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# Modelling multiple term structures of defaultable bonds with common and idiosyncratic state variables ☆

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**Abstract**

In this paper we develop a multi-factor "reduced-form" model that is general enough to capture simultaneously the dynamics of multiple term structures of corporate bonds, each with a different credit rating. In this way, we are able to fully incorporate a number of "stylised facts", reported on a number of previous empirical studies. More specifically, we are able to estimate the different degrees of covariation between the term structure of each credit rating and the default-free yield curve. Furthermore, we report the differing sensitivities of the credit curves to a number of observable macro-factors that reflect changes in credit conditions, both domestic and international. Finally, the dependence of each credit curve on a number of idiosyncratic state-variables is also documented. Our results are based on two special cases of the model, estimated using US and UK corporate bond data.


*JEL classification:* G1; E4; C5
*Keywords:* Corporate bonds; Credit spreads; Affine interest rate models

## 1. Introduction

Models of the term structure of defaultable corporate bond yields and other contingent claims subject to default risk have developed in two distinct directions. The first direction is based on the

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seminal work of Merton (1974), according to which the price of the equity of the firm can be modelled as a call option on the total value of its assets, struck at the face value of the debt. According to this approach, the probability that a firm defaults, over a period of time, is a function of the stochastic process followed by the value of the firm's assets and the distance between the value of these assets and the value of the firm's liabilities. Subsequent extensions of the Merton framework have been put forward by, among others, Black and Cox (1976), Geske (1977), Leland and Toft (1996), Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). Because default in these models is, in a sense, a function of the structure of the firm's balance-sheet, this approach is usually referred to as the "structural" approach.

Despite their sound theoretical foundations, these structural models have met limited empirical support. The balance-sheet of a modern corporation appears to be far too complex to be accurately captured by a tractable structural model. In addition, the complicated seniority structure of the payoffs to all firm's liabilities should ideally be identified by the model. As a result the structural models often provide a poor fit to the empirical data (see Jones et al., 1984; Eom et al., 2002).

An alternative approach, put forward by Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Lando (1998) and Duffie and Singleton (1999), models the time to default of a bond as the time of the first jump of a Poisson process with possible time varying intensity. Contrary to the structural approach, these "reduced-form" models provide no intuition as to the causes of default. Instead, default is treated as an exogenous event. The reason that these models are so appealing is their ability to price default-risky bonds by discounting (under the risk-neutral probability measure) future payoffs with a default-adjusted instantaneous short rate, i.e. the default-free short-term rate augmented by the process that determines the likelihood of default. This means that the vast array of the modelling techniques that have been developed for the default-free term structure of interest rates and for the pricing of fixed income instruments are easily transferable to credit risk modelling.

In addition to the development of theoretical models for pricing defaultable bonds, a series of econometric studies has sought to document the stylised features of default risk and corporate bond spreads. A number of studies by Fons, Carty and Kaufman (1994), Wilson (1997a,b) and Altman (1990) have all found significant links between the stage of the economic business cycle and the default rates in corporate bonds. Moreover, Elton, Gruber, Agrawal and Mann (2001) and Pedrosa and Roll (1998) have found a significant systematic component that affects credit spreads, irrespective of credit quality, within the economy. Finally, Collin-Dufresne, Goldstein and Martin (2001) report that, in addition to macroeconomic and financial variables, a substantial part of the variation of corporate bond spreads can be attributed to a corporate bond market idiosyncratic factor.

Previous attempts to estimate "reduced-form" credit risk models include Duffee (1999) and Bakshi, Madan and Zhang (2001). Duffee (1999) models the default intensity as a function of three variables; two determining the dynamics of the default-free term structure and one idiosyncratic to each obligor. Bakshi et al. (2001) also use a two-factor model for the default-free term structure (albeit a different one from that of Duffee, 1999) and introduce an observable firm specific factor such as leverage, profitability and book-to-market value ratio to capture the excess variation in corporate bond yields.

While our paper is closely related to the research of Duffee (1999) and Bakshi et al. (2001), it makes a number of contributions to the existing literature. Instead of modelling one defaultable term structure at a time, in isolation of all other corporate bonds or credit ratings, our aim is to model the joint evolution of a collection of term structures of defaultable bond yields with different credit ratings and at the same time capture the most significant stylised facts reported in the econometric literature. The vector of default-adjusted short rates of different credit ratings is modelled as an affine function of three components. The first component consists of the state variables determining the dynamics of the underlying default-free term structure of interest rates. The second component consists of a set of observable, domestic and international, economy-wide factors. An innovative feature of our model is that we do not employ macro-variables, such as GDP or industrial production, as proxies for these economy-wide factors, because these variables do not always capture the cyclical variations in the credit conditions in the economy. Instead, we estimate a credit cycles indicator that reflects, more accurately, changes in the financing costs of the corporate sector and the risk premia associated with them. Contrary to previous studies we do not employ principal components analysis to extract the common factor from a set of financial variables. We introduce a dynamic factor model that allows for a richer specification of the credit cycles dynamics. Finally, we explore further the significance of residual factors, not related to either the default-free state variables or to credit cycles, by introducing a set of factors idiosyncratic to each credit category.

