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It takes a model to beat a model: Volatility bounds ☆

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Abstract

Models like the CAPM and Fama–French three-factor models are commonly used as benchmarks for calculating cost of capital and evaluating portfolio performance, despite the empirical evidence to reject them. For many practical purposes, "it takes a model to beat a model." In this paper we derive restrictions on models that could "beat" a bench-mark model but might still be misspecified. In these "takes-a-model-to-beat-a-model" (TMBM) bounds, model A beats model B if model A's quadratic form of pricing errors is smaller. The bounds generalize the Hansen–Jagannathan bound and distance measure. We use the TMBM bounds to evaluate various linear factor models and consumption-based models. The failure of the power utility model is much less extreme when it is compared with the CAPM and Fama–French model. For reasonable utility curvature, the Ferson–Constantinides model and Epstein–Zin model perform best among the consumption-based models, beating the model of Campbell and Cochrane, in which model the value of the persistence parameter that matches the time-series properties of aggregate stock market returns seems too low for cross-sectional asset pricing.

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Keywords: Asset Pricing Model; Stochastic discount factor; Volatility bound; Hansen-Jagannathan; GMM

1. Introduction

Various models have been developed to explain asset returns, for example, the Capital Asset Pricing Model (CAPM) by Sharpe (1964) and Lintner (1965), the intertemporal CAPM by Merton (1973), the consumption CAPM (CCAPM) by Rubinstein (1976), Lucas (1978) and Breeden (1979), the Arbitrage Pricing Theory by Ross (1976), and the Fama–French three-factor model (Fama and French, 1993, 1996). The literature on empirical asset pricing has tested these models extensively. Almost all the models so far are found to be rejected using stock market data.

On the practical side, popular models like the CAPM and Fama–French model are commonly used as benchmarks for evaluating portfolio performance and calculating cost of capital, despite the fact that they fail to pass certain statistical

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tests. No model is perfect, and for many purposes, "it takes a model to beat a model". It is therefore meaningful for researchers to develop models that might not work perfectly but could beat popular benchmark models. In this paper we investigate the restrictions on such models. We represent models in the form of stochastic discount factors (SDFs) and derive bounds on the volatility of their SDFs similar to Hansen and Jagannathan (1991), who derive bounds for models that exactly price asset returns. We call our bounds "takes-a-model-to-beat-a-model" (TMBM) bounds.

Previous studies use the Hansen–Jagannathan bound as a diagnostic tool to test for perfect models. For example, Hansen and Jagannathan (1991) provide a characterization of the equity premium puzzle (Mehra and Prescott, 1985) by showing that the SDF of the basic power utility model cannot lie above the Hansen–Jagannathan bound unless the relative risk aversion (RRA) coefficient is unreasonably large.[1] The TMBM bounds are designed for *comparing* models. For example, we revisit the equity premium puzzle and evaluate the power utility model relative to the CAPM and Fama–French model. We find that it takes a substantially smaller RRA coefficient for the SDF to lie above the TMBM bounds than the Hansen–Jagannathan bound. Hence, the failure of the power utility model is much less extreme when we examine its performance relative to other models.

The TMBM bounds also generalize the Hansen–Jagannathan distance measure. Previous studies compare models using the distance measure developed by Hansen and Jagannathan (1997).[2] The distance between a model's TMBM bounds and the Hansen–Jagannathan bound is a measure of the model's misspecification. Hence, models can be directly compared using their TMBM bounds. In Fig. 1 we plot the TMBM bounds for the CAPM, Fama–French three-factor model, Carhart four-factor model, and the linearized version of the CCAPM.[3] The first apparent feature is that all the TMBM bounds are well below the Hansen–Jagannathan bound and the differences among the TMBM bounds appear small compared with their distances from the Hansen–Jagannathan bound, suggesting that all the models have a long way to go in explaining asset returns. Furthermore, Fig. 1 shows that the Carhart model and Fama–French model dominate the other models. The improvement of the Fama–French model over the CAPM is much more evident than the improvement of the Carhart model over the Fama–French model. In all the panels, the performance of the linearized CCAPM is the worst.

This paper also makes a methodological contribution by generalizing Hansen's (1982) asymptotic distribution for the Generalized Method of Moment (GMM) estimator. Hansen's (1982) theory is based on the assumption that certain population moment conditions hold at the true parameter values. In the context of testing asset pricing models, this amounts to assuming that the models are correct. In our context of comparing misspecified models, however, this is inappropriate. We derive the asymptotic distribution without assuming that the population moment conditions hold and show that Hansen's (1982) asymptotic distribution is a special case.

Our extension of Hansen and Jagannathan's (1991) analysis is different from all previous attempts. The previous studies *tighten* the Hansen–Jagannathan bound using additional information (e.g., Balduzzi and Kallal, 1997; Kan and Zhou, 2003; Ferson and Siegel, 2003; Bekaert and Liu, in press).[4] Tighter bounds present greater challenges for asset pricing models and would eliminate more candidate models. Given that many economically interesting models fail the original Hansen–Jagannathan bound, we take the opposite approach. We essentially *loosen* instead of *tighten* the Hansen–Jagannathan bound.

For illustration, we compare various consumption-based models using the TMBM bounds. Apart from the power utility model, we also consider the non-expected-utility model developed by Epstein and Zin (1989), and the habit-formation models by Abel (1990), Ferson and Constantinides (1991), and Campbell and Cochrane (1999). We find that the Ferson–Constantinides model and Epstein–Zin model perform best and the outperformance is more evident when

[1] See Mehra and Prescott (1985), Kocherlakota (1996), Cochrane (2001), and Mehra and Prescott (2002) for surveys on the equity premium puzzle. Other examples that test asset pricing models using the Hansen–Jagannathan bound include Cochrane and Hansen (1992), Burnside (1994), Cecchetti et al. (1994), Hansen et al. (1995), Heaton (1995), and Ferson and Siegel (2003).

[2] Hansen and Jagannathan (1997) compare some consumption-based models and linear factor models using the distance measure. Hodrick and Zhang (2001) conduct a more comprehensive comparison of linear factor models. Other such examples of using the distance measure to compare models include Jagannathan and Wang (1996), Campbell and Cochrane (2000), Lettau and Ludvigson (2001), Dittmar (2002), Farnsworth et al. (2002), Kan and Zhou (2002). In particular, Jagannathan and Wang (1996) derive the asymptotic distribution for the Hansen–Jagannathan distance measure. Kan and Zhou (2002) derive the exact distribution in finite samples and provide geometric interpretations.

[3] The three factors in the Fama–French model are the excess return on a market portfolio, the difference between the returns on a small-firm portfolio and big-firm portfolio, and the difference between the returns on a high-book/market-ratio portfolio and low-book/market-ratio portfolio. The Carhart model has an additional momentum factor. The linearized CCAPM has the consumption growth rate as the only factor.

[4] Balduzzi and Kallal (1997) use information from risk premiums for economic variables. Kan and Zhou (2003) improve the bounds when SDFs are functions of observable state variables. Ferson and Siegel (2003) and Bekaert and Liu (in press) show how conditioning information might be incorporated to tighten the bounds. Snow (1991) takes a different approach in extending Hansen and Jagannathan's (1991) analysis. He derives the restrictions on higher-order norms of an SDF.

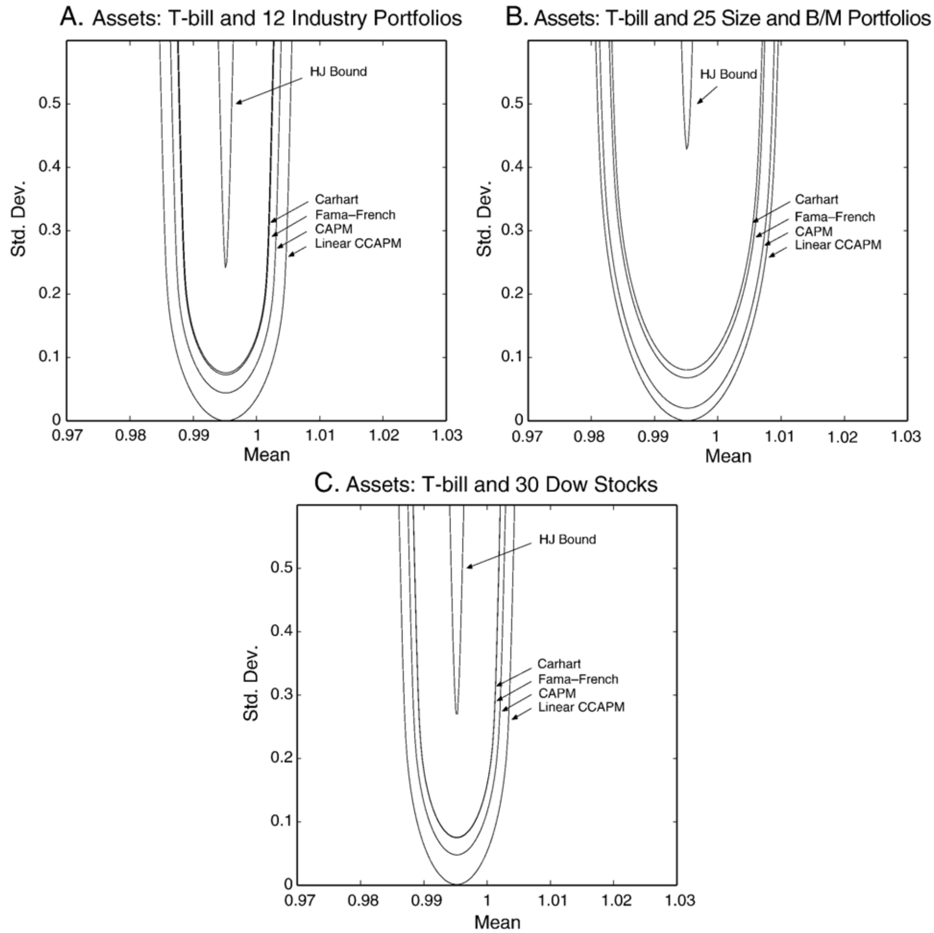

Fig. 1. TMBM bounds for linear factor models. The weighting matrix is $\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the covariance matrix or the asset returns. w is 0.0015 in Panel A, 0.001 in Panel B, and 0.000925 in Panel C. β, the time discount factor for the power utility model, is chosen so as to minimize $\alpha'\mathbf{V}^{-1}\alpha$ for each value of γ, the utility curvature parameter. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in all panels, we use gross return for the one-month T-bill and excess returns for the risky assets.

the utility curvature parameters of all the models are restricted to reasonable values. We explore further the Campbell–Cochrane model, which is designed to explain some time-series properties of the aggregate stock market. We find that the degree of persistence for the surplus consumption ratio is very important for cross-sectional pricing. The value that matches the serial correlation of the price/dividend ratio for a stock index implies a level of risk premium that is too high for a good fit to the cross-section of expected returns.

The rest of the paper is organized as follows. In Section 2 we derive the TMBM bounds and discuss their properties. Section 3 describes the econometric framework. Section 4 describes the data. Section 5 presents the empirical results. We conclude the paper in Section 6.

2. Derivation of the TMBM bounds

Our derivation of the TMBM bounds is based on the SDF representation of asset pricing models. Let $\mathbf{R}_t$ denote the vector of the payoffs or returns on a given set of assets. An SDF that prices the assets exactly assigns a vector of prices $\mathbf{p}$ to $\mathbf{R}_t$, i.e.,

$$E(M_t\mathbf{R}_t) = \mathbf{p}. \tag{1}$$

The price for any asset, $\mathbf{p}$, is 1 if its $\mathbf{R}_t$ is a gross return and 0 if an excess return. Let μ and Ω denote the mean vector and covariance matrix of $\mathbf{R}_t$, respectively.

Hansen and Jagannathan (1991) show that for any SDF that prices the assets exactly and has a mean of v, there is a lower bound on its standard deviation:

$$\sigma_{\mathrm{HJ}}(v) = [(v\mu - \mathbf{p})'\boldsymbol{\Omega}^{-1}(v\mu - \mathbf{p})]^{\frac{1}{2}}. \tag{2}$$

By varying v, Hansen and Jagannathan trace out a cup in the mean-standard-deviation space for SDFs. Any perfect SDF must lie above the cup, i.e., the Hansen–Jagannathan bound.

We define α, the vector of pricing errors for any SDF M_t as[5]

$$\alpha \equiv E(M_t\mathbf{R}_t) - \mathbf{p}. \tag{3}$$

Empirical studies typically evaluate asset pricing models by aggregating their α's into quadratic forms such as $\alpha'\mathbf{V}^{-1}\alpha$, where $\mathbf{V}$ is a symmetric and positive-definite matrix.

The quadratic form $\alpha'\mathbf{V}^{-1}\alpha$ is extensively studied in the empirical asset pricing literature. It is the key component of many test statistics, for example, the F statistic in MacKinlay (1987), Jobson and Korkie (1982) and Gibbons et al. (1989), the likelihood ratio test statistic in Jobson and Korkie (1982) and the cross-sectional regression test statistic in Shanken (1985). The quadratic form in the F statistic and likelihood ratio is $\alpha_p'\Sigma^{-1}\alpha_p$, where α_p is the vector of alphas calculated with respect to a market index p, and Σ is the residual covariance matrix in the market model. Shanken's (1985) test is based on $\alpha_e'\Omega^{-1}\alpha_e$, where α_e is the deviation from the security market line calculated with respect to a market index.[6]

It is useful to start by constructing "projected" SDFs. For any candidate SDF M_t, let v, σ, and α denote its mean, standard deviation and pricing errors, respectively. It is clear from Eq. (3) that the component of M_t that is orthogonal to $\mathbf{R}_t$ does not affect α, but would increase σ. Proposition 1 presents the "projected" version of M_t, a random variable that has the same mean and pricing error as M_t but the smallest possible volatility.

Proposition 1. *For any given v and α, the SDF*

$$M_{p,t} = v + (\alpha + \mathbf{p} - v\mu)'\boldsymbol{\Omega}^{-1}(\mathbf{R}_t - \mu) \tag{4}$$

has mean v and pricing errors α, and the minimum variance among all SDFs with mean v and pricing errors α.

A formal proof is given in the Appendix. Intuitively, $M_{p,t}$ is the projection of M_t onto a constant and $\mathbf{R}_t$. For any asset pricing model, its projected SDF summarizes all its pricing implications.

The standard deviation of the projected SDF $M_{p,t}$ is

$$\sigma_p = [(\alpha + \mathbf{p} - v\mu)'\boldsymbol{\Omega}^{-1}(\alpha + \mathbf{p} - v\mu)]^{\frac{1}{2}}. \tag{5}$$

σ_p can be interpreted as proportional to a maximum Sharpe ratio. Re-write Eq. (5) as

$$\frac{\sigma_p}{v} = \left[\left(\mu - \frac{\alpha}{v} - \frac{\mathbf{p}}{v}\right)'\boldsymbol{\Omega}^{-1}\left(\mu - \frac{\alpha}{v} - \frac{\mathbf{p}}{v}\right)\right]^{\frac{1}{2}}. \tag{6}$$

α/v are the pricing errors for expected returns (described in Footnote 5), and $\mu-\alpha/v$ are the expected returns predicted by the model. When $\mathbf{R}_t$ is excess returns, $\mathbf{p}$ is a vector of zeros and $\mu-\alpha/v-\mathbf{p}/v$ would be the expected excess returns predicted by the model. Hence, σ_p/v is the maximum Sharpe ratio obtained in a world in which investors believe that the model is right.[7] The same interpretation applies when $\mathbf{R}_t$ is gross returns.[8]

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[5] Alternatively, one can define the pricing errors in terms of the expected return predicted by the SDF M_t, i.e., $\alpha^{\mathbf{R}}=E(\mathbf{R}_t)-[\mathbf{p}-\mathrm{cov}(M_t, \mathbf{R}_t)]/E(M_t)$. The two definitions are closely related as $\alpha^{\mathbf{R}}=\alpha/E(M_t)$. Hansen and Jagannathan (1997) discuss the relation between their distance measure which is based on α and the Shanken's (1987) error bound which is based on $\alpha^{\mathbf{R}}$. We derive the TMBM bounds based on $\alpha^{\mathbf{R}}$ in the Appendix.

