Series Models*. Princeton University Press.
- Hansen, L., 1982. Large sample properties of generalized method of moments estimators. *Econometrica* 50, 1029–1054.
- Hommes, C., 2001. Financial markets as nonlinear adaptive evolutionary systems. *Quantitative Finance* 1, 149–167.
- Hommes, C., 2002. Modeling the stylized facts in finance through simple nonlinear adaptive systems. *Proceedings of the National Academy of Sciences* 99, 7221–7228.
- Hommes, C., 2006. Heterogeneous agent models in economics and finance. In: Tesfatsion, L., Judd, K. (Eds.), *Handbook of Computational Economics*, vol. 2: Agent-Based Computational Economics. Elsevier/North Holland, Amsterdam, pp. 1109–1186.
- Jegadeesh, N., Titman, S., 2001. Profitability of momentum strategies: an evaluation of alternative explanations. *Journal of Finance* 56, 699–720.
- Kirman, A., 1991. Epidemics of opinion and speculative bubbles in financial markets. In: Taylor, M. (Ed.), *Money and Financial Markets*. Macmillan, New York.
- LeBaron, B., 2001a. Evolution and time horizons in an agent based stock market. *Macroeconomic Dynamics* 5, 225–254.
- LeBaron, B., 2001b. Empirical regularities from interacting long and short horizon investors in an agent-based stock market. *IEEE Transactions on Evolutionary Computation* 5, 442–455.
- LeBaron, B., 2006. Agent-based computational finance. In: Tesfatsion, L., Judd, K. (Eds.), *Hand-book of Computational Economics*, vol. 2: Agent-based Computational Economics. Elsevier/North Holland, Amsterdam, pp. 1187–1233.
- LeBaron, B., Arthur, W., Palmer, R., 1999. Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control* 23, 1487–1516.
- Lux, T., 1995. Herd behavior, bubbles and crashes. *Economic Journal* 105, 881–896.
- Lux, T., Marchesi, M., 2000. Volatility clustering in financial markets: a microsimulation of interacting agents. *International Journal of Theoretical and Applied Finance* 3, 675–702.
- Manski, C., McFadden, D., 1981. *Structural Analysis of Discrete Data with Econometric Applications*. MIT Press, Cambridge.
- Manzan, S., 2003. Essays in nonlinear economic dynamics. PhD thesis Tinbergen Institute Research Series; 317; University of Amsterdam.
- Müller, U., Dacorogna, M., Dave, R., Olsen, R., Pictet, O., von Weizsäcker, J., 1997. Volatility of different time resolutions — analyzing the dynamics of market components. *Journal of Empirical Finance* 4, 213–239.
- Muth, J., 1961. Rational expectations and the theory of price movements. *Econometrica* 29, 315–335.
- Pagan, A., 1996. The econometrics of financial markets. *Journal of Empirical Finance* 3, 15–102.
- Pagan, A., Schwert, G., 1990. Alternative models for conditional stock volatility. *Journal of Econometrics* 45, 267–290.

- Press, W., Flannery, B., Teukolsky, S., Vetterling, W., 1992. *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, New York.
- Routledge, B., 1999. Adaptive learning in financial markets. *Review of Financial Studies* 12, 1165–1202.
- Sargent, T., 1993. *Bounded Rationality in Macroeconomics*. Clarendon Press, Oxford.
- Shefrin, H., 2000. *Beyond Greed and Fear: Understanding Behavioral Finance and the Psychology of Investing*. Harvard Business School Press, Boston.
- Teräsvirta, T., 2004. Smooth transition regression modelling. In: Lutkepohl, H., Krätzig, K. (Eds.), *Applied Time Series Econometrics*. Cambridge University Press, Cambridge.
- Vigfusson, R., 1997. Switching between chartists and fundamentals: a Markov regime-switching approach. *International Journal of Finance and Economics* 2, 291–305.
- Wang, K.-L., Fawson, C., Barrett, C., McDonald, J., 2001. A flexible parametric GARCH model with an application to exchanges rates. *Journal of Applied Econometrics* 16, 521–536.
- Westerhoff, F., Reitz, S., 2003. Nonlinearities and cyclical behavior: the role of chartists and fundamentalists. *Studies in Nonlinear Dynamics and Econometrics* 7 (4) (article no. 3).
- Winker, P., Gilli, M., 2001. Indirect estimation of the parameters of agent based models of financial markets. FAME Research Paper, vol. 38. University of Geneva.
- Yoon, G., 2003. A simple model that generates stylized facts of returns. Pusan National University, Korea, UCSD Working Paper. San Diego, CA.