This framework is flexible enough to allow us to examine several important features of the empirical data. More specifically, we examine the ability of different specifications of the model to fit the default-free term structure along with the term structures of AA, A and BBB corporate bonds using US and UK data. The existence of common factors between the default-free and corporate term structures allows us to examine the covariation between default-free and defaultable bond yields documented by Duffee (1998). In addition, we study the impact of domestic credit cycles on the term structures of corporate bonds, as well as the interlinkages between the US and UK corporate bond markets. We estimate the relative significance of these factors based on the impact that shocks to these factors have on the term structures of corporate bond spreads across the ratings and maturity spectrum.[1] The final issue to discuss is the selection of the dataset we employ to estimate our credit risk model. One option is to estimate a default intensity for each obligor available and subsequently aggregate the results according to the credit quality of the obligor. We have not followed this approach for several reasons: The first is the obvious difficulty of compiling such a dataset for the US and UK. The second reason is related to the problem of parameter instability that arises when the credit quality of the underlying obligors changes (either upgraded or downgraded) during the sample period.[2] Last but not least, the use of individual bonds would complicate significantly the specification and interpretation of the risk premia in our model. As Jarrow, Lando and Yu (2002) and Yu (2002) point out, if the default-event risk is non-systematic then investors of diversified bond portfolios should require compensation for systematic variations in the overall level of default risk, but should not require an additional risk premium for the event of default itself. The default-event premium is difficult to estimate and if ignored it can distort the estimates of the remaining risk premia (see Jarrow et al., 2002). If the assumption of diversifiable default-event risk holds, then, by estimating our models using corporate bond indices with constant credit quality, the risk premia associated with our default intensities will reflect investors' compensation for systematic variations in the default risk of corporate bonds, but not for the default-event itself.

[1] The emphasis in this paper is to develop a framework that allows for the joint evolution of default intensities, through the existence of common factors. However, our model is not a ratings-based model, in the form of Lando (1998), Huge and Lando (1999), Arvanitis et al. (1999) and Li (2000). As a result our default intensities are determined by the dynamics of probabilities of default and do not reflect the potential of upgrades or downgrades of the underlying bonds, which would result in a shift of the default intensity to that of the new rating. The use of constant credit quality corporate bond indices to estimate the dynamics of the default intensities is consistent with our theoretical formulation.

[2] For additional discussion of this issue see Duffee (1999).

The rest of the paper is organised as follows. In Section 2, we discuss the models for the default-free and defaultable term structures of interest rates. We also specify two special cases of our model. The first case allows for the vector of default intensities to depend on the domestic credit conditions and a credit rating idiosyncratic factor. The second case models the default intensities as a function of both domestic and international credit conditions. Section 3 presents the data and the Kalman filter methodology used to estimate the affine term structure models. Section 4 discusses the construction of the credit cycles indicators. Sections 5 and 6) present the results for the credit cycles dynamics and the default-free, and defaultable term structure models. Finally, Section 7 concludes.

## 2. Theoretical model

### 2.1. Default-free term structure

Our formulation of the risk-free term structure of interest rates is based on the "affine" class of interest rate models put forward by Duffie and Kan (1996) and recently extended and generalised by Dai and Singleton (2000). Under the equivalent martingale measure, $Q$, the price, at time $t$, of a default-free zero-coupon bond maturing at time $T=t+\tau$ is given by

$$P(t,\tau)=E_t^Q\left\{\exp\left(-\int_t^{t+\tau}r_u\mathrm{d}u\right)\right\} \tag{1}$$

where $r_t$ is the instantaneous risk-free interest rate.[3] An $N$-factor affine interest rate model is obtained by allowing $r_t$ to be an affine function of the $N$ underlying factors $X_{it}$, $i=1, \ldots, N$

$$r_t=\delta_1X_{1t}+\delta_2X_{2t}+\ldots+\delta_NX_{Nt}\equiv\delta_X^{'}X_t \tag{2}$$