[6] When asset pricing models are tested using Hansen's (1982) GMM, the J statistic is $\alpha'\mathbf{S}^{-1}\alpha$, where the pricing errors α are the moment conditions and $\mathbf{S}$ is the spectral density matrix at frequency zero for the moment conditions. $(\alpha'\boldsymbol{\Omega}^{-1}\alpha)^{1/2}$ is one version of Hansen and Jagannathan's (1997) distance measure.

[7] Alternatively, if an econometrician estimates mean returns using the asset pricing model rather than sample mean returns (for example, Jorion, 1991, and MacKinlay and Pastor, 2000), the sample version of σ_p/v would be the maximum Sharpe ratio based on these mean estimates.

[8] In this case $\mathbf{p}$ is a vector of ones. Note that $1/v$ is the risk-free rate predicted by the model. Again $\mu-\alpha/v-\mathbf{p}/v$ is the expected excess returns predicted by the model.

When the model prices the assets exactly, α is zero and Eq. (5) becomes

$$\sigma_p = [(v\mu - \mathbf{p})' \boldsymbol{\Omega}^{-1}(v\mu - \mathbf{p})]^{\frac{1}{2}}, \tag{7}$$

equal to the Hansen–Jagannathan bound. Hence, the projected SDFs for perfect models must lie exactly on the Hansen–Jagannathan bound, and the Hansen–Jagannathan bound is proportional to the maximum Sharpe ratio.

2.1. The TMBM bounds

Let $M_{b,t}$ denote the SDF for a benchmark model and M_t the SDF for a candidate model that could "beat" the benchmark model by having a smaller quadratic form of pricing errors, i.e.,

$$\alpha' \mathbf{V}^{-1} \alpha \leq \alpha_b' \mathbf{V}^{-1} \alpha_b, \tag{8}$$

where α_b is the vector of pricing errors produced by $M_{b,t}$. The TMBM bounds represent the restrictions on M_t's mean and standard deviation, v and σ.

The TMBM bounds are obtained by searching for the maximum and minimum values for the volatility of the projected SDF in Eq. (5), subject to the constraint in Eq. (8). Formally,

$$\max_{\alpha}(\alpha + \mathbf{p} - v\mu)' \boldsymbol{\Omega}^{-1}(\alpha + \mathbf{p} - v\mu) \tag{9}$$

and

$$\min_{\alpha}(\alpha + \mathbf{p} - v\mu)' \boldsymbol{\Omega}^{-1}(\alpha + \mathbf{p} - v\mu), \tag{10}$$

both subject to

$$\alpha' \mathbf{V}^{-1} \alpha \leq \alpha_b' \mathbf{V}^{-1} \alpha_b. \tag{11}$$

The solution to such optimization problem with both quadratic objective and constraint is discussed by Gander (1981).[9] Before presenting the solutions in Proposition 2, we first define a few quantities to simplify the notation. Let

$$e_{\max} = \max_{\lambda \in \Lambda} f(\lambda) \tag{12}$$

$$e_{\min} = \min_{\lambda \in \Lambda} f(\lambda), \tag{13}$$

where

$$f(\lambda) = (v\mu - \mathbf{p})' [\mathbf{V}(\mathbf{V} + \lambda\boldsymbol{\Omega})^{-1} - \mathbf{I}]' \boldsymbol{\Omega}^{-1} [\mathbf{V}(\mathbf{V} + \lambda\boldsymbol{\Omega})^{-1} - \mathbf{I}](v\mu - \mathbf{p}) \tag{14}$$

and $\boldsymbol{\Lambda}$ is the set of the roots λ for the following equation:

$$(v\mu - \mathbf{p})' (\mathbf{V} + \lambda\boldsymbol{\Omega})^{-1} \mathbf{V} (\mathbf{V} + \lambda\boldsymbol{\Omega})^{-1} (v\mu - \mathbf{p}) = \alpha_b' \mathbf{V}^{-1} \alpha_b. \tag{15}$$

$f(\lambda)$ is the value of the objective function $(\alpha + \mathbf{p} - v\mu)' \boldsymbol{\Omega}^{-1}(\alpha + \mathbf{p} - v\mu)$ at optimal α, which is substituted out as a function of the Lagrangian multiplier λ. Eq. (15) results from the first-order conditions with α substituted out. Eventually we will solve for λ from Eq. (15).

[9] I would like to thank an anonymous referee for pointing to this strand of literature.

Proposition 2. *For any projected SDF that has mean v and pricing errors α with $\alpha' V^{-1}\alpha \leq \alpha_b' V^{-1}\alpha_b$, its standard deviation is bounded between σ_U and σ_L, where the upper bound is*

$$\sigma_U(v) = e_{\max}^{\frac{1}{2}} \tag{16}$$

and the lower bound is

$$\sigma_L(v) = \begin{cases} e_{\min}^{\frac{1}{2}} & for(v\mu - \mathbf{p})'\mathbf{V}^{-1}(v\mu - \mathbf{p}) \geq \alpha_b'\mathbf{V}^{-1}\alpha_b \\ 0 & for(v\mu - \mathbf{p})'\mathbf{V}^{-1}(v\mu - \mathbf{p}) < \alpha_b'\mathbf{V}^{-1}\alpha_b \end{cases}. \tag{17}$$

The proof is in the Appendix.

In Eq. (17), the lower TMBM bound $\sigma_L(v)$ is zero if $(v\mu-\mathbf{p})'\mathbf{V}^{-1}(v\mu-\mathbf{p})$ is smaller than $\alpha_b'\mathbf{V}^{-1}\alpha_b$. $(v\mu-\mathbf{p})'\mathbf{V}^{-1}(v\mu-\mathbf{p})$ is the quadratic form produced by a constant SDF, $M_t=v$, which holds in a risk neutral world.[10] If the constant SDF can beat the benchmark model, it must be at or above the lower TMBM bound. The volatility of the constant SDF is zero. Hence, the lower TMBM bound must also be zero.

To solve for the TMBM bounds, we need to find the Lagrangian multipliers λ from Eq. (15), which is a polynomial of order of up to $2n$, where n is the number of assets.[11] For a general weighting matrix $\mathbf{V}$, the solution is only available numerically.[12] In our empirical analysis, we consider $\mathbf{V}$ of the form:

$$\mathbf{V} = wE(\mathbf{R}_t\mathbf{R}_t') + (1-w)\boldsymbol{\Omega}. \tag{18}$$

In the Appendix, we show that with this specification, Eq. (15) is a fourth-order polynomial in λ and the TMBM bounds can be solved in closed form, as described in the Appendix. We discuss some additional reasons for choosing this specification later.

Proposition 2 shows that the projected SDFs for models that could beat a benchmark model must lie between an upper and a lower bound, $\sigma_U(v)$ and $\sigma_L(v)$, respectively. The intuition for this result can be obtained by using the interpretation of Eq. (5) as proportional to the maximum Sharpe ratio implied by the model. If a candidate model fits the data well and its quadratic form of pricing errors is smaller than the benchmark model's, the maximum Sharpe ratio that it implies should not be too high or too low relative to the maximum Sharpe ratio in the data.

The TMBM bounds in Proposition 2 are for projected SDFs. For general SDFs, Proposition 3 shows that there is only a lower bound and it is the same as the lower TMBM bound in Proposition 2. The absence of the upper bound should not be surprising given Propositions 1 and 2. For a given projected SDF, we can always construct a new SDF by adding to it a random variable that is orthogonal to $\mathbf{R}_t$ and has arbitrarily large variance. The new SDF would have the same pricing errors as the projected SDF but arbitrarily large variance.

Proposition 3. *For any SDF that has mean v and pricing errors α with $\alpha' V^{-1}\alpha \leq \alpha_b' V^{-1}\alpha_b$, its standard deviation is at least as great as that in Eq.* (17).

2.2. The TMBM bounds and Hansen–Jagannathan bound

Since by definition a perfect model prices all the assets correctly and can beat any benchmark model, except in the case that the benchmark model is also perfect, the projected SDFs for perfect models must lie between the upper and lower TMBM bounds. We have shown that the projected SDFs for perfect models must lie exactly on the Hansen–Jagannathan bound. Thus we have

[10] A constant SDF with mean v is just v itself, i.e., $M_t=v$. Its pricing error is $E(M_t\mathbf{R}_t)-\mathbf{p}=v\mu-\mathbf{p}$, and thus its quadratic form is $(v\mu-\mathbf{p})'\mathbf{V}^{-1}(v\mu-\mathbf{p})$.

[11] Gander (1981) showed that using some singular value decomposition techniques equations like Eq. (15) can be written more explicitly as a rational function of λ, which could be then transformed into a polynomial of order of up to $2n$. I would again like to thank the referee for pointing this issue out.

[12] Golub and von Matt (1991) proposed some numerical solutions for such problem.

Corollary 1. *The Hansen–Jagannathan bound lies between the upper and lower TMBM bounds, i.e.,*

$$\sigma_L(v) \leq \sigma_{HJ}(v) \leq \sigma_U(v).$$

Cecchetti et al. (1994), Burnside (1994) and Heaton (1995) test asset pricing models by comparing the volatility of their SDFs with the Hansen–Jagannathan bound. We can test whether a candidate model can beat a benchmark model by comparing the volatility of the candidate SDF with the TMBM bounds for the benchmark model. If the volatility is smaller than the lower TMBM bound, we can conclude that the candidate model cannot beat the benchmark model. Previous studies find that the Hansen–Jagannathan bound eliminates many economically interesting models such as the power utility model with reasonable RRA coefficients (see, for example, Hansen and Jagannathan, 1991). Corollary 1 shows that the lower TMBM bound is below the Hansen–Jagannathan bound. This suggests that models that do not survive the Hansen–Jagannathan bound might survive the lower TMBM bound.

When the benchmark model is perfect, any candidate model that could beat it must also be perfect. Hence, the projected SDFs for such candidate models can only lie exactly on the Hansen–Jagannathan bound and the TMBM bounds are the same as the Hansen–Jagannathan bound.

Corollary 2. *The upper and lower TMBM bounds for a perfect benchmark model are the same as the Hansen–Jagannathan bound, i.e.,*

$$\sigma_U(v) = \sigma_L(v) = \sigma_{HJ}(v).$$

Hence we conclude that the TMBM bounds generalize the Hansen–Jagannathan bound and reduce to it when the benchmark model is perfect.

2.3. The TMBM bounds and Hansen–Jagannathan distance measure

The next two corollaries go a step further and show that the TMBM bounds also generalize the Hansen–Jagannathan distance measure.

Corollary 3. *When $V = \Omega$ and assuming the benchmark model can beat the constant SDF,*

$$\sigma_{\mathrm{U}}(v) - \sigma_{HJ}(v) = \sigma_{HJ}(v) - \sigma_L(v) = (\alpha_b' \, \boldsymbol{\Omega}^{-1}\alpha_b)^{\frac{1}{2}}. \tag{19}$$

Corollary 4. *When $\boldsymbol{V} = wE(\boldsymbol{R}_t\boldsymbol{R}_t') + (1-w)\boldsymbol{\Omega}$, $\mathbf{p}$ is a vector of zeros, and $v = 1$, assuming the benchmark model can beat the constant SDF,*[13]

$$\sigma_U(1) - \sigma_{HJ}(1) = \sigma_{HJ}(1) - \sigma_L(1) = (\alpha_b' \, \mathbf{V}^{-1}\alpha_b)^{\frac{1}{2}}[1 + w(\mu' \boldsymbol{\Omega}^{-1}\mu)]^{\frac{1}{2}}. \tag{20}$$

Hansen and Jagannathan (1997) show that, if the benchmark model can correctly price constant payoffs, $(\alpha_b{}' \boldsymbol{\Omega}^{-1}\alpha_b)^{\frac{1}{2}}$ measures the distance between the SDF of the benchmark model and the set of SDFs for correct models. Eqs. (19) and (20) show that the distances between the TMBM bounds and the Hansen–Jagannathan bound are closely related to the benchmark model's Hansen–Jagannathan distance measure. The greater the distance measure, the lower the lower TMBM bound and the higher the upper TMBM bound. In particular, in Corollary 3, since the distance measure $(\alpha_b' \, \boldsymbol{\Omega}^{-1}\alpha_b)^{\frac{1}{2}}$ does not depend on the mean of the SDF v, the TMBM bounds and Hansen–Jagannathan bound must be parallel to one another. Hence, we can judge two models' performance simply by comparing their TMBM bounds. A similar intuition applies to a general weighting matrix $\mathbf{V}$. Hence, the TMBM bounds are a useful tool for comparing models under a general weighting matrix, especially when a weighting matrix other than $\boldsymbol{\Omega}$ is used, in which cases the Hansen–Jagannathan distance measure cannot be directly used.

[13] The case that $\mathbf{p}$ is a vector of zeros and $v = 1$ could arise when econometricians are not concerned about models' ability to price correctly the risk-free rate and only examine models' ability to price excess returns, for example, Balduzzi and Kallal (1997) and Kirby (1998). In these studies, the returns on the test assets are excess returns and hence $\mathbf{p}$ is a vector of zeros. As a result, the means of SDFs cannot be identified from the data. So the econometricians usually normalize the SDFs such that their means are 1, i.e., $v = 1$.

3. The econometric framework

We estimate asset pricing models using the Generalized Method of Moment (GMM). However, to avoid assuming that models are correct, we derive a generalization of Hansen's (1982). In this section, we first discuss the choice of the weighting matrix and then show how estimation and hypothesis testing are conducted.

3.1. The weighting matrix

For comparing asset pricing models, the second moment matrix and covariance matrix of the returns on test assets, i.e., $E(\mathbf{R}_t\mathbf{R}_t')$ and $\mathbf{\Omega}$, are two common choices for the weighting matrix.[14] Hansen and Jagannathan (1997) show that $\alpha' E(\mathbf{R}_t\mathbf{R}_t')^{-1}\alpha$ measures the minimum squared distance between the SDF and the set of perfect SDFs, and $\alpha'\mathbf{\Omega}^{-1}\alpha$ has similar interpretation when the SDF has the same mean as the perfect SDFs.

The difference between $E(\mathbf{R}_t\mathbf{R}_t')$ and $\mathbf{\Omega}$ is perhaps subtle but important for comparing models. It is well documented in the literature that the power utility models fail to price correctly both the market risk premium and risk-free rate, i.e., the equity premium puzzle and risk-free puzzle. Mehra and Prescott (1985) find that the power utility model is unable to match the observed market equity premium unless the RRA coefficient is unreasonably high.

Weil (1989) points out that the power utility model with a high RRA coefficient would predict a risk-free that is too low. Later studies develop different utility functions, aiming to resolve both the puzzles. For example, the non-expected utility by Epstein and Zin (1989) separates risk aversion from the elasticity of intertemporal substitution in the hope that the model can match both the risk-free rate and equity risk premium.