where the affine state vector $X_t=(X_{1t}, \ldots, X_{Nt})'$ satisfies

$$\mathrm{d}X_t=K_X(\Theta_X-X_t)\mathrm{d}t+\Sigma_X\sqrt{S_{X_t}}\mathrm{d}W_{X_t} \tag{3}$$

where $K$ is an $N\times N$ matrix that contains the mean reversion coefficients that determine the rate to which the state variables $X_{1t}$ to $X_{Nt}$ revert to their long-term mean $\Theta$, where $\Theta=(\vartheta_{X_1},\ldots,\vartheta_{X_N})'$. $\Sigma_X$ and $S_{X_t}$ are also $N\times N$ matrices that determine the correlation and the volatility of the $N$ state variables and $\mathrm{d}W_{X_t}$ is the increment of a standard $N$-dimensional Brownian motion under the empirical probability measure. The transition from the empirical to the risk-neutral measure, $Q$, used for pricing in Eq. (1) is completed by specifying the prices of risk for the underlying factors as $\sqrt{S_{X_t}}\Xi_X$ where $\Xi_X=(\xi_{X_1},\ldots,\xi_{X_N})'$ is a vector of constants.

Eq. (3) is the most general representation of an affine $N$-factor model and as Dai and Singleton (2000) and de Jong (2000) point out not all possible specifications are admissible or identifiable.[4] In our model we allow $X_t$ to follow a multivariate independent square-root process. As a result $K$ is a diagonal matrix with $k_{X_1}, \ldots, k_{X_N}$ elements on its main diagonal, $\Sigma_X$ is an identity matrix and

[3] For a more rigorous discussion of the technical conditions under which Eq. (1) holds see Duffie (1992).

[4] A model is admissible if it leads to well defined bond prices and it is identifiable if all of its parameters can be separately estimated using discretely sampled data. For a more detailed discussion see Dai and Singleton (2000).

$S_{X_t}$ is a $N \times N$ diagonal matrix with $X_{1t}, \ldots, X_{Nt}$ on its diagonal. Under these assumptions the price of the $\tau$-maturity zero-coupon bond is given by

$$P(t,\tau) = A(\tau)e^{-B'(\tau)X_t} \tag{4}$$

where $A(\tau)$ and $B(\tau)=(B_1(\tau), \ldots, B_N(\tau))'$ are solutions to ordinary differential equations. In the case of $N$ independent square-root processes, the closed form solutions for $A(\tau)$ and $B(\tau)$ are given by Cox, Ingersoll and Ross (1985).

The $\tau$-maturity yield, $r(t, \tau)$, of the default-free zero-coupon bond is

$$\begin{aligned} r(t,\tau) &= -\frac{1}{\tau}\ln P(t,\tau) \\ &= -\frac{A(\tau)}{\tau} + \frac{B(\tau)}{\tau}X_t \end{aligned} \tag{5}$$

Eq. (5) characterises the entire term structure of default-free interest rates.

### 2.2. Term structure of corporate bonds

Our model of corporate bonds falls within the reduced form category of Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), and Duffie and Singleton (1999). In this category of models the default process of the $j$-th obligor is controlled by the default intensity, or hazard process, $\lambda_{jt}$. Let $\tau_d$ represent the random time at which default occurs. Then the probability of no-default (survival) over a period from $t$ to $T$, given no prior default, is

$$p_j(\tau_d > T) = E_t\left[e^{-\int_t^T \lambda_{ju}\mathrm{d}u}\right]$$

and the probability of default over a small time interval d$t$, conditional on no prior default is $\lambda_{jt}$d$t$.

Under technical conditions discussed in Duffie and Singleton (1999), the price of a $\tau$-maturity, zero-recovery, zero-coupon bond is given by

$$V_j(t,\tau) = E_t^Q\left\{\exp\left(-\int_t^{t+\tau}(r_u + \lambda_{ju})\mathrm{d}u\right)\right\} \tag{6}$$

where $j$ denotes the $j$-th class of corporate bonds. In our general framework a credit class can be interpreted as an indicator of credit quality, such as a rating category assigned to bonds by a rating agency such as Moody's or Standard & Poor's. Alternatively, it can be used as an indicator of all bonds issued by the $j$-th obligor. Hence, $\lambda_{jt}$ is the intensity of a Cox process used to model the event of default of the $j$-th credit class.

We assume that there exist $K$ corporate bond classes, i.e. $j=1, \ldots, K$. An affine term structure model for the term structures of all corporate bonds can be recovered if we allow the vector $\Lambda_t = (\lambda_{1t}, \lambda_{2t}, \ldots, \lambda_{Kt})'$ of all default intensities to be an affine function of the underlying state variables. The vector of default intensities, $\lambda_t$, is modelled as a function of:

(i) the $N$-state variables that determine the dynamics of the default-free government bond yields, $X_t=(X_{1t}, \ldots, X_{Nt})'$,
(ii) a set of $L$ observable factors, $C_t=(C_{1t}, \ldots, C_{Lt})'$, that summarise fluctuations in the credit conditions in the economy and are common to all defaultable term structures, and

(iii) a set of $K$ factors, $Z_t=(Z_{1t}, \ldots, Z_{Kt})'$, each idiosyncratic to a particular class of defaultable corporate bonds. Hence,

$$\Lambda_t = \Lambda_0 + \Gamma X_t + HC_t + \Pi Z_t \tag{7}$$

where $\Lambda_0$ is a $K\times 1$ vector of constants, $\Gamma$ is a $K\times N$ matrix of sensitivities of default intensities to the default-free term structure, $H$ a $K\times L$ matrix of sensitivities to the observable credit cycles indicators and $\Pi$ a diagonal $K\times K$ matrix of factor loadings to the idiosyncratic factors.

In Section 4, we discuss extensively the dynamic factor methodology we employ in order to extract the credit cycles indicators from a large collection of financial variables. One feature of the methodology is that the indicators are extracted under the assumption that they follow a normally distributed, zero-mean process $AR(1)$ process. Hence, in continuous time, $C_t$ follows a Gaussian diffusion process

$$\mathrm{d}C_t = K_C(\Theta_C - C_t)\mathrm{d}t + \Sigma_C \mathrm{d}W_{C_t} \tag{8}$$

with $K_C$ and $\Sigma_C$ diagonal $L\times L$ matrices, $\Theta_C$ an $L\times 1$ vector with the restriction that $\Theta_C\equiv 0$ and $\mathrm{d}W_{C_t}$ $L$-dimensional increments of a Brownian motion under the empirical measure. Notice that we are able to identify the elements of both $H$ and $\Sigma_C$ simply because $C_t$ is a vector of observable factors. Otherwise, one of these matrices should have been normalised to the identity matrix.[5]

Finally, we model the vector of idiosyncratic factors, $Z_t$, as a square-root process

$$\mathrm{d}Z_t = K_Z(\Theta_Z - Z_t)\mathrm{d}t + \Sigma_Z\sqrt{S_{Z_t}}\mathrm{d}W_{Z_t} \tag{9}$$

with $K_Z$ a diagonal $K\times K$ matrix, $\Theta_K$ a $K\times 1$ vector and $\mathrm{d}W_{Z_t}$ $K$-dimensional increments of a Brownian motion under the empirical measure. $\Sigma_Z$ is an identity matrix and $S_{Z_t}$ is a $N\times N$ diagonal matrix with $Z_{1t}, \ldots, Z_{Kt}$ in its diagonal.

The vector of $K$ default-adjusted short rates, $Y_t=(y_{1t}, y_{2t}, \ldots, y_{kt})'$, where $y_{jt}=r_t+\lambda_{jt}$, can be expressed as

$$\begin{aligned} Y_t &= \mathbf{1}\delta_X' X_t + \Lambda_t \\ &= \mathbf{1}\delta_X' X_t + \Lambda_0 + \Gamma X_t + HC_t + \Pi Z_t \\ &= \Lambda_0 + (\mathbf{1}\delta_X' + \Gamma)X_t + HC_t + \Pi Z_t \end{aligned} \tag{10}$$

where $\mathbf{1}$ is a $K\times 1$ vector of ones. The $j$-th credit class zero-coupon bond with zero recovery at default is given by

$$V_j(t,\tau) = A_j(\tau)\exp\{-B_{j1}(\tau)X_t - B_{j2}(\tau)C_t - B_{j3}(\tau)Z_t\} \tag{11}$$

where $B_{j1}(\tau)=(b_{11}(\tau), \ldots, b_{1N}(\tau))'$, $B_{j2}(\tau)=(b_{21}(\tau), \ldots, b_{2L}(\tau))'$, and $B_{j3}(\tau)=(b_{31}(\tau), \ldots, b_{3K}(\tau))'$. In a way similar to the default-free case the transition from the empirical probability measure, used for econometric estimation, to the risk-neutral measure used for pricing in Eq. (6) is completed by specifying the prices of risk for $C_t$, $\Xi_C=(\xi_{C_1}, \ldots \xi_{C_L})'$, and for $Z_t$, $\sqrt{S_{Z_t}}\Xi_Z$, where $\Xi_Z=(\xi_{Z_1}, \ldots, \xi_{Z_K})'$, is a vector of constants. This specification of the risk premia follows the analysis of Jarrow, Lando and Yu (2002) who show that if default events are non-systematic then investors who hold a well

[5] From a theoretical point of view the use of the Gaussian process for the dynamics of $z_t$ is undesirable because it permits negative default intensities. Nevertheless, the Gaussian process is de facto imposed from the dynamic factor model specification for the estimation of the credit cycles indicators.

diversified bond portfolio should demand compensation, in the form of risk premia, for systematic variation in the credit quality of bonds that is introduced by the state variables, but not for the event of default itself.