We argue that the choice of the weighting matrix $\mathbf{V}$ represents the econometrician's tradeoff between his desire to resolve the equity premium puzzle and risk-free rate puzzle. In this paper we use the one-month T-bill as a proxy for risk-free asset. When $\mathbf{\Omega}$ is the weighting matrix, the T-bill with its small volatility receives a large weight. Consider an example of only two test assets, the T-bill and market portfolio. The standard deviation of the excess return on the market portfolio is around 4.37% per month, about 20 times as large as that of the T-bill return (0.22% per month). Ignoring their correlation, the T-bill would be 20 times as important as the market portfolio in determining $\alpha'\mathbf{\Omega}^{-1}\alpha$ for the same size pricing errors. Thus, by setting $\mathbf{V}=\mathbf{\Omega}$, asset pricing models would be evaluated almost entirely on their ability to price the T-bill. Whether or not they assign the right risk premium to risky assets correctly becomes inessential. On the other hand, when $E(\mathbf{R}_t\mathbf{R}_t')$ is the weighting matrix, the weights for the T-bill and market portfolio are much more comparable, and asset pricing models can be evaluated based on their ability to match both the risk-free rate and market risk premium.

We consider the weighting matrix $\mathbf{V}$ of the following form:

$$V = wE(\mathbf{R}_t\mathbf{R}_t') + (1-w)\mathbf{\Omega} \tag{21}$$

for two reasons. First, as discussed before, the TMBM bounds can be solved in closed form with this specification. Second, we can select a desired level of tradeoff between the equity premium puzzle and risk-free rate puzzle through the choice of w. A w close to 0 would represent a strong emphasis on resolving the risk-free rate puzzle and little on the equity premium puzzle. As w increases, the risky assets receive larger weights and the emphasis on the equity premium puzzle gets stronger.

To see more specifically how important the T-bill is for a given w, we follow an approach used by Cochrane (1996). We decompose $\mathbf{V}^{-1}$ as $\mathbf{V}^{-1}=\mathbf{Q\Lambda Q}'$, where $\mathbf{Q}$ is the orthonormal matrix with the eigenvectors of $\mathbf{V}^{-1}$ in its columns and $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues. We can form a set of mutually uncorrelated portfolios of $\mathbf{R}_t$, $\mathbf{R}_t^*=\mathbf{Q}'\mathbf{R}_t$. Their pricing errors are $\alpha_t^*=\mathbf{Q}'\alpha_t$. The quadratic form $\alpha'\mathbf{V}^{-1}\alpha=\alpha^{*\prime}\mathbf{\Lambda}\alpha^*$. Since usually the first (largest) eigenvalue dominates the rest, the first portfolio in $\mathbf{R}_t^*$, i.e. $\mathbf{Q}_1'\mathbf{R}_t$, where $\mathbf{Q}_1$ is the first column in $\mathbf{Q}$, is the most important in determining the quadratic form. We can assess the importance of the T-bill by examining its weight in this portfolio.

[14] The studies that *test* asset pricing models often use the GMM optimal weighting matrix $\mathbf{S}$, the spectral density matrix at frequency zero for the moment conditions. Typically, the moment conditions are the models' pricing errors α. However, as Hansen and Jagannathan (1997) point out, $\alpha'\mathbf{S}^{-1}\alpha$ is not an appropriate metric for *comparing* asset pricing models. $\mathbf{S}$ is different for different models. A model might be strongly favored under this metric by blowing up $\mathbf{S}$ rather than having small pricing errors.

Table 1
Weight of the T-bill in $\mathbf{Q}_1'\mathbf{R}_t$

w	Weight
A: Assets: T-bill and 12 industry portfolios	
0	0.9967
0.0001	0.9971
0.001	0.0946
0.0015	0.0764
0.01	0.0474
0.1	0.0430
0.5	0.0426
1	0.0426
B: Assets: T-bill and 25 size and B/M portfolios1	
0	0.9937
0.0001	4.9777
0.001	0.0383
0.01	0.0093
0.1	0.0067
0.5	0.0064
1	0.0064
C: Assets: T-bill and 30 Dow stocks	
0	0.9945
0.0001	0.9949
0.000925	0.0316
0.001	0.0186
0.01	0.0070
0.1	0.0076
0.5	0.0076
1	0.0076

This table reports the weight of the T-bill in $\mathbf{Q}_1'\mathbf{R}_t$, where $\mathbf{R}_t$ are the returns on the test assets and $\mathbf{Q}_1$ is the first eigenvector of $\mathbf{V}^{-1}$. $\mathbf{V}$ is the weighting matrix in $\alpha'\mathbf{V}^{-1}\alpha$ and $\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\mathbf{\Omega}$. The weights are obtained by rescaling $\mathbf{Q}_1$ so that it sums up to 1. We use the monthly data for 1960:1-1999:12 (480 months). For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets.

Table 1 reports the T-bill's weight under different values of w. We rescale $\mathbf{Q}_1$ so that all its elements sum up to 1 and can be interpreted as portfolio weights. We find that the T-bill's weight varies a lot as w varies. When $w=0$, the weight is close to 1 and thus the T-bill dominates the quadratic form. As w increases, the weight decreases sharply. For example, as w reaches 0.01, the weight drops below 5% in Panel A, 1% in Panel B, and 2% in Panel C.

The choice of w is up to the econometrician. One sensible choice is to view the T-bill as of similar importance to an average risky asset.[15] This suggests that we choose the value of w at which the T-bill's weight is roughly equal to the inverse of the total number of test assets. For example, in Panel A there are 13 test assets (the T-bill and 12 industry portfolios). We choose w to be 0.0015, at which the T-bill's weight is 0.0764, roughly 1/13. We choose w to be 0.001 in Panel B, where the test assets are the T-bill and 25 size and book/market portfolios, and 0.000925 in Panel C, where the test assets are the T-bill and 30 Dow stocks.

In Fig. 2 we plot the TMBM bounds for the CAPM with different w and also the Hansen–Jagannathan bounds. The shapes of the TMBM bounds are very similar across the panels. When $w=1$, the TMBM bounds are very flat. In this case $\mathbf{V}$ is equal to the second moment matrix of asset returns. As discussed before, the quadratic form places a relatively small weight on the T-bill. Consequently, models that have the wrong means receive little punishment and their SDFs can easily lie above the TMBM bounds. When $w=0$, $\mathbf{V}$ is equal to the covariance matrix. The bounds are parallel to the Hansen–Jagannathan bounds, as predicted by Corollary 3. The TMBM bounds are so narrow that an SDF can only lie above them if its mean is in the neighborhood of 0.995, which is roughly equal to the inverse of the mean gross return

[15] This is essentially the view taken by Cochrane (1996), who estimates asset pricing models using the identity matrix in the quadratic form metric that GMM tries to minimize. However, Hodrick and Zhang (2001) find that an identity matrix leads to very imprecise parameter estimates in their GMM estimation of linear asset pricing models.

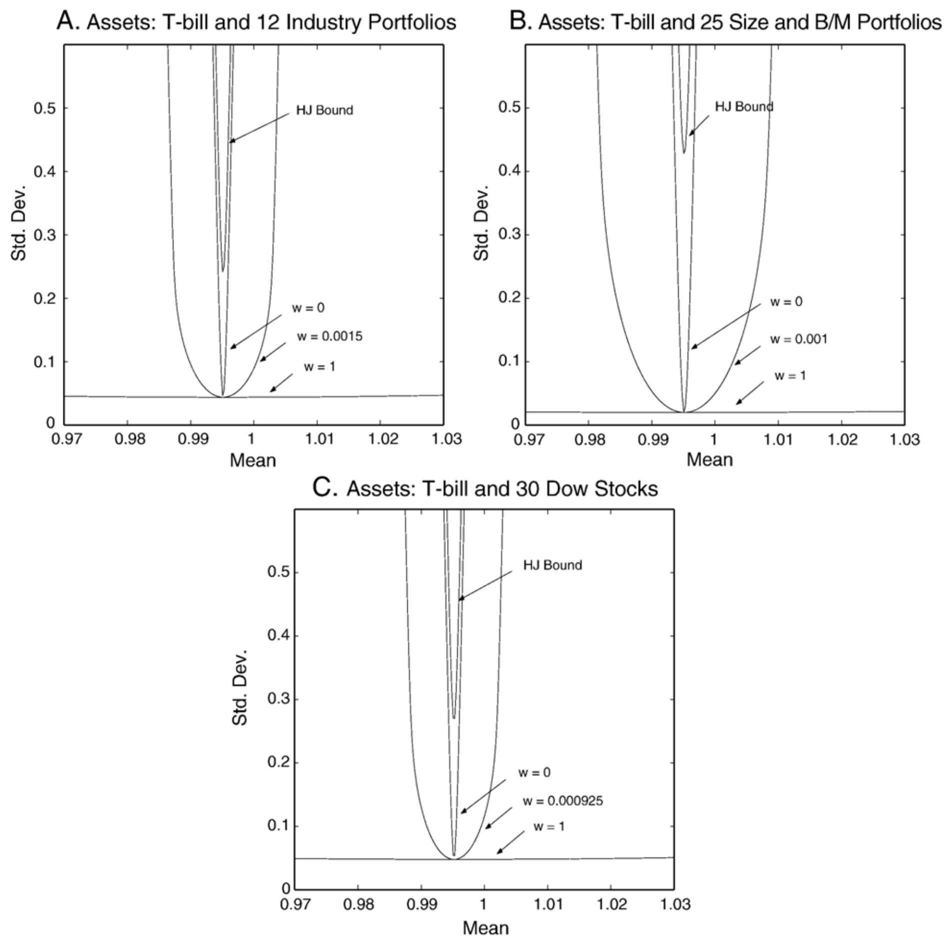

Fig. 2. TMBM bounds for CAPM under different weighting matrices. The time discount factor for the power utility model, β, is set to 1. γ is the utility curvature parameter. The weighting matrix is $\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\boldsymbol{\Omega}$. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets.

on the T-bill. When w is chosen such that the T-bill's weight is roughly the same as an average risky asset, the TMBM bounds lie between the bounds with w of 0 and 1.

3.2. Estimating the TMBM bounds

With $\mathbf{V}$ given by Eq. (21), the TMBM bounds can be solved analytically. The derivations are shown in the Appendix. The expressions for the TMBM bounds involve the roots of a fourth-order polynomial. However, for a given value of v, they are functions of the following four quantities: $\mu'\boldsymbol{\Omega}^{-1}\mu$, $\mu'\boldsymbol{\Omega}^{-1}\mathbf{p}$, $\mathbf{p}'\boldsymbol{\Omega}^{-1}\mathbf{p}$, and $\alpha_b'\mathbf{V}^{-1}\alpha_b$.

The first three quantities can be viewed as the SDF counterparts to the familiar mean-variance efficient-set constants. Roll (1977) shows that the mean-variance efficient frontier can be constructed from the constants $\mu'\boldsymbol{\Omega}^{-1}\mu$, $\mu'\boldsymbol{\Omega}^{-1}\mathbf{1}$, and $\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}$. As discussed earlier, Hansen and Jagannathan (1991) reveal a duality between the mean-variance efficient frontier and the Hansen–Jagannathan bound. Therefore, for a given v, $\mu'\boldsymbol{\Omega}^{-1}\mu$, $\mu'\boldsymbol{\Omega}^{-1}\mathbf{p}$, and $\mathbf{p}'\boldsymbol{\Omega}^{-1}\mathbf{p}$ are sufficient to construct the Hansen–Jagannathan bound. The fourth quantity, $\alpha_b'\mathbf{V}^{-1}\alpha_b$, is the quadratic form of pricing errors for the benchmark model.

To estimate the TMBM bounds, we simply estimate the four quantities by GMM. The standard errors for the TMBM bounds can be obtained using the delta method. An obvious strategy to estimate the four quantities is to estimate μ, $\boldsymbol{\Omega}$, and α_b separately using GMM, which is the approach taken by Cecchetti et al. (1994) and Burnside (1994). However this approach is undesirable in our context since the GMM system would involve a large number of moment conditions, mainly due to the estimation of $\mathbf{\Omega}$. For example, with 26 test assets (e.g., 25 size and book/market portfolios plus the T-bill), 351 moments would be needed to estimate $\mathbf{\Omega}$. With a sample of 40 years of monthly data (480 months), $\mathbf{S}$ would be nearly singular and its inversion would be extremely unstable and inaccurate. As a result, the GMM standard errors could be highly unreliable.

We bypass this problem by defining the following quantities:

$$\eta_1 = \boldsymbol{\Pi}^{-1}\mu \tag{22}$$

$$\eta_2 = \boldsymbol{\Pi}^{-1}\mathbf{p} \tag{23}$$

$$\eta_3 = \boldsymbol{\Pi}^{-1}\alpha_b \tag{24}$$

$$\zeta_1 = \mu'\boldsymbol{\Pi}^{-1}\mu, \tag{25}$$

where $\mathbf{\Pi}=E(\mathbf{R}_t\mathbf{R}_t')$. Now, $\mu'\mathbf{\Omega}^{-1}\mu$, $\mu'\mathbf{\Omega}^{-1}\mathbf{p}$, $\mathbf{p}'\mathbf{\Omega}^{-1}\mathbf{p}$, and $\alpha_b'\mathbf{V}^{-1}\alpha_b$ can be written as functions of α_b, η_1, η_2, η_3, and ζ_1 as[16]

$$\mu'\boldsymbol{\Omega}^{-1}\mu = \frac{\zeta_1}{1-\zeta_1} \tag{26}$$

$$\mu'\boldsymbol{\Omega}^{-1}\mathbf{p} = \frac{\eta_1'\,\mathbf{p}}{1-\zeta_1} \tag{27}$$

$$p'\boldsymbol{\Omega}^{-1}\mathbf{p} = \eta_2'\,\mathbf{p} + \frac{(\eta_1'\,\mathbf{p})^2}{1-\zeta_1} \tag{28}$$

$$\alpha_b'\,\mathbf{V}^{-1}\alpha_b = \eta_3'\,\alpha_b + \frac{(1-w)(\eta_1'\,\alpha_b)^2}{1-(1-w)\zeta_1}. \tag{29}$$

α_b, η_1, η_2, η_3, and ζ_1 can be estimated from the following moment conditions:

$$g_{1T} = E_T(M_{bt}\mathbf{R}_t - \mathbf{p}) \tag{30}$$

$$g_{2T} = E_T(\mathbf{R}_t\mathbf{R}_t'\,\eta_1 - \mathbf{R}_t) \tag{31}$$

$$g_{3T} = E_T(\mathbf{R}_t\mathbf{R}_t'\,\eta_2 - \mathbf{p}) \tag{32}$$

$$g_{4T} = E_T(\mathbf{R}_t\mathbf{R}_t'\,\eta_3 - M_{bt}\mathbf{R}_t + \mathbf{p}) \tag{33}$$

$$g_{5T} = E_T(\eta_1'\,\mathbf{R}_t\mathbf{R}_t'\,\eta_1 - \zeta_1), \tag{34}$$

where E_T denotes sample average, i.e., $\frac{1}{T}\sum_t(\cdot)$. Eq. (30) is α_b. Eqs. (31)–(34) exactly identify η_1, η_2, η_3, and ζ_1. The dimension of this GMM system is much smaller than what it would have been if we estimated μ, $\mathbf{\Omega}$, and α_b separately.