Eq. (11) provides the prices of the $j$-th class corporate bond assuming zero-recovery. Nevertheless, research shows that in almost all bankruptcies investors manage to recover a significant amount of their invested funds. There exist several alternative ways to incorporate recovery-at-default in our framework. The simplest way is to assume that investors recover a fraction of par at default. Alternatively we can follow Jarrow et al. (1997) and assume that after default the bond trades at a percentage of an equivalent default-free bond. Duffie and Singleton (1997, 1999) assume that investors receive a fraction of the value of the bond immediately prior to default. Finally, Jarrow and Turnbull (2000) assume that investors recover a fraction of their legal claim, that consists of principal plus implicit accrued interest until the time of default. We proceed by following Jarrow et al. (1997) and Duffee (1999) and assume that, at default, investors recover a fraction $\mu_j$, of the value of an equivalent default-free bond. No arbitrage dictates that prior to default the value of the $j$-th class corporate bond with recovery $\mu_j$ is given by

$$V_j(t,\tau,\mu_j) = \mu_j P(t,\tau) + (1-\mu_j)V_j(t,\tau) \tag{12}$$

In our case we allow the recovery rate to vary with the credit-class of the bond. Altman and Kishore (1998) find that $\mu_{\mathrm{AA}}=\mu_{\mathrm{A}}=60\%$ and $\mu_{\mathrm{BBB}}=49.5\%$.

### 2.3. Special cases

The model in its most general form, given by Eq. (10) can be considered as a conceptual framework that helps in understanding the potential risk-drivers that affect the dynamics of, and the covariation between, all term structures of defaultable yields. This framework can be specialised to accommodate several models as particular cases. Which one will be implemented depends upon data availability and the research question at hand. We proceed by estimating two special cases for both the US and UK, using data for AA, A and BBB corporate bonds, i.e. $Y_t=(y_{\mathrm{AA}t}, y_{\mathrm{A}t}, y_{\mathrm{BBB}t})'$. In the first case, we look at the corporate bond markets of each country in isolation and we try to examine the ability of this class of models to describe the dynamics of corporate bond yields and the relative importance of the various domestic factors, both observed and unobserved. In the second case, we focus more on the international linkages between corporate bond markets and examine the significance of foreign vs. domestic factors on each term structure of corporate bonds.

#### 2.3.1. Case I: domestic credit cycle

In Case I, the specification of the model allows us to examine, in a more rigorous and systematic framework, the relative significance of "stylised" features of corporate bond spreads documented in previous empirical studies. First, based on the results of Duffee (1998), we model the default intensities as a function of the default-free state variables. Following Pedrosa and Roll (1998) and Elton et al. (2001), we also explore the dependence of default risk upon an economy-wide systematic factor. Finally, we augment the model with a number of additional factors, each idiosyncratic to each credit class, in order to investigate the existence of additional factors argued for by Collin-Dufresne et al. (2001).

More specifically, we model the term structures of AA, A and BBB corporate bond yields as a function of two, default-free state variables, i.e. $N=2$, $X_t=(X_{1t}, X_{2t})'$, the domestic credit cycle indicator, i.e. $L=1$, so that $C_t=C_{1t}$, and three state variables each idiosyncratic to one of the three credit classes, i.e. $Z_t=(Z_{AAt}, Z_{At}, Z_{BBBt})'$.

### 2.3.2. Case II: domestic and international credit cycles

In Case II, we shift our attention to the international interlinkages between default risk in different economies and the impact that both domestic and international credit conditions have on the term structures of corporate bond yields. We achieve that by augmenting the dimensions of $C_t$ to include both US and UK credit cycles indicators. In this case, we model AA, A and BBB term structures as a function of two default-free state variables, i.e. $N=2$, $X_t=(X_{1t}, X_{2t})'$, and the two observable variables that capture the fluctuation of the domestic and foreign credit conditions, i.e. $L=2$, $C_t=(C_{1t}, C_{2t})'$.

## 3. Data and estimation method

### 3.1. Data

Our primary data source is the "fair-value" par-coupon corporate bond yields, provided by Bloomberg Financial Services.[6] We use weekly data for US and UK Industrial corporate bond yields with 6-month, 1-, 3-, 5-, 7-, 10-, 20- and 30-year maturities for the period from 28/March/1993 to 1/November/2002. The credit ratings available are AAA, AA, A and BBB. The "fair-value" yields are estimates of newly issued par-coupon bond yields for a given rating, adjusted for the value of embedded options.

We use the 1-, 5-, 10- and 30-year bond yields to estimate the parameters of our models. The remaining maturities are used to assess the out-of-sample pricing performance of the models. An additional issue is the choice of the reference default-free term structure used for the estimation of the default-free state variables. The most obvious choice would be the government term structure. Nevertheless, the use of government bond yields is usually problematic given the liquidity distortions on the government bond market, created by reduced issuance of government bonds, in both the US and the UK, during the 1990s, problems with on-the-run vs. off-the-run Treasury bonds and inefficiencies in the repo-market. As a consequence, Houweling and Vorst (2001) and Jarrow, Lando and Yu (2002) report that using the government term structure as a reference curve to calibrate default intensities results in unrealistically high default probabilities.

There are two main alternatives: the swap yield curve and the AAA yield curve. Houweling and Vorst (2001) report that the use of the swap yield curve as a reference curve produced a much better fit to the observed default probabilities. Nevertheless, the available interest rate swap data have an average AA credit rating. This might induce a downward bias of our estimates of corporate bond default probabilities, especially for the high quality bonds. For that reason we follow Jarrow, Lando and Yu (2002) and employ the AAA corporate bond yields as the reference term structure. Our choice is further supported by the findings of Duffee (1999) who reports a very bad fit of default intensities to AAA bond yields implying a negligible probability of default.

A linear relationship between bond yields and the state variables exists only for zero-coupon bonds. For that reason, we use the Svensson (1994, 1995) methodology to estimate zero-coupon yields from the par-coupon rates (see Appendix A for more details). The estimation of zero-coupon yields out of coupon-paying bond prices will introduce some approximation error. The alternative, used by Duffee (1999), is to allow for a non-linear mapping between the coupon bond

[6] This is the same data source employed by Pedrosa and Roll (1998) and Duffie and Singleton (2003).

prices and the state variables. In that case, some form of linear approximation will have to be introduced in the later stages of the econometric estimation of the model (the extended Kalman filter methodology uses a Taylor approximation in order to linearize the relationship between the observed and state variables). We have chosen to use the Svensson methodology because the properties of the Svensson's parametric discount function are better understood and the quality of the fit to the data is easier to assess.[7]

### 3.2. Econometric methodology

The affine term structure models express the universe of zero-coupon bonds as an affine function of a small number of state variables. Although several (or all) of the state variables might be unobserved, by using as many bond yields as unobserved factors, the term structure Eq. (5), can be inverted to provide estimates for the unobserved factors and their dynamics. In our estimation we include more bonds than factors and assume that all bonds are observed with some error. In this way we increase the cross-sectional information used to estimate the models and we avoid the issue of which of the bonds available are better assumed to be observed with no error. This approach has also been employed in the past for the estimation of affine term structure models by Pennacchi (1991), Ball and Torous (1996), Chen and Scott (1995), de Jong (2000) and Gong and Remolona (1996a,b). Finally, it has to be mentioned that we do not impose any assumptions regarding the identity or nature of the underlying factors. Once the estimation is completed, their identity is extracted by examining their impact along the term structures of default-free and defaultable interest rates.

One advantage of assuming independence of the state variables is that the estimation of our model can be separated into three stages. In the first stage, we assume that four default-free zero coupon bonds with 1-, 5-, 10-, and 30-year maturity are all observed with a measurement error. We then employ the Kalman filter methodology, similar to Duffee (1999), de Jong (2000) and Duan and Simonato (1999), to estimate the vector of state variables $X_t$, and its dynamics. The second stage is to estimate the dynamics of the CCI process $C_t$, which is straightforward given that $C_t$ is observable and follows a Gaussian $AR(1)$ process. In the third stage, we assume that the inferred state vector $X_t$ and the parameters governing the evolution of $X_t$ and $C_t$ are the true ones and proceed to estimate the vector of idiosyncratic state variables $Z_t$ and the prices of risk for the CCI factors.

#### 3.2.1. Default-free parameter estimation

In both Case I and Case II, the default-free state vector follows a bivariate independent square-root process. In order to make the presentation more transparent we transform the short rate process

$$r_t = \delta_1 X_{1t} + \delta_2 X_{2t} = X^*_{1t} + X^*_{2t} \tag{13}$$

where $X^*_{it} = \delta_i X_{it}$, $i = 1, 2$. This is equivalent to setting $[\Sigma_{X^*}]_{ii} = \sqrt{\delta_i}$ and $[\Theta_{X^*}]_i = \delta_i [\Theta_X]_i$.