[16] The derivation is straightforward and based on the facts that

$$\boldsymbol{\Omega}^{-1} = \boldsymbol{\Pi}^{-1} + \frac{1}{1-\mu'\boldsymbol{\Pi}^{-1}\mu}\boldsymbol{\Pi}^{-1}\mu\mu'\boldsymbol{\Pi}^{-1}$$
$$\mathbf{V}^{-1} = \boldsymbol{\Pi}^{-1} + \frac{1-w}{1-(1-w)(\mu'\boldsymbol{\Pi}^{-1}\mu)}\boldsymbol{\Pi}^{-1}\mu\mu'\boldsymbol{\Pi}^{-1}.$$

For example, after obtaining the GMM estimates for α_b, η_1, η_2, η_3, and ζ_1, we can plug them into Eqs. (26)–(29) and compute the value of the TMBM bounds for any v. To trace out the whole bounds in the mean-standard-deviation space, we vary v over a plausible range of values.

As a byproduct, we can also estimate the Hansen–Jagannathan bound using η_1, η_2, η_3, and ζ_1. For a given v, the Hansen–Jagannathan bound can be written as

$$(v\mu-\mathbf{p})'\boldsymbol{\Omega}^{-1}(v\mu-\mathbf{p}) = v^2\zeta_1+\eta_2'\mathbf{p}-2v\eta_1'\mathbf{p}+\frac{(v\zeta_1-\eta_1'\mathbf{p})^2}{1-\zeta_1}. \tag{35}$$

3.3. Estimating the benchmark SDF

If there are unknown parameters in the benchmark SDF M_{bt}, we can estimate them by minimizing the quadratic form $\mathbf{g}_{1T}'\mathbf{V}^{-1}\mathbf{g}_{1T}$. Thus, the parameter estimates are chosen such that the benchmark model fits the data as best as it can.[17]

3.4. Testing with the TMBM bounds

Propositions 2 and 3) provide the basis for testing whether a candidate model could beat a benchmark model. We use the GMM framework to test whether the projected and original SDFs for the candidate model lie above or below the TMBM bounds.

For any given candidate asset pricing model, let v_M and σ_M denote the mean and standard deviation of its SDF M_t, respectively. Let σ_p in Eq. (5) denote the standard deviation of its projected SDF $M_{p,t}$. We need to compare σ_p and σ_M with $\sigma_{\mathrm{L}}(v_M)$, the value of the lower TMBM bound at v_M.

We use the following moment conditions in addition to Eqs. (30)–(34):

$$g_{6T} = E_T(M_t\mathbf{R}_t-\mathbf{p}) \tag{36}$$

$$g_{7T} = E_T(M_t-v_m) \tag{37}$$

$$g_{8T} = E_T(M_t^2-v_M^2-\sigma_M^2). \tag{38}$$

Eq. (36) is α, the pricing errors of M_t. Eq. (36) allows us to estimate the unknown parameters in M_t. Eqs. (37) and (38) exactly identify v_M and σ_M. To estimate σ_p, we also define

$$\eta_4 = \boldsymbol{\Pi}^{-1}\alpha. \tag{39}$$

σ_p in Eq. (5) can be rewritten as a function of η_1, η_2, η_4, and α as

$$\sigma_p^2 = \eta_4'\alpha+\eta_2'p+2\eta_4'\mathbf{p}+\frac{(\eta_1'\alpha)^2+(\eta_1'\mathbf{p})^2+v^2\zeta_1+2(\eta_2'\mathbf{p})(\eta_1'\alpha)-2\eta_1'\alpha-2v\eta_1'\mathbf{p}}{1-\zeta_1}. \tag{40}$$

We append the following moment conditions to identify η_4:

$$g_{9T} = E_T(\mathbf{R}_t\mathbf{R}_t'\eta_4-M_t\mathbf{R}_t+\mathbf{p}). \tag{41}$$

[17] Hansen and Jagannathan (1997) show that when SDFs are linear in some factors, minimizing the Hansen–Jagannathan distance measure amounts to requiring the models to price correctly the mimicking portfolios for the constant 1 and the factors. Farnsworth et al. (2002) find that restricting SDFs to price correctly the risk-free return and factors, when the factors are traded, can increase estimation efficiency. We experiment with the linearized CCAPM, CAPM, Fama–French three-factor model, and Carhart four-factor model. We find that requiring the models to price correctly the risk-free return gives almost the same result as minimizing the Hansen–Jagannathan distance measure. This is because we include the T-bill in the test assets. The mimicking portfolio for 1, constructed from these test assets, is very much like the T-bill return, which has a very small variance. Thus, whether restricting SDFs to price correctly the risk-free return or the mimicking portfolio for 1 makes little difference. We also find that restricting SDFs to price correctly the factors, when the factors are traded, reduces the standard errors of the parameter estimates, confirming the finding of Farnsworth et al. (2002). However, we are more interested in giving the benchmark model best fit for the data than obtaining better parameter estimates. So we choose to estimate the parameters by minimizing the quadratic form. When the SDFs are linear in the parameters, the parameter estimates can be solved in closed form.

To compare $M_{p,t}$ and M_t with the TMBM bounds, we consider the differences

$$\Delta_1(\vartheta) = \sigma_p - \sigma_{\mathrm{L}}(v_M) \tag{42}$$

$$\Delta_2(\vartheta) = \sigma_M - \sigma_{\mathrm{L}}(v_M), \tag{43}$$

where ϑ denotes the vector of all the unknown parameters. If the estimators are asymptotically normal, i.e.,

$$\sqrt{T}(\hat{\vartheta} - \vartheta) \xrightarrow{d} N(0, \Sigma_\vartheta), \tag{44}$$

we could use the delta method and obtain that

$$\sqrt{T}(\Delta_i(\hat{\vartheta}) - \Delta_i(\vartheta)) \xrightarrow{d} N(0, \sigma^2_{\Delta_i}), \tag{45}$$

where

$$\sigma^2_{\Delta_i} = \left(\frac{\partial \Delta_i}{\partial \vartheta'}\right) \Sigma_\vartheta \left(\frac{\partial \Delta_i}{\partial \vartheta'}\right)' \tag{46}$$

and $i=1, 2$.

Our estimators $\hat{\vartheta}$ are obtained by minimizing the GMM criterion $\mathbf{g}_T'\mathbf{W}_T\mathbf{g}_T$. Under the null hypothesis that the population moment conditions hold, the consistency and asymptotic normality of the estimators follow directly from Hansen (1982). However, our purpose is to evaluate asset pricing models that are not correct. Hence, we cannot assume that Eqs. (30) and (36) hold in population and use Hansen's (1982) asymptotic theory. Instead we utilize the asymptotic theory for extreme estimators. Hansen (1982) defines the true parameter ϑ as the values at which the population moment conditions hold, i.e., $\mathbf{g}(\vartheta)=\mathbf{0}$. We define the true parameter as the values at which the population criterion is minimized. In the Appendix we prove that the estimators are still consistent and asymptotically normal. The consistency is established by using an argument in Hansen (1982) with a different set of regularity conditions. The asymptotic normality is established by using the usual Mean Value Theorem and writing the estimators as linear combinations of three sets of asymptotically normal random variables: the sample moment conditions, the first-order derivatives of the sample moment conditions, and the random elements in the weighting matrix $\mathbf{W}_T$. As a result, the asymptotic covariance matrix is determined by the variances of and covariances between these random variables. Our asymptotic distribution generalizes Hansen's (1982). When the population moment conditions hold, which is the case discussed by Hansen (1982), the second and third sets of random variables vanish and the asymptotic covariance matrix reduces to that derived by Hansen (1982).

4. Data

The data for the factors in the Fama–French and Carhart models are taken from Kenneth French's web site. We use the first factor in the Fama–French model, the excess market return, as the market factor for the CAPM. For consumption data, we use the personal consumption expenditures for consumer nondurable goods (Citibase GMCN) plus services (GMCS) divided by the total U.S. civilian noninstitutional population (Citibase P16). For the market return in the Epstein–Zin model, we use the value-weighted index of stocks traded on NYSE from CRSP.

We choose three sets of asset returns. The first two sets are the returns on portfolios grouped based on firm characteristics. They are 12 industry portfolios and 25 size and book/market portfolios. Both are provided by Kenneth French and have been repeatedly used to evaluate asset pricing models by previous studies. The third set contains the returns on the 30 stocks in the Dow Jones Industrial Average (DJIA). Over the past few decades, there have been quite some changes in the composition of the DJIA. To avoid the survivor bias and back-filling bias, at each point in time, we use the returns on the 30 stocks that were concurrently in the index. For example, if stock B replaced stock A on date t, we concatenate the returns on stock A before date t with the returns on stock B on and after date t. The returns on individual stocks are obtained from CRSP. In all three sets of asset returns, we also include the one-month T-bill return obtained from Ibbotson Associates.

Since monthly consumption data are not available before something of 1959, we only consider the 40-year sample period for January 1960 through December 1999.

5. Empirical results

5.1. The equity premium puzzle revisited

Mehra and Prescott (1985) document the equity premium puzzle, i.e., the failure of the basic power utility model to explain the equity risk premium. Hansen and Jagannathan (1991) provide a characterization for the puzzle using their volatility bound. They show that the SDF of the power utility model does not lie above the Hansen–Jagannathan bound unless the RRA coefficient is unreasonably large. The failure of the power utility model is perhaps not so embarrassing given the fact that other models also fail in asset pricing tests. In this section, we evaluate the performance of the power utility model relative to other models using the TMBM bounds.

We use the CAPM and Fama–French model as the benchmark models and compare their TMBM bounds with the volatility of the (projected) SDF of the power utility model with different RRA coefficients. The parameters in the SDFs of the CAPM and Fama–French model and the time discount factor β for the power utility model are estimated as described above. The volatility of the (projected) SDF of the power utility model is increasing in γ. In Table 2 we report the largest values of γ for which the (projected) SDF lies below the TMBM bounds both when the sampling errors are ignored and considered.

The equity premium puzzle is clearly shown in Table 2. In all the panels, the Hansen–Jagannathan bounds eliminate the power utility model with a RRA coefficient less than 90, a number that definitely seems too high to be reasonable. Although the failure of the power utility model is indisputable, it is much less extreme when the model is compared with the CAPM and Fama–French model. In all the panels, the power utility model with a RRA coefficient of 5 can survive the TMBM bounds for the Fama–French model. The power utility model with a RRA coefficient of 1 can survive the TMBM bounds for the CAPM.

5.2. Comparing consumption-based models

In this section we compare consumption-based models. We consider five consumption-based models. Their utility function specifications and SDFs are listed in the appendix. The first model is the basic power utility model studied by Hansen and Singleton (1982) and others. The second model is based on the non-expected utility function proposed by Epstein and Zin (1989, 1991) and Weil (1989). The SDF for the Epstein–Zin model contains the return on a market portfolio.[18] The other three models are based on habit formation, and the utility functions are of the form $U(C_t, X_t)$, where X_t is the time-varying habit or subsistence level. We consider Abel's (1990) "catching up with the Joneses" model and the habit formation models considered by Ferson and Constantinides (1991) and Campbell and Cochrane (1999). The habit X_t enters the utility function as a ratio X_t/C_t in the Abel model and as a difference $C_t - X_t$ in the other two models. X_t is a linear function of past consumption in the Ferson–Constantinides model and a nonlinear function in the Campbell–Cochrane model. Campbell and Cochrane (1999) model the log of the consumption surplus ratio $S_t \equiv X_t/C_t$ as a heteroscedastic AR(1) process. We follow Li (2001) and calculate S_t from actual consumption data. We assume the consumption growth $q_t = \ln(C_t/C_{t-1})$ follows an AR(1) process:

$$q_t = \mu_q(1-\omega) + \omega q_{t-1} + \sigma_\epsilon \epsilon_t,$$

where ϵ_t is an i.i.d. standard normal random variable. To estimate μ_q, ω, and σsubϵ, we use the following exactly-identified moment conditions in addition to $\mathbf{g}_{iT}$, $i=1,\ldots,9$:

$$\mathbf{g}_{10T} = E_T(q_t - \mu_q) \tag{47}$$

$$\mathbf{g}_{11T} = E_T\left(q_t^2 - \frac{\sigma_\epsilon^2}{1-\omega^2} - \mu_q^2\right) \tag{48}$$

$$\mathbf{g}_{12T} = E_T\left(q_t q_{t-1} - \frac{\omega\sigma_\epsilon^2}{1-\omega^2} - \mu_q^2\right). \tag{49}$$

[18] Rubinstein (1976) derives the SDF for a model based on the power utility. He shows that when the consumption growth rate follows a random walk, the consumption growth rate is proportional to the return on the market portfolio. Hence, the consumption growth rate can be substituted out using the market portfolio return. The resulting SDF can be viewed as a special case of that of the Epstein–Zin model.

Table 2
SDF for power utility model and TMBM bounds for CAPM and Fama–French models

Model	Original SDF		Projected SDF	
	γ^{-}	γ^{-*}	γ^{-}	γ^{-*}
A. Assets: T-bill and 12 industry portfolios				
Hansen–Jagannathan	48	33	152	104
CAPM	9	N/A	31	N/A
Fama–French	15	2	49	N/A
B. Assets: T-bill and 25 size and B/M portfolios				
Hansen–Jagannathan	79	64	181	131
CAPM	3	N/A	11	N/A
Fama–French	14	4	38	N/A
C. Assets: T-bill and 30 Dow stocks				
Hansen–Jagannathan	53	38	132	90
CAPM	9	N/A	29	N/A
Fama–French	15	4	45	N/A

We test whether the SDF of the power utility model lies above or below the TMBM bounds for various linear SDF models. "Original SDF" means that the test is based on the volatility of the original SDF, and "Projected SDF" means that the test is based on the volatility of the projected SDF of the power utility model. The second and fourth columns (γ^{-}) list the biggest values for the curvature parameter of the power utility model at which the projected/original SDF lies below the TMBM bounds when sampling error is ignored. The third and fourth columns (γ^{-*}) list those at which the projected/original SDF lies below the TMBM bounds significantly at 5% confidence level. The tests for the Hansen–Jagannathan bounds are based on Hansen's (1982) GMM. The tests for the TMBM bounds are based on the generalization of the GMM. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets.

We plot the TMBM bounds in Fig. 3. Similar to the linear factor models, all the TMBM bounds are far below the Hansen–Jagannathan bounds, suggesting substantial pricing errors. The distances among the TMBM bounds of all five models are fairly small compared with their distances from the Hansen–Jagannathan bounds. Among the five models, the Ferson–Constantinides model and Epstein–Zin model perform best and the power utility model does worst.

To illustrate how the performance of the consumption-based models compares with the linear factor models, we transform the heights of the TMBM bounds into Sharpe ratios and list them in Tables 3 and 4. It is clear from the comparison between the two tables that the consumption-based models as a whole do much worse than the linear factor models. The only exception is the Ferson–Constantinides model, which even performs even better than the linear factor models on the industry portfolios. The performance of the Epstein–Zin model is comparable to that of the CAPM. Recall that the SDF of the Epstein–Zin model contains a market return.

To understand why the consumption-based models perform poorly and the Ferson–Constantinides model and Epstein–Zin model do better than the other models, we present the parameter estimates and some diagnostic statistics for the estimated SDFs. Previous studies of the equity premium puzzle typically characterize the failure of consumption-based models by their inability to generate a high enough risk premium, because the volatility of their SDFs is too low. Panel A in Table 5 shows that the habit formation introduced in the Ferson–Constantides model is very effective in generating high risk premiums. With a utility curvature parameter of only 0.6, the volatility of its SDF is about 1.30, much higher than that of the other models.