The closed form solution of a zero-coupon bond price is

$$P(t,\tau) = A_{X_1^*}(\tau) A_{X_2^*}(\tau) \exp\left\{-B_{X_1^*}(\tau) X^*_{1t} - B_{X_2^*}(\tau) X^*_{2t}\right\} \tag{14}$$

where $A^*_{X_1}(\tau)$, $A^*_{X_2}(\tau)$, $B^*_{X_1}(\tau)$, $B^*_{X_2}(\tau)$ are given by Cox, Ingersoll and Ross (1985) for the transformed set of parameters.

[7] More information about the estimated parameters of the Svensson discount function and the fitting error is available from the author upon request.

At time $t$, we observe the 1-, 5-, 10-, and 30-year zero-coupon default-free yields, $R_t=(r_{1t}, r_{5t}, r_{10t}, r_{30t})'$. The observation equation of the state-space system is

$$R_t = A_{X^*}(\tau) + B_{X^*}(\tau)X_t^* + \varepsilon_t \tag{15}$$

where $A_X^*(\tau)=-(\ln\ A_{X_1}^*(\tau)+\ln\ A_{X_2}^*(\tau))/\tau$, $B_X^*(\tau)=(B_{X_1}^*(\tau)/\tau,\ B_{X_2}^*(\tau)/\tau)$ and $\tau=1$, 5, 10 and 30 years. The measurement errors covariance matrix, $\Sigma_{\varepsilon,t}\equiv E(\varepsilon_t, \varepsilon_t')$, is a $4\times4$ diagonal matrix with $[\Sigma_\varepsilon]_i$, $i=1, \ldots, 4$ on its main diagonal.

The transition equation depends on the conditional mean and variance of the state variables

$$X_t^* = \Phi_1 + \Phi_2 X_{t-h}^* + \eta_t \tag{16}$$

where $h$ is the time interval between two observations, $\Phi_1 = \begin{pmatrix} \vartheta_{X_1^*}(1-e^{-\kappa_{X_1^*}h}) \\ \vartheta_{X_2^*}(1-e^{-\kappa_{X_2^*}h}) \end{pmatrix}$ and $\Phi_2 = \begin{pmatrix} e^{-\kappa_{X_1^*}h} & 0 \\ 0 & e^{-\kappa_{X_2^*}h} \end{pmatrix}$ and the covariance matrix of the state vector, $\Sigma_{\eta,t}\equiv E(\eta_t, \eta_t^*)$, is a $2\times2$ diagonal matrix with

$$X_{it}^* \frac{\sigma_{X_i^*}^2}{\kappa_{X_i^*}}(e^{-\kappa_{X_1^*}h}-e^{-2\kappa_{X_1^*}h}) + \vartheta_{X_1^*}\frac{\sigma_{X_i^*}^2}{2\kappa_{X_i^*}}(1-e^{-\kappa_{X_1^*}h})^2 \tag{17}$$

$i=1$, 2 on its diagonal.

Based on the observation Eq. (15) and the transition Eq. (16) the Kalman filter methodology, described in Hamilton (1994), is employed to estimate the parameters and the unobserved state variables (for a detailed description of the Kalman filter algorithm see Appendix D). A well known drawback of using a linear Kalman filter to estimate the parameters of square-root models, where the error terms, $\eta_t$, are not normally distributed and their conditional variance, given by Eq. (17), depends upon the unknown realisation of the unknown factors, is that both maximum likelihood and quasi-maximum likelihood (QML) are inefficient. The bias of QML is discussed extensively by Duan and Simonato (1999), de Jong (2000) and Lund (1997). They employ Monte Carlo simulations to demonstrate that for one-factor square-root models the bias introduced by QML is negligible. In Appendix E, we extend their simulations to the case of our two-factor models.[8] The results, reported in Table 12, show that, in accordance with the results for the one-factor models, no serious bias exists, with one exception of the case with the mean reversion parameter very close to zero. Nevertheless, the estimation in this case of borderline non stationarity is biased even in the Gaussian example and is not clear that this bias can be attributed to the use of QML.