The table also shows that the low volatility is not the whole story for the failure of consumption-based models. The fifth column (R^2) lists the R-squares in the cross-sectional regressions of the means excess returns of the risky assets onto their betas with respect to the estimated SDFs. If the SDFs price correctly the risky assets, the mean excess returns would be perfectly aligned with the betas and hence high R-squares would be expected. However, the column shows that the mean excess returns are poorly aligned with the betas. The relative success of the Ferson–Constantinides model and Epstein–Zin model can be at least partly illustrated through this beta effect. Most noticeably, the R-square generated by the Ferson–Constantinides model in Panel A is as high as 0.52. The sixth column ($\mathrm{Corr}(M_t, \mathbf{R}_t)$) shows that the SDF of the Epstein–Zin model has higher correlations with the returns on the risky assets than the other models.

Table 5 also reveals that in many cases the parameter estimates are economically unreasonable. The estimates for the utility curvature parameters are generally large, except for the Ferson–Constantinides model, where the estimates are

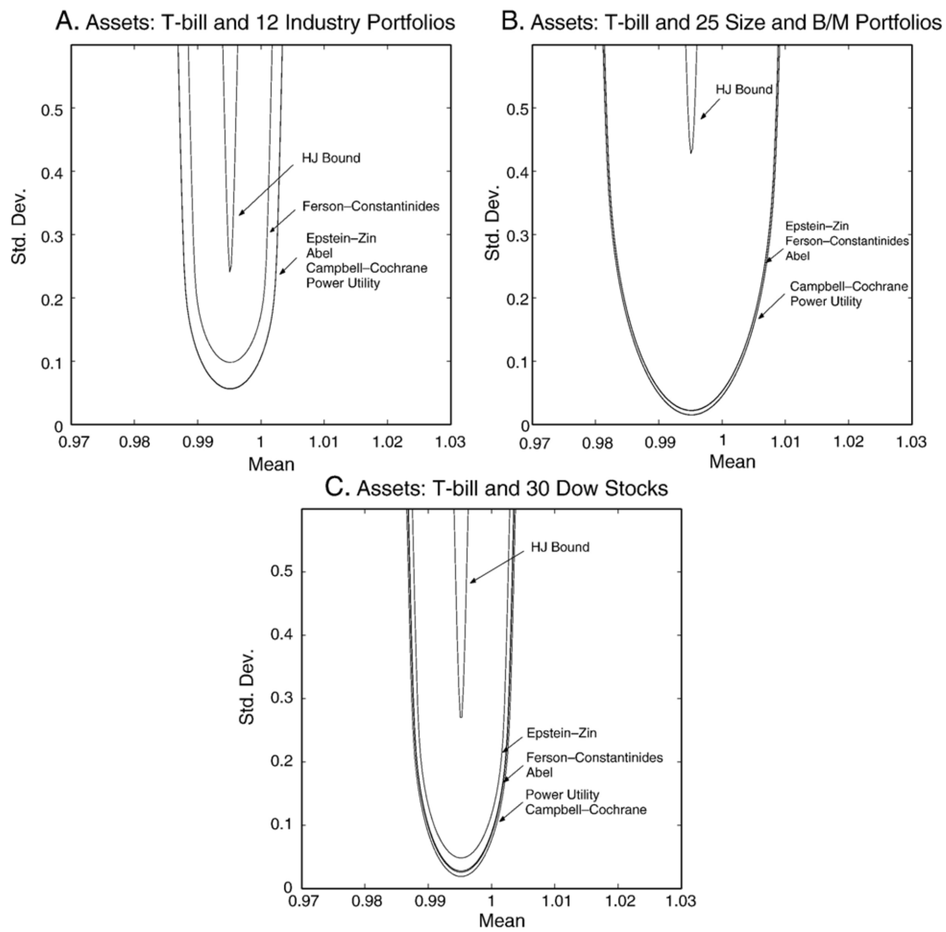

Fig. 3. TMBM bounds for consumption-based models. The weighting matrix is $\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\mathbf{\Omega}$, where $\mathbf{\Omega}$ is the covariance matrix or the asset returns. w is 0.0015 in Panel A, 0.001 in Panel B, and 0.000925 in Panel C. All parameters other than γ in the consumption-based models are chosen so as to minimize $\alpha'\mathbf{V}^{-1}\alpha$. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets, we use gross return for the one-month T-bill and excess returns for the risky assets.

around 1 in Panel A and C. The power utility model cannot price assets well unless its RRA coefficient takes on economically unreasonable values. The Epstein–Zin, Abel, Ferson–Constantinides, and Campbell–Cochrane models were developed largely to improve upon the power utility model. However, Table 5 indicates that they do not seem to resolve the puzzle except the Ferson–Constantinides model.

To see whether these models can improve upon the power utility model for given (lower) values of utility curvature parameters, we conduct another exercise. We fix the utility curvature parameters of all the models to 10, a value widely accepted as reasonable, and compare their performance. The other parameters of the SDFs are estimated as previously described. We plot the TMBM bounds based on the resulting parameter estimates.[19] The improvement of the other models over the power utility model becomes more evident. In particular, the Ferson–Constantinides model and Epstein–Zin model continue to perform well, with their TMBM bounds clearly on top of those for the other models. With a utility curvature parameter of 10, the volatility of the SDF of the Epstein–Zin model is very low. However, the correlations between its SDF and the risky asset returns are fairly high, enabling the model to fit the data better than the other models.

[19] The figures and tables for this section are available upon request.

Table 3
TMBM bounds for linear SDF models

Model	Sharpe ratio
A. Assets: T-bill and 12 industry portfolios	
Hansen–Jagannathan	0.24
Linear CCAPM	0.00
CAPM	0.04
Fama–French	0.07
Carhart	0.08
B. Assets: T-bill and 25 size and B/M portfolios	
Hansen–Jagannathan	0.43
Linear CCAPM	0.00
CAPM	0.02
Fama–French	0.07
Carhart	0.08
C. Assets: T-bill and 30 Dow stocks	
Hansen–Jagannathan	0.27
Linear CCAPM	0.00
CAPM	0.05
Fama–French	0.08
Carhart	0.08

We express the heights of the TMBM bounds and the Hansen–Jagannathan bound in terms of Sharpe ratio. Let v be the inverse of the mean gross return on the one-month T-bill. The Sharpe ratios are computed as the values of the bounds at v divided by v. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets.

Table 4
TMBM bounds for consumption-based models

Model Sharpe	Ratio
A. Assets: T-bill and 12 industry portfolios	
Hansen–Jagannathan	0.24
Power utility	0.06
Epstein–Zin	0.06
Abel	0.06
Ferson–Constantinides	0.10
Campbell–Cochrane	0.06
B. Assets: T-bill and 25 size and B/M portfolios	
Hansen–Jagannathan	0.43
Power utility	0.02
Epstein–Zin	0.02
Abel	0.02
Ferson–Constantinides	0.02
Campbell–Cochrane	0.02
C. Assets: T-bill and 30 Dow stocks	
Hansen–Jagannathan	0.27
Power utility	0.02
Epstein–Zin	0.05
Abel	0.03
Ferson–Constantinides	0.03
Campbell–Cochrane	0.02

We express the heights of the TMBM bounds and the Hansen–Jagannathan bound in terms of Sharpe ratio. Let v be the inverse of the mean gross return on the one-month T-bill. The Sharpe ratios are computed as the values of the bounds at v divided by v. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets.

Table 5
Estimated SDFs for the consumption-based models

Model	$(\alpha'\mathbf{V}^{-1}\alpha)^{\frac{1}{2}}$	μ_M	σ_M	R^2	Corr(M_t, $\mathbf{R}_t$)	β	γ	Other parameter
A. Assets: T-bill and 12 industry portfolios								
Power utility	0.182	0.995	0.821	0.14	−0.155	0.981	124.255	
						(0.070)	(42.598)	
Epstein–Zin	0.182	0.995	0.841	0.14	−0.149	0.979	126.320	0.009
						(0.076)	(70.196)	(0.017)
Abel	0.182	0.995	0.811	0.13	−0.147	0.941	116.135	0.218
						(0.462)	(79.477)	(2.294)
Ferson–Constantinides	0.140	0.995	1.295	0.52	−0.068	0.907	0.606	0.981
						(0.041)	(0.494)	(0.000)
Campbell–Cochrane	0.182	0.995	0.821	0.14	−0.155	0.981	124.231	0.997
						(0.053)	(13.896)	(0.000)
B. Assets: T-bill and 25 size and B/M portfolios								
Power utility	0.412	0.995	0.435	0.01	−0.183	1.031	79.788	
						(0.037)	(48.905)	
Epstein–Zin	0.404	0.995	0.194	0.11	−0.626	0.982	30.231	−0.053
						(0.072)	(103.074)	(0.271)
Abel	0.405	0.995	0.434	0.00	−0.178	1.096	74.977	−0.725
						(0.064)	(87.096)	(1.518)
Ferson–Constantinides	0.405	0.995	0.433	0.00	−0.178	1.096	128.461	−0.716
						(0.064)	(73.054)	(1.507)
Campbell–Cochrane	0.411	0.995	0.430	0.01	−0.186	0.992	51.563	0.997
						(0.086)	(1.490)	(0.003)
C. Assets: T-bill and 30 Dow stocks								
Power utility	0.245	0.995	0.406	0.00	−0.103	1.032	75.617	
						(0.024)	(45.602)	
Epstein–Zin	0.215	0.995	0.161	0.13	−0.554	1.003	−1.461	0.504
						(0.090)	(168.277)	(16.876)
Abel	0.238	0.995	0.496	0.07	−0.074	0.931	70.955	0.688
						(0.139)	(38.800)	(0.860)
Ferson–Constantinides	0.236	0.995	0.460	0.11	−0.070	0.926	1.343	0.973
						(0.075)	(2.053)	(0.025)
Campbell–Cochrane	0.245	0.995	0.406	0.00	−0.103	1.032	75.593	0.998
						(0.021)	(7.076)	(0.000)

The parameters are estimated by using GMM and minimizing $\alpha'\mathbf{V}^{-1}\alpha$. The weighting matrix is $\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\mathbf{\Omega}$, where w is 0.0015 in Panel A, 0.001 in Panel B, and 0.000925 in Panel C. μ_M and σ_M are the means and standard deviations of the estimated SDFs, respectively. R^2 is the R-square in the cross-sectional regression of mean excess returns of the risky assets on their betas with the SDFs. Corr(M_t, $\mathbf{R}_t$) is the average correlation between the SDFs and the returns on the risky assets. In parenthesis we report the Newey–West standard errors based on three lags. "Other Parameter" denotes ψ for the Epstein–Zin model, κ for the Abel model, h for the Ferson–Constantinides model, and ϕ for the Campbell–Cochrane model. For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets. We use the monthly data for 1960:1–1999:12 (480 months).

5.3. The Campbell–Cochrane model: time series versus cross section

Campbell and Cochrane (1999) add a slow-moving external habit to the power utility and the model successfully explains a number of time-series properties of the aggregate stock market. However, the model's ability to explain cross-sectional returns has not been thoroughly examined. It is especially interesting to explore the role of the persistence parameter for the habit process ϕ, in determining the cross-section of asset returns.

We first repeat Campbell and Cochrane's (1999) calibration using our data. We set $\phi=0.89$, which Campbell and Cochrane (1999) choose to match the serial correlation of price/dividend ratio, and search for the values for γ and β with which the model correctly prices the excess return on the market portfolio and the T-bill return. The optimizer finds $\gamma=7.4265$ and $\beta=0.5342$. We call the model with these parameter values "the calibrated model". We are interested in examining how the calibrated model does in pricing the cross-section of asset returns. To isolate the effect of ϕ on cross-sectional asset pricing from that of γ, we conduct another exercise. We hold γ fixed at 7.4265 and look

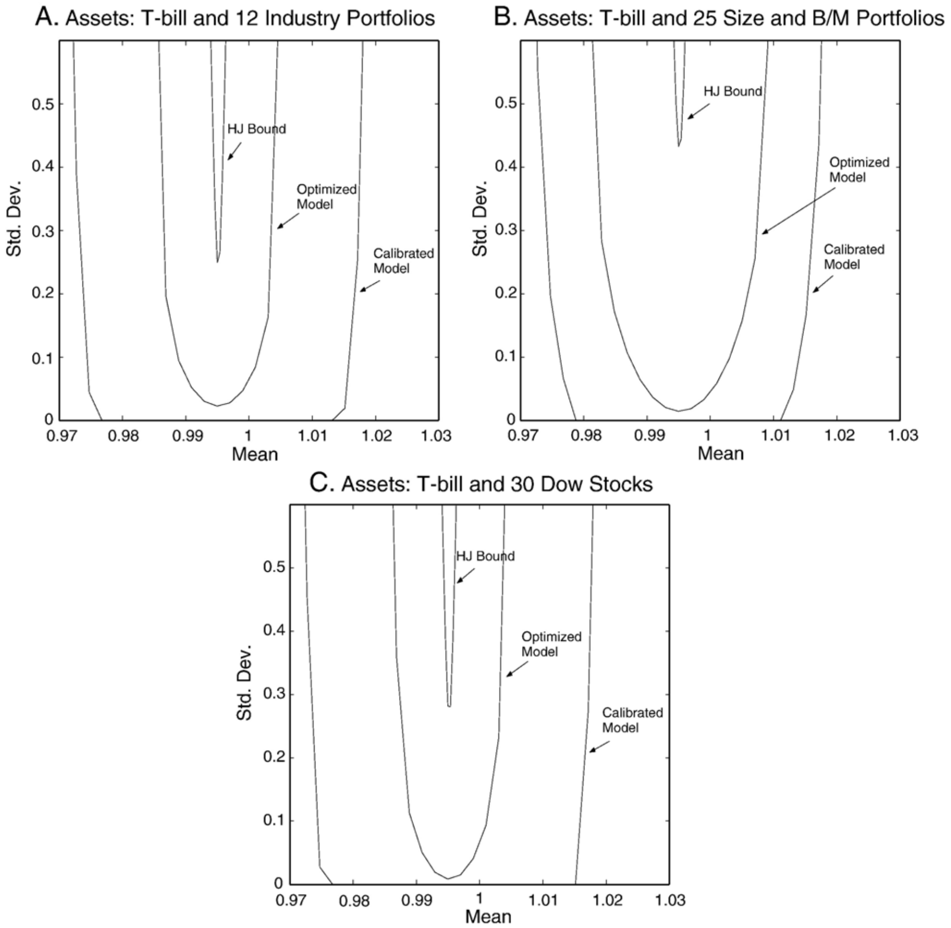

Fig. 4. TMBM bounds for the Campbell–Cochrane model under different calibrations. The weighting matrix is $\mathbf{V} = wE(\mathbf{R}_t\mathbf{R}_t') + (1-w)\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the covariance matrix or the asset returns, w is 0.0015 in Panel A, 0.001 in Panel B, and 0.000925 in Panel C. β for power utility is chosen so as to minimize $\alpha'\mathbf{V}^{-1}\alpha$ for each value of γ, the utility curvature parameter. For "calibrated models", β is 0.5342, γ is 7.4265, and ϕ is 0.89. For "optimized models", γ is also set to 7.4265 and β and ϕ are chosen such that the quadratic forms of pricing errors are minimized. β is 0.9494, 0.9311, and 0.9852 in Panel A, B, and C, respectively. ϕ is 0.9888, 0.9855, and 0.9958 in Panel A, B, and C, respectively. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in all panels, we use gross return for the one-month T-bill and excess returns for the risky assets.

for the values for ϕ and β that minimize the quadratic form. The optimizer finds higher values of ϕ for all three sets of returns, 0.9888 for the industry portfolios, 0.9855 for the size and book/market portfolios, and 0.9958 for the Dow stocks. We call the resulting models "the optimized models".

The effect of ϕ on pricing expected returns is mainly through λ_t, the sensitivity of the log consumption surplus ratio to the shock to the consumption growth. Campbell and Cochrane (1999) argue that in their model there is a second state variable representing recession, $(S_t/S_{t-1})^{\gamma}$, in addition to the consumption growth factor $(C_t/C_{t-1})^{\gamma}$. The risk premium it commands varies over time and is λ_t times the risk premium commanded by the consumption growth factor. In other words, the risk premium of the Campbell–Cochrane model is approximately $1+\lambda_t$ times that of the power utility model. We find that the values of ϕ in the calibrated model and optimized model result in dramatically different λ_t. The sample average of λ_t in the calibrated model is around 26, and those in the optimized model are around 8, 9 and 4 for the industry portfolios, the size and book/market portfolios, and the Dow stocks, respectively. Campbell and Cochrane specify the functional form of λ such that it is inversely related to ϕ.[20] Out of several motivations for the functional form, one is to make the risk-free rate constant. Both the log consumption surplus ratio s_t and λ affect the log risk-free

[20] See the Appendix for the expressions.

rate.[21] s_t does so through intertemporal substitution. If s_t is low today, the marginal utility of consumption is high. The marginal utility is expected to fall as s_t reverts to its long-run mean. The consumer would then like to borrow, driving up the equilibrium interest rate. Hence the volatility of the risk-free rate is affected by that of s_t. Since ϕ controls the speed of the mean reversion, the smaller its value is, the larger the effect of s_t on the risk-free rate. As λ_t is directly related to the conditional volatility of s_t, it affects the risk-free rate through precautious saving. As uncertainty increases, the consumer would save more, driving down the equilibrium interest rate. Campbell and Cochrane specify the functional form of λ_t so that its effect on the risk-free rate offsets that s_t's.

We plot the TMBM bounds for both the calibrated and optimized models in Fig. 4. If Campbell and Cochrane's calibration, which is meant to match the time-series properties of the aggregate stock market, also captures well the cross-sectional variation in expected returns, we would expect the two bounds to be close to each other. Fig. 4 shows that the calibrated model performs poorly. In all the plots, its TMBM bounds are zero, suggesting that even a constant SDF can beat it. The performance of the optimized models is much better. Thus, it appears that although $\phi=0.89$ explains the time-series behavior of the aggregate stock return, it is too low for pricing correctly the cross-section of expected returns, assigning too large risk premium to the recession state variable.

6. Conclusions

Previous studies, starting form Hansen and Jagannathan (1991), focus on deriving volatility bounds on SDFs for asset pricing models that exactly price assets. We derive bounds for models that might misprice assets but to a lesser degree than a benchmark model. The volatility bounds we derive, the "takes-a-model-to-beat-a-model" (TMBM) bounds, generalize the Hansen–Jagannathan bound and distance measure. While the Hansen–Jagannathan bound is often used to *test* models, the TMBM bounds are used to *compare* imperfect models. The TMBM bounds are less stringent than the Hansen–Jagannathan bound in eliminating candidate models. This distinguishes the TMBM bounds from the previous extensions of the Hansen–Jagannathan bound which derive more stringent bounds by introducing additional restrictions such as conditioning information.

We apply the TMBM bounds to various linear factor models and consumption-based models. We find that the failure of the basic power utility model seems less extreme when it is compared with popular benchmark models like the CAPM and Fama–French model. We also find that the differences among various consumption-based models are small when no restriction is placed on their parameters. Among the consumption-based models we consider, the Ferson–Constantinides model and Epstein–Zin model seem to perform better than the other models and especially so when we restrict the utility curvature parameters to economically reasonable values. We also explore Campbell and Cochrane's (1999) external habit formation model. We find that the degree of persistence in surplus consumption plays an important role and there is a tradeoff between matching the time-series properties of the aggregate stock market and the cross-section of asset returns.

We also make a methodological contribution by deriving the asymptotic distribution for the GMM when the population moment conditions might not hold. This is important given our focus on analyzing imperfect models. We show that our asymptotic distribution generalizes Hansen's (1982) for GMM.

Appendix A. Proofs

Proof of Proposition 1. It is straightforward to show that for SDF $M_{p,t}=\nu+(\alpha+\mathbf{p}-\nu\mu)'\mathbf{\Omega}^{-1}(\mathbf{R}_t-\mu)$, we must have $E(M_{p,t})=\nu$ and $E(M_{p,t}\mathbf{R}_t)-\mathbf{p}=\alpha$. To prove that $M_{p,t}$ has the minimum variance among all SDFs with the same mean and pricing errors, consider any such SDF y_t, i.e., $E(y_t)=\nu$ and $E(y_t\mathbf{R}_t)-\mathbf{p}=\alpha$. We must have

$$E(y_t - M_{p,t}) = 0 \tag{50}$$

$$E[(y_t - M_{p,t})\mathbf{R}_t] = 0. \tag{51}$$

By Eqs. (50) and (51), we must have $E[(y_t-M_{p,t})M_{p,t}]=0$ and thus $\mathrm{cov}(y_t-M_{p,t}, M_{p,t})=0$. Hence, we have $\mathrm{var}(y_t)=\mathrm{var}(M_{p,t})+\mathrm{var}(y_t-M_{p,t})\geq\mathrm{var}(M_{p,t})$. □

[21] See Eq. (8) in Campbell and Cochrane (1999) for the expression of the risk-free rate.

Proof of Proposition 2. The variance of the projected SDF $M_{p,t}=v+(\alpha+\mathbf{p}-v\mu)'\boldsymbol{\Omega}^{-1}(\mathbf{R}_t-\mu)$ is given by

$$\sigma_p^2 = (\alpha+\mathbf{p}-v\mu)'\boldsymbol{\Omega}^{-1}(\alpha+\mathbf{p}-v\mu). \tag{52}$$

The lower and upper TMBM bounds are solved from following optimization problems:

$$\min_{\alpha}(\alpha+\mathbf{p}-v\mu)'\boldsymbol{\Omega}^{-1}(\alpha+\mathbf{p}-v\mu) \tag{53}$$

and

$$\max_{\alpha}(\alpha+\mathbf{p}-v\mu)'\boldsymbol{\Omega}^{-1}(\alpha+\mathbf{p}-v\mu), \tag{54}$$

both subject to

$$\alpha'\mathbf{V}^{-1}\alpha \leq \alpha_b'\mathbf{V}^{-1}\alpha_b. \tag{55}$$

Since both the objective function and constraint are ellipses in the vector α, the maximum and minimum values must be obtained on the boundary of the constraint set, i.e., when the equality holds in Eq. (55). We form the lagrangian as

$$\mathcal{L}(\alpha) = (\alpha+\mathbf{p}-v\mu)'\boldsymbol{\Omega}^{-1}(\alpha+\mathbf{p}-v\mu)+\lambda(\alpha'\mathbf{V}^{-1}\alpha-\alpha_b'\mathbf{V}^{-1}\alpha b).$$

The first-order condition is

$$\boldsymbol{\Omega}^{-1}(\alpha+\mathbf{p}-v\mu)+\lambda\mathbf{V}^{-1}\alpha = 0.$$

α is solved as

$$\alpha = \mathbf{V}(V+\lambda\boldsymbol{\Omega})^{-1}(v\mu-\mathbf{p}). \tag{56}$$

Eq. (52) now becomes

$$(v\mu-\mathbf{p})'[\mathbf{V}(V+\lambda\boldsymbol{\Omega})^{-1}-I]'\boldsymbol{\Omega}^{-1}[\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}-I](v\mu-\mathbf{p}).$$

To solve for λ, we plug Eq. (56) into the equality constraints (55) and obtain

$$(v\mu-\mathbf{p})'(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}(v\mu-\mathbf{p}) = \alpha_b'\mathbf{V}^{-1}\alpha_b. \tag{57}$$

Let Λ denote the set of the roots of this equation. The maximum and minimum values for σ_p^2 in Eq. (52) are given by

$$e_{\max} = \max_{\lambda\epsilon\Lambda}(v\mu-\mathbf{p})'[\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}-I]'\boldsymbol{\Omega}^{-1}[\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}-I](v\mu-\mathbf{p}) \tag{58}$$

$$e_{\min} = \min_{\lambda\epsilon\Lambda}(v\mu-\mathbf{p})'[\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}-I]'\boldsymbol{\Omega}^{-1}[\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}-I](v\mu-\mathbf{p}), \tag{59}$$

respectively. □

Proof of Proposition 3. Proposition 3 is implied directly by Propositions 1 and 2. □

Proof of Corollary 1. Consider the SDF $M_t=v+(\mathbf{p}-v\mu)'\boldsymbol{\Omega}^{-1}(\mathbf{R}_t-\mu)$. Its mean is v and standard deviation is equal to the Hansen–Jagannathan bound $\sigma_{\mathrm{HJ}}(v)$. It has zero pricing errors and thus can beat any benchmark model. Since the TMBM bound $\sigma_{\mathrm{L}}(v)$ is the smallest standard deviation obtainable for candidate SDFs that can beat the benchmark, we must have $\sigma_{\mathrm{L}}(v)\leq\sigma_{\mathrm{HJ}}(v)$. Similarly, we have $\sigma_{\mathrm{HJ}}(v)\leq\sigma_{\mathrm{U}}(v)$. □

Proof of Corollary 2. When the benchmark model prices assets correctly, i.e., $\alpha_b=0$, by the positive definiteness of $\mathbf{V}^{-1}$, any candidate SDF that can beat the benchmark model must also have $\alpha=0$. From Eq. (52) it is trivial that $\sigma_{\mathrm{U}}(v)=\sigma_{\mathrm{L}}(v)=\sigma_{\mathrm{HJ}}(v)$. □

Proof of Corollary 3. When $\mathbf{V}=\boldsymbol{\Omega}$, the solutions to Eq. (57) are

$$\pm\frac{[\alpha_b'\boldsymbol{\Omega}^{-1}\alpha_b]^{\frac{1}{2}}}{[(v\mu-\mathbf{p})'\boldsymbol{\Omega}^{-1}(v\mu-\mathbf{p})]^{\frac{1}{2}}}-1.$$

Plugging them into Eqs. (58) and (59), we have

$$e_{\max}=\sigma_{\mathrm{HJ}}(v)+(\alpha_b'\boldsymbol{\Omega}^{-1}\alpha_b)^{\frac{1}{2}}$$
$$e_{\min}=|\sigma_{\mathrm{HJ}}(v)-(\alpha_b'\boldsymbol{\Omega}^{-1}\alpha_b)^{\frac{1}{2}}|.$$

□

Proof of Corollary 4. Available upon request. □

Appendix B. Solving for the TMBM bounds

The key to solving for the TMBM bounds is to find the roots of Eq. (15). We can rewrite Eq. (18) as

$$\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\boldsymbol{\Omega}=\boldsymbol{\Omega}+w(\mu\mu'), \tag{60}$$

and thus

$$\mathbf{V}+\lambda\boldsymbol{\Omega}=(1+\lambda)\boldsymbol{\Omega}+w(\mu\mu'). \tag{61}$$

From Eq. (61), we can obtain

$$(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}=\frac{1}{1+\lambda}\left[\boldsymbol{\Omega}^{-1}-\frac{w}{(1+\lambda)+w(\mu'\boldsymbol{\Omega}^{-1}\mu)}\boldsymbol{\Omega}^{-1}\mu\mu'\boldsymbol{\Omega}^{-1}\right]. \tag{62}$$

To simplify our derivation, let $x=1+\lambda$ and $\mathbf{a}=v\mu-\mathbf{p}$. We plug Eq. (62) into Eq. (15). After some tedious algebra, we have

$$\begin{aligned}&(v\mu-\mathbf{p})(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}(v\mu-\mathbf{p})\\&=\frac{1}{x^2}\left\{\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a}+w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2+\frac{w^2(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2(\mu'\boldsymbol{\Omega}^{-1}\mu)[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]}{[x+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]^2}-\frac{2w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]}{x+w(\mu'\boldsymbol{\Omega}^{-1}\mu)}\right\}.\end{aligned} \tag{63}$$

To simplify the notation further, we make a second change of variable, $z=\frac{1}{x+w(\mu'\boldsymbol{\Omega}^{-1}\mu)}$. Hence $x=\frac{1}{z}-w(\mu'\boldsymbol{\Omega}^{-1}\mu)$ and $\frac{1}{x^2}=\frac{z^2}{[1-w(\mu'\boldsymbol{\Omega}^{-1}\mu)z]^2}$. Eq. (63) can be rewritten as

$$\begin{aligned}&(v\mu-\mathbf{p})(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}\mathbf{V}(\mathbf{V}+\lambda\boldsymbol{\Omega})^{-1}(v\mu-\mathbf{p})=\frac{z^2}{[1-w(\mu'\boldsymbol{\Omega}^{-1}\mu)z]^2}\{\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a}+w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2\\&\quad+w^2(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2(\mu'\boldsymbol{\Omega}^{-1}\mu)[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]z^2-2w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]z\}.\end{aligned} \tag{64}$$

Hence, Eq. (15) becomes

$$\begin{aligned}&\frac{z^2}{[1-w(\mu'\boldsymbol{\Omega}^{-1}\mu)z]^2}\{\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a}+w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2+w^2(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2(\mu'\boldsymbol{\Omega}^{-1}\mu)[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]z^2\\&\quad-2w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]z\}=\alpha_b'\mathbf{V}^{-1}\alpha_b.\end{aligned} \tag{65}$$

Equivalently,

$$z^2\{\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a}+w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2+w^2(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2(\mu'\boldsymbol{\Omega}^{-1}\mu)[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]z^2-2w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)]z\}$$
$$=(\alpha_b'\mathbf{V}^{-1}\alpha_b)[1-w(\mu'\boldsymbol{\Omega}^{-1}\mu)z]^2. \tag{66}$$

Combining terms, we have

$$c_4z^4+c_3z^3+c_2z^2+c_1z+c_0=0, \tag{67}$$

where

$$c_4=w^2(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2(\mu'\boldsymbol{\Omega}^{-1}\mu)[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)] \tag{68}$$

$$c_3=-2w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2[1+w(\mu'\boldsymbol{\Omega}^{-1}\mu)] \tag{69}$$

$$c_2=\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a}+w(\mathbf{a}'\boldsymbol{\Omega}^{-1}\mu)^2-w^2(\alpha_b'\mathbf{V}^{-1}\alpha_b)(\mu'\boldsymbol{\Omega}^{-1}\mu)^2 \tag{70}$$

$$c_1=2w(\alpha_b'\mathbf{V}^{-1}\alpha_b)(\mu'\boldsymbol{\Omega}^{-1}\mu) \tag{71}$$

$$c_0=-\alpha_b'\mathbf{V}^{-1}\alpha_b. \tag{72}$$

This equation is a fourth-order polynomial. We find the roots by using the function *roots* in MATLAB.

Appendix C. GMM

Let $\{x_t\!:\!1\le t\}$ denote the data and ϑ denote all the parameters to be estimated. Let $g_T(\vartheta)=\frac{1}{T}\sum_{t=1}^{T}g(x_t,\vartheta)$ denote the vector of the sample moments as functions of ϑ. The GMM estimator $\hat{\vartheta}$ is obtained by minimizing $\mathbf{g}_T'(\vartheta)\mathbf{W}_T\mathbf{g}_T(\vartheta)$ or equivalently setting some linear combinations of $\mathbf{g}_T(\vartheta)$ to zero:

$$\mathbf{a}_T(\vartheta)\mathbf{g}_T(\vartheta)=0, \tag{73}$$

where $\mathbf{a}_T=\dfrac{\partial\mathbf{g}_T'(\vartheta)}{\partial\vartheta}\mathbf{W}_T^{-1}$. [22]

C.1. Consistency

To prove the consistency, we need to make a few assumptions. Let Θ denote the parameter space.

Assumption 1. $\{x_t\}$ is stationary and ergodic.

Assumption 2. Θ is a compact subset of R^n, where n is the number of parameters in ϑ.

Assumption 3. $\mathbf{g}(x_t,\vartheta)$ is continuous in ϑ for all x_t.

Assumption 4. $\mathbf{g}(x_t,\vartheta)$ is measurable in x_t for all ϑ in Θ.

[22] When the parameters in the benchmark SDF M_{bt}, i.e. $\mathbf{b}$, are unknown, we estimate $\mathbf{b}$ by minimizing $\mathbf{g}_{1T}'\mathbf{V}^{-1}\mathbf{g}_{1T}$. This amounts to setting linear combinations of $\mathbf{g}_{1T}$ to zero, i.e.,

$$\mathbf{a}_{1T}\mathbf{g}_{1T}=0,$$

where $\mathbf{a}_{1T}=\frac{\partial g_{1T}'}{\partial b}\mathbf{V}^{-1}$. When Mbt is linear in b, for example, Mbt=F0 b,

$$\mathbf{a}_{1T}=E_T(\mathbf{F}_t\mathbf{R}_t')\mathbf{V}^{-1}.$$

The parameter estimates can be analytically solved as

$$\hat{\mathbf{b}}=[E_T(\mathbf{F}_t\mathbf{R}_t')\mathbf{V}^{-1}E_T(\mathbf{R}_t\mathbf{F}_t')]^{-1}[E_T(\mathbf{F}_t\mathbf{R}_t')\mathbf{V}^{-1}-\mathbf{p}].$$

Similarly, any unknown parameter in the candidate SDF M_t is estimated from

$$\mathbf{a}_{6T}\mathbf{g}_{6T}=0,$$

where $\mathbf{a}_{6T}=\frac{\partial\mathbf{g}_{6T}'}{\partial b}V^{-1}$. We use identity matrices as $\mathbf{a}_{iT}$ for all the exactly identified moment conditions. The linear combinations of all the moment conditions $\mathbf{g}_T$ are $\mathbf{a}_T$=diag ($\mathbf{a}_{1T}$,..., $\mathbf{a}_{9T}$), where diag (...) concatenate all the matrices in the parenthesis diagonally.

Assumption 5. $E\,[\mathbf{g}(x_t, \vartheta)]$ exists and is finite for all ϑ in Θ.

We now obtain the consistency result.

Proposition 4. *Suppose* Assumptions 1–5 *are satisfied. Suppose that the symmetric matrix* $\boldsymbol{W}_T$ *converges in probability to a symmetric positive definite matrix* $\boldsymbol{W}$. *Suppose, further, that*

(a) $E\,[\mathbf{g}(x_t, \vartheta)]'\boldsymbol{W}E\,[\mathbf{g}(x_t, \vartheta)]$ *is uniquely minimized at* $\vartheta_0 \in \Theta$,
(b) $E\,[\sup_{\vartheta\in\Theta} ||\mathbf{g}(x_t, \vartheta)||] < \infty$,
then the GMM estimator $\hat{\vartheta} \xrightarrow{p} \vartheta_0$.

Proof. By Assumptions 1, 4, and 5, the Ergodic Theorem implies that $\mathbf{g}_T(\vartheta)$ converges in probability to $E\,[\mathbf{g}(x_t, \vartheta)]$ pointwise. With 1–4, and Condition (b), the Uniform Law of Large Numbers (Lemma 2.4 in Newey and McFadden, 1994) implies that $\mathbf{g}_T(\vartheta)$ also converges in probability to $E\,[\mathbf{g}(x_t, \vartheta)]$ uniformly, and thus $\mathbf{g}_T'(\vartheta)\mathbf{W}_T\mathbf{g}_T(\vartheta)$ converges to $E\,[\mathbf{g}(x_t, \vartheta)]'\,\mathbf{W}E\,[\mathbf{g}(x_t, \vartheta)]$ uniformly (see Newey and McFadden, 1994, p. 2132). The consistency then follows from Theorem 4.1.1 in Amemiya (1985). □

Condition (a) defines the parameter values which the estimator $\hat{\vartheta}$ converges to. Assumption 2.4 in Hansen (1982) defines ϑ_0 such that $E\,[\mathbf{g}(x_t, \vartheta_0)] = \mathbf{0}$, i.e., the population moment conditions hold. In our context, we recognize that the population moment conditions might not hold at any parameter value, and the GMM estimator $\hat{\vartheta}$ converges to the parameter values at which $E\,[\mathbf{g}(x_t,\vartheta)]'\,\mathbf{W}E\,[\mathbf{g}(x_t, \vartheta)]$ is minimized. The definition for $\hat{\vartheta}_0$ in Condition (a) is more general than Hansen's definition. As long as $\mathbf{W}$ is positive definite, if there exists a unique ϑ_0 such that $E\,[\mathbf{g}(x_t, \vartheta_0)] = \mathbf{0}$, it must also uniquely minimize $E\,[\mathbf{g}(x_t, \vartheta)]'\mathbf{W}E\,[\mathbf{g}(x_t, \vartheta)]$.

C.2. Asymptotic normality

To establish the asymptotic normality, we need some additional assumptions.

Assumption 6. ϑ_0 is in the interior of Θ.

Assumption 7. $\mathbf{g}(x_t, \vartheta)$ is continuously differentiable in ϑ for each x_t.

Assumption 8. $E\left[\frac{\partial \mathbf{g}(x_t,\vartheta_0)}{\partial\vartheta'}\right]$ has full column rank.

Now we obtain the asymptotic normality result.

Proposition 5. *Suppose* Assumptions 1–5 *are satisfied so that the GMM estimator* $\hat{\vartheta}$ *is consistent. In addition, suppose* Assumptions 6–8 *are satisfied. Suppose, further, that*

(a) $\sqrt{T}\{\mathbf{g}_T(\vartheta_0) - E[\mathbf{g}(x_t, \vartheta_0)]\}$ *converges in distribution to zero-mean normal random variables,*
(b) $\sqrt{T}[\mathbf{d}_T(\vartheta_0) - \mathbf{d}(\vartheta_0)]$ *converges in distribution to zero-mean normal random variables,* $\mathbf{d}_T(\vartheta) = \frac{\partial \mathbf{g}_T(\vartheta)}{\partial\vartheta'} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial \mathbf{g}(x_t,\vartheta)}{\partial\vartheta'}$ *and* $\mathbf{d}(\vartheta) = E\left[\frac{\partial \mathbf{g}(x_t,\vartheta)}{\partial\vartheta'}\right]$,
(c) $\sqrt{T}(\mathbf{W}_T - \mathbf{W})$ *converges in distribution to zero-mean normal random variables, where* $\boldsymbol{W}$ *is a symmetric positive definite matrix,*
(d) for some neighborhood $\mathcal{N}$ of ϑ_0,

$$E\left[\sup_{\vartheta\in\mathcal{N}} ||\frac{\mathbf{g}(x_t,\vartheta)}{\partial\vartheta\partial\vartheta'}||\right] < \infty,$$

so that for any consistent estimator $\tilde{\vartheta}$,

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \mathbf{g}(x_t,\tilde{\vartheta})}{\partial\vartheta\partial\vartheta'} \xrightarrow{p} E\left[\frac{\partial \mathbf{g}(x_t,\vartheta_0)}{\partial\vartheta\partial\vartheta'}\right],$$

then $\sqrt{T}(\hat{\vartheta}-\vartheta_0)$ *converges in distribution to zero-mean normal random variables with the asymptotic covariance matrix*

$$A\mathrm{Cov}(\hat{\vartheta}) = 4\left[\frac{\partial Q(\vartheta_0)}{\partial\vartheta\partial\vartheta'}\right]^{-1}\mathbf{C}A\mathrm{Cov}(\mathbf{A})\mathbf{C}'\left[\frac{\partial Q(\vartheta_0)}{\partial\vartheta\partial\vartheta'}\right]^{-1}, \tag{78}$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{d}_T'(\vartheta_0)\mathbf{W}_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \otimes \{E[\mathbf{g}(x_t,\vartheta_0)]'\mathbf{W}_T'\} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}'(\vartheta_0)\{I_n \otimes E[\mathbf{g}(x_t,\vartheta_0)]'\} \end{pmatrix} \tag{79}$$

$$A = \begin{pmatrix} \sqrt{T}\{\mathbf{g}_T(\vartheta_0) - E[\mathbf{g}(x_t,\vartheta_0)]\} \\ \mathrm{vec}\{\sqrt{T}[\mathbf{d}_T'(\vartheta_0) - \mathbf{d}'(\vartheta_0)]\} \\ \mathrm{vec}[\sqrt{T}(W_T - W)] \end{pmatrix}. \tag{80}$$

Let m be the number of elements in the vector ϑ *and n be that in the vector* $\mathbf{g}_T(\vartheta)$.

Proof. We first define

$$Q_T(\vartheta) = \mathbf{g}_T'(\vartheta)\mathbf{W}_T\mathbf{g}_T(\vartheta)$$

and

$$Q(\vartheta) = E[\mathbf{g}(x_t,\vartheta)]'\mathbf{W}E[\mathbf{g}(x_t,\vartheta)].$$

The GMM estimator $\hat{\vartheta}$ is such that

$$\frac{\partial Q_T(\hat{\vartheta})}{\partial\vartheta} = \mathbf{0}. \tag{81}$$

We expand Eq. (81) around ϑ_0 using the Mean Value Theorem:

$$\mathbf{0} = \frac{\partial Q_T(\vartheta_0)}{\partial\vartheta} + \frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}(\hat{\vartheta}-\vartheta_0),$$

where ϑ_1 is between ϑ_0 and $\hat{\vartheta}$. It easily follows that

$$\begin{aligned}\sqrt{T}(\hat{\vartheta}-\vartheta_0) &= -\left[\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}\right]^{-1}\sqrt{T}\frac{\partial Q_T(\vartheta_0)}{\partial\vartheta} \\ &= -2\left[\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}\right]^{-1}\sqrt{T}\mathbf{d}_T{}'(\vartheta_0)\mathbf{W}_T\mathbf{g}_T(\vartheta_0) \\ &= -2\left[\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}\right]^{-1}\Big\{\mathbf{d}_T'(\vartheta_0)\mathbf{W}_T\sqrt{T}\{\mathbf{g}_T(\vartheta_0) - E[\mathbf{g}(x_t,\vartheta_0)]\} + \sqrt{T}[\mathbf{d}_T'(\vartheta_0) \\ &\quad -\mathbf{d}'(\vartheta_0)]\mathbf{W}_T E[\mathbf{g}(x_t,\vartheta_0)] + \mathbf{d}'(\vartheta_0)\sqrt{T}(\mathbf{W}_T-\mathbf{W})E[\mathbf{g}(x_t,\vartheta_0)]\Big\}.\end{aligned}$$

The ergodic stationarity of $\{x_t\}$ and consistency of $\hat{\vartheta}$ and Conditions (b) and (d) imply that

$$\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'} \xrightarrow{p} \frac{\partial Q(\vartheta_0)}{\partial\vartheta\partial\vartheta'}.$$

Conditions (b) and (c) imply that $\mathbf{d}_T(\vartheta_0) \xrightarrow{p} \mathbf{d}(\vartheta_0)$ and $\mathbf{W}_T \xrightarrow{p} \mathbf{W}$. Since Conditions (a), (b), and (c) say that $\sqrt{T}\{\mathbf{g}_T(\vartheta_0) - E[\mathbf{g}(x_t,\vartheta_0)]\}$, $\sqrt{T}[\mathbf{d}_T(\vartheta_0) - \mathbf{d}(\vartheta_0)]$, and $\sqrt{T}(\mathbf{W}_T - \mathbf{W})$ all converge in distribution to normal random variables with mean zero, $\sqrt{T}(\hat{\vartheta}-\vartheta_0)$ must also do so.

The asymptotic covariance matrix is determined by the asymptotic covariances matrices of $\sqrt{T}\{\mathbf{g}_T(\vartheta_0)-E[\mathbf{g}(x_t,\vartheta_0)]\}$, $\sqrt{T}[\mathbf{d}_T'(\vartheta_0)-\mathbf{d}'(\vartheta_0)]$, and $\sqrt{T}(\mathbf{W}_T-\mathbf{W})$. To derive the asymptotic covariance matrix explicitly, for any matrix $\mathbf{x}$, let us first define vec($\mathbf{x}$) as the vector formed by stacking all the row vectors of $\mathbf{x}$. Eq. (82) can be rewritten as

$$\begin{aligned}\sqrt{T}(\hat{\vartheta}-\vartheta_0) &= -2\left[\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}\right]^{-1}\Big\{\mathbf{d}_T'(\vartheta_0)\mathbf{W}_T\sqrt{T}\{\mathbf{g}_T(\vartheta_0)-E[\mathbf{g}(x_t,\vartheta_0)]\} \\ &\quad +\{I_m\otimes\{E[\mathbf{g}(x_t,\vartheta_0)]'\mathbf{W}_T'\}\}\mathrm{vec}\{\sqrt{T}[\mathbf{d}_T'(\vartheta_0)-\mathbf{d}'(\vartheta_0)]\} \\ &\quad +\mathbf{d}'(\vartheta_0)\{I_n\otimes E[\mathbf{g}(x_t,\vartheta_0)]'\}\mathrm{vec}\left[\sqrt{T}(\mathbf{W}_T-\mathbf{W})\right]\Big\} \\ &= -2\left[\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}\right]^{-1}\mathbf{CA},\end{aligned} \tag{83}$$

where $\mathbf{C}$ and $\mathbf{A}$ are defined in Eqs. (79) and (80), respectively. The asymptotic covariance matrix for $\vartheta\hat{}$ in Eq. (78) then follows directly. □

We can provide a set of sufficient conditions for Conditions (a) and (b). Let

$$\mathbf{f}_t = \left(\mathbf{g}'(x_t,\vartheta),\mathrm{vec}\left(\frac{\partial\mathbf{g}(x_t,\vartheta)}{\partial\vartheta'}\right)'\right)'$$

and

$$\mathbf{e}_{tj} = E(f_t|\mathbf{f}_{t-j},\mathbf{f}_{t-j-1},\ldots)-E(\mathbf{f}_t|\mathbf{f}_{t-j-1},\mathbf{f}_{t-j-2},\ldots).$$

Conditions (a) and (b) can be replaced by

Assumption 9. *$E(f_tf_t')$ exists and is finite.*

Assumption 10. *$E(\boldsymbol{f}_t|\boldsymbol{f}_{t-j}, f_{t-j-1},\ldots)$ converges in mean square to zero.*

Assumption 11. *$\sum_{j=0}^{\infty}[E(\mathbf{e}_{tj}\mathbf{e}_{tj}')]^{\frac{1}{2}}$ is finite.*

These assumptions are the so-called Gordin's conditions and similar to Assumption 3.5 in Hansen (1982).

The next proposition shows that the asymptotic distribution we derive generalizes Hansen's (1982).

Proposition 6. *If E [$\mathbf{g}(x_t,\vartheta_0)$] = $\mathbf{0}$, the GMM estimator $\hat{\vartheta}$ is asymptotically normal with the asymptotic covariance matrix*

$$[\mathbf{d}'(\vartheta_0)\mathbf{Wd}(\vartheta_0)]^{-1}\mathbf{d}'(\vartheta_0)\mathbf{WSWd}(\vartheta_0)[\mathbf{d}'(\vartheta_0)\mathbf{Wd}(\vartheta_0)]^{-1}, \tag{84}$$

where $\boldsymbol{S}$ is the asymptotic covariance matrix of $\mathbf{g}T(\vartheta_0)$.

Proof. When E [$\mathbf{g}(x_t,\vartheta_0)$] = $\mathbf{0}$, the last two terms in the parenthesis in Eq. (82) vanish and $\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}$ converges in probability to $2\mathbf{d}'(\vartheta_0)\mathbf{Wd}(\vartheta_0)$. Hence the asymptotic covariance matrix is given by Eq. (84), which is the same as that derived in Theorem 3.1 in Hansen (1982). □

The difference between the asymptotic covariance matrices when E [$\mathbf{g}(x_t,\vartheta_0)$] = 0 does not hold and when it does, the case analyzed in Hansen (1982), is due to the stochastic variation in $\sqrt{T}[\mathbf{d}_T'(\vartheta_0)-\mathbf{d}'(\vartheta_0)]$, $\sqrt{T}(\mathbf{W}_T-\mathbf{W})$ and the fact that in general $\frac{\partial Q_T(\vartheta_1)}{\partial\vartheta\partial\vartheta'}$ does not converge to $2\mathbf{d}'(\vartheta_0)\mathbf{Wd}(\vartheta_0)$.[23]

Appendix D. Consumption-based models

We list the utility functions and SDFs for various consumption-based models. In all the models, β is the time discount factor and γ is the utility curvature parameter. The power utility model is the basic consumption-CAPM. The

[23] Empirically we find that the difference is mainly due to the third source, and the contributions to the asymptotic covariance matrix by $\sqrt{T}[\mathbf{d}_T'(\vartheta_0)-\mathbf{d}'(\vartheta_0)]\mathbf{W}_T E\mathbf{g}(x_t,\vartheta_0)$ and $\sqrt{T}(\mathbf{W}_T-\mathbf{W})E\mathbf{g}(x_t,\vartheta_0)$ are very cumbersome to compute. So we assume both terms vanish to simplify the empirical analysis.

(EIS). $R_{m,t}$ is the return on a market portfolio and $\theta=(\gamma-1)/(1/\psi-1)$. The Abel, Ferson–Constantinides, and Campbell–Cochrane models are all based on external habit formation (for simplicity, we consider the external version of the Ferson–Constantinides model). X_t is the habit level. In the Abel model,

$$X_t = C_{t-1}^{\kappa}$$

where κ governs the degree of time-nonseparability. In the Ferson–Constantinides model,

$$X_t = hC_{t-1},$$

where h also governs the degree of time-nonseparability. In the Campbell–Cochrane model, the log surplus consumption ratio $s_t \equiv \ln(S_t) \equiv \ln\left(\frac{C_t - X_t}{C_t}\right)$ evolves as a heteroskedastic AR(1) process:

$$s_{t+1} = (1-\phi)\overline{s} + \phi s_t + \lambda(s_t)\sigma_\epsilon \epsilon_{t+1},$$

where $\overline{s}$ is the steady state of the process, ϕ controls the speed at which s_t reverts to $\overline{s}$. $\sigma_\in \in_{t+1}$ is the innovation to the consumption growth process and $\in_{t+1}$ is an i.i.d. standard normal random variable. $\lambda(s_t)$ is called the sensitivity function and varies over time with s_t. Campbell and Cochrane specify the above parameters as follows:

$$\overline{s} = \ln(\overline{S})$$
$$\overline{S} = \sigma_\epsilon \sqrt{\frac{\gamma}{1-\phi}}$$
$$\lambda(s_t) = \begin{cases} \frac{1}{S}\sqrt{1-2(s_t-\overline{s})} - 1 & s_t \le s_{\max} \\ 0 & s_t \ge s_{\max} \end{cases}$$
$$s_{\max} = \overline{s} + \frac{1}{2}(1-\overline{S}^2).$$

Epstein–Zin model is based on nonexpected utility defined recursively. ψ is the elasticity of intertemporal substitution

Model	Utility function	SDF
Power utility	$E_t\left[\sum_{j=0}^{\infty} \beta^j \frac{C_{t+j}^{1-\gamma}-1}{1-\gamma}\right]$	$\beta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}$
Epstein–Zin	$\left[(1-\beta)C_t^{1-\frac{1}{\psi}} + \beta\left(E_t U_{t+1}^{1-\gamma}\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$	$\left[\beta\left(\frac{C_t}{C_{t-1}}\right)^{\frac{1}{\psi}}\right]^{\theta}\left(\frac{1}{1+R_{m,t}}\right)^{1-\theta}$
Abel	$E_t\left[\sum_{j=0}^{\infty} \beta^j \frac{(C_{t+j}/X_{t+j})^{1-\gamma}-1}{1-\gamma}\right]$	$\beta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\left(\frac{C_{t-1}}{C_{t-2}}\right)^{\kappa(\gamma-1)}$
Ferson–Constantinides	$E_t\left[\sum_{j=0}^{\infty} \beta^j \frac{(C_{t+j}-X_{t+j})^{1-\gamma}-1}{1-\gamma}\right]$	$\beta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{\left(1-h\frac{C_{t-1}}{C_t}\right)^{-\gamma}}{\left(1-h\frac{C_{t-2}}{C_{t-1}}\right)^{-\gamma}}$
Campbell–Cochrane	$E_t\left[\sum_{j=0}^{\infty} \beta^j \frac{(C_{t+j}-X_{t+j})^{1-\gamma}-1}{1-\gamma}\right]$	$\beta\left(\frac{S_t}{S_{t-1}}\frac{C_t}{C_{t-1}}\right)$

Appendix E. Further discussions

E.1. Pricing errors of expected returns

The pricing error vector α in Eq. (3) is defined in terms of the prices assigned by the SDF M_t to $\mathbf{R}_t$. It can be alternatively defined in terms of expected returns, i.e.,

$$\alpha^{\mathbf{R}} \equiv E(\mathbf{R}_t) - \frac{\mathbf{p}}{E(M_t)} + \frac{\operatorname{cov}(M_t, \mathbf{R}_t)}{E(M_t)}. \tag{85}$$

It is easy to verify that $\alpha^{\mathbf{R}}$ and α differ only by a scalar $E(M_t)$,

$$\alpha^{\mathbf{R}} = \frac{1}{E(M_t)}\alpha. \tag{86}$$

This difference has interesting implications for model comparisons.

According to Eq. (8), a candidate SDF M_t with mean v beats a benchmark SDF $M_{b,t}$ with mean v_b if

$$\alpha'\mathbf{V}^{-1}\alpha \le \alpha_b'\mathbf{V}^{-1}\alpha_b.$$

Substituting out α with $\alpha^{\mathbf{R}}$ using Eq. (86),

$$v^2(\alpha^{\mathbf{R}\prime}\mathbf{V}^{-1}\alpha^{\mathbf{R}}) \le v_b^2(\alpha_b^{\mathbf{R}\prime}\mathbf{V}^{-1}\alpha_b^{\mathbf{R}}). \tag{87}$$

Eq. (87) suggests that candidate SDFs with low means, ceteris paribus, have higher chances of beating the benchmark model. Thus models like the power utility model with a high RRA coefficient might be favored, since a high RRA would imply a high risk-free rate and thus a low mean for the model's SDF. To circumvent this issue, we can change the definition for a candidate model to beat the benchmark model to[24]

$$\alpha^{\mathbf{R}\prime}\mathbf{V}^{-1}\alpha^{\mathbf{R}} \le \alpha_b^{\mathbf{R}\prime}\mathbf{V}^{-1}\alpha_b^{\mathbf{R}}. \tag{88}$$

The TMBM bounds based on Eq. (88), $\sigma_{\mathrm{L}}^{\mathbf{R}}(v)$, can be easily solved using the existing methodology. We can rewrite Eq. (88) as

$$\alpha'\mathbf{V}^{-1}\alpha \le \left(\frac{v^2}{v_b^2}\right)\alpha_b'\mathbf{V}^{-1}\alpha_b.$$

The following proposition shows the relationship between the two sets of the TMBM bounds, $\sigma_{\mathrm{L}}(v)$ and $\sigma_{\mathrm{L}}^{\mathbf{R}}(v)$.

Proposition 7.

$$\sigma_{\mathrm{L}}^{\mathbf{R}}(v)\begin{cases} >\sigma_{\mathrm{L}}(v) & \textit{if} \quad v<v_b \\ =\sigma_{\mathrm{L}}(v) & \textit{if} \quad v=v_b \\ <\sigma_{\mathrm{L}}(v) & \textit{if} \quad v>v_b \end{cases} \tag{89}$$

If the mean of the candidate SDF is smaller than that of the benchmark SDF, the TMBM bounds based on $\alpha^{\mathbf{R}}$ are higher than those based on α, and vice versa. In an unreported figure, we plot the TMBM bounds for the CAPM based on both definitions of pricing errors under different weighting matrices.[25] We find that overall the two versions of the TMBM bounds are very close to each other.

E.2. Finite-sample bias

Previous studies on the Hansen–Jagannathan bound document finite-sample biases in estimates of the bound, for example, Burnside (1994), Cecchetti et al. (1994), and Ferson and Siegel (2003). In particular, Ferson and Siegel (2003) provide a bias correction based on a normality assumption. We take a nonparametric approach and provide bootstrap bias correction for the estimates of the Hansen–Jagannathan bounds and distance measures, and TMBM bounds.[26]

Fig. 5 plots the bias-corrected Hansen–Jagannathan bounds and TMBM bounds for the CAPM together with the unadjusted TMBM bounds. First of all, the bias-corrected Hansen–Jagannathan bounds are much lower than the

[24] An alternative is to include the risk-free return as a test asset so that a candidate SDF with a unreasonably low mean would misprice the risk-free return and incur greater pricing errors. This motivates our inclusion of the one-month T-bill return as a test asset in the previous empirical exercises.

[25] Available upon request.

[26] The details for the bootstrap methodology are provided in the Appendix.

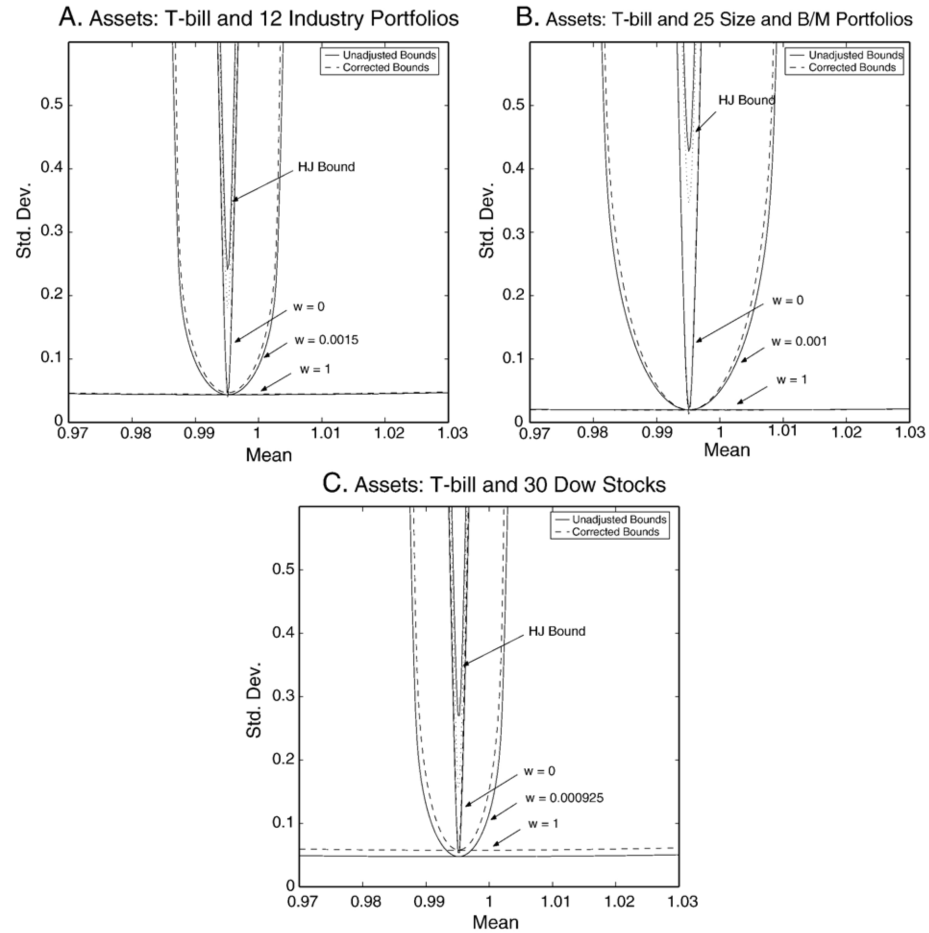

Fig. 5. Biased-corrected TMBM bounds for CAPM. The weighting matrix is $\mathbf{V}=wE(\mathbf{R}_t\mathbf{R}_t')+(1-w)\mathbf{\Omega}$, where $\mathbf{\Omega}$ is the covariance matrix or the asset returns. We use the monthly data for 1960:1–1999:12 (480 months). For the test assets in Panel A, B, and C, we use gross return for the one-month T-bill and excess returns for the risky assets.

bounds without the correction, confirming the upward finite-sample bias in the Hansen–Jagannathan bound documented by Burnside (1994), Cecchetti et al. (1994), and Ferson and Siegel (2003). In particular, Ferson and Siegel (2003) find that the corrected Hansen–Jagannathan bounds are even negative sometimes, losing their economic content.

The bias-corrected TMBM bounds are very close to the original bounds, for all the choices of the weighting matrices. This indicates that the finite-sample bias might not be a big concern for the TMBM bounds. To understand why the finite-sample bias for the TMBM bounds is small, we consider the special case of the TMBM bound when the weighting matrix is the covariance matrix of the asset returns, i.e., $\mathbf{V}=\mathbf{\Omega}$. This case is discussed in Corollary 3. The expression for the lower TMBM bound is given in Eq. (19) and is the difference between the Hansen–Jagannathan bound and distance measure. We also compute the bias-corrected Hansen–Jagannathan distance measure. The bias correction also substantially adjusts downward the GMM estimates for the distance measures.[27] Since the sample estimates for the Hansen–Jagannathan bound and distance measure are both upward biased, their difference is much less biased.

[27] This suggests that tests based on uncorrected distance measures might tend to over-reject asset pricing models, consistent with Ahn and Gadaroski (1999), who conduct Monte Carlo studies on the finite-sample properties of GMM estimates for the Hansen–Jagannathan distance measure.

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