### 3.2.2. Estimation of CCI parameters

The details of the construction of the credit cycles indicators, $C_{lt}$, $l=1, \ldots, L$, using a dynamic factor model, is discussed in the following section. Given these indicators, it is straightforward to estimate the coefficients of Eq. (8) via maximum likelihood estimation. The conditional mean and variance are given by[9]

$$E(C_{lt+h}|C_{lt}) = \vartheta_{C_l} + e^{-\kappa_{C_l}h}(C_{lt}-\vartheta_{C_l}) \tag{18}$$

[8] We would like to thank the referee for suggesting this idea to us.

[9] The conditional mean and variance are estimated using an exact discretisation of the Gaussian process, instead of the more intuitive Euler discretisation scheme that we discuss in footnote (11), in order to avoid the discretisation bias introduced by the Euler approximation.

and

$$\mathrm{Var}(C_{lt+h}|C_{lt}) = \sigma^2_{C_l}\left(\frac{1-e^{-2\kappa_{C_l}h}}{2\kappa_{C_l}}\right) \tag{19}$$

Then the value of the log-likelihood function, $f_{Ct}$, is

$$-2\ln f_{Ct} = \ln\left\{\sigma^2_{C_l}\left(\frac{1-e^{-2\kappa_{C_l}h}}{2\kappa_{C_l}}\right)\right\} - \left\{\frac{C_{lt}-\vartheta_{C_l}-e^{-\kappa_{C_l}h}(C_{lt}-\vartheta_{Cl})}{\sigma^2_{C_l}\left(\frac{1-e^{-2\kappa_{C_l}h}}{2\kappa_{Cl}}\right)}\right\}^2. \tag{20}$$

*3.2.3. Case I and Case II model estimation*

Again we begin by transforming the dynamics of the vector of default adjusted short rate

$$Y_t = \Lambda_0 + (\mathbf{1}\delta'_X + \Gamma)X_t + HC_t + \Pi Z_t = \Lambda_0 + X^*_t + C^*_t + Z^*_t \tag{21}$$

where $X^*_t = (\mathbf{1}\delta'_X + \Gamma)X_t$, $C^*_t = HC_t$ and $Z^*_t = \Pi Z_t$. Based on this specification for the vector of the defaultable short rate, the observation and transition equation can be formulated in a manner similar to the default-free model. The only difference is that we consider the estimated parameters $K_X$, $\Theta_X$, $\delta_X$, $\Xi_X$, $K_C$, $\Sigma_C$ and the implied vector of state variables $X_t$ as the true ones. As Duffee (1999) points out, this results in some loss of efficiency but it significantly simplifies the maximisation problem. We estimate the parameters $\Lambda_0$, $\Gamma$, $H$, $\Xi_C$ and in Case I, also $\Pi$, $K_Z$, $\Theta_Z$, $\Xi_Z$ and $Z_t$, using a Kalman filter approach similar to that used for the estimation of the default-free term structure.

## 4. Credit cycles indicators

An innovative feature of our model is that it allows the default intensities to depend upon a set of variables that capture the dynamics of the credit conditions in the economy as they fluctuate with the business cycle. As will be discussed in the following sections, a number of variables have been found in the literature to be able to describe several aspects of the credit conditions in the economy. In order to be able to keep the dimensions of our model to a manageable level, we proceed by estimating credit cycles indicators for the US and UK economies and use these as proxies for $C_t$ in our model. These indicators are able to summarise general changes in the cost of borrowing for corporate sectors and investors' perceptions of the risks associated with the business cycles in the two countries.

There are two main methodologies that can be employed to estimate the credit cycles indicator: factor analysis and state-space models. According to factor analysis, each observable variable under examination consists of a *common* and a *unique* component. The common component is that part of the variable's variation that is shared with the other variables in the system and can be attributed to a small number of latent (unobserved) factors, whereas the unique component is idiosyncratic to the variable. The estimation of factor models is ultimately based on the correlation matrix of the underlying variables, which is assumed constant. Because of that, factor analysis is frequently criticised on the basis that, while it is successful in measuring static interrelationships between observable variables, it often fails to capture the dynamics of the underlying system.

Stock and Watson (2002) have developed a model that is more finely tuned towards capturing the dynamics of the underlying variables. In their first paper, Stock and Watson (1989) express the dynamic factor model in state-space form and use Kalman filtering techniques to identify one unobserved factor and estimate its dynamics. One disadvantage of this methodology is that the model is estimated by maximising the log-likelihood function of the state-space system, which is possible only when a small number of observed variables are included in the system.[10] In subsequent work Stock and Watson (2002) propose a factor model that is estimated through principal components analysis. This allows for the addition of a large number of observable variables in the system, but the model is dynamic only in its asymptotic limit i.e. when the number of time series tends to infinity.

Our research strategy is to extract a dynamic CCI based on a factor model that avoids the shortcomings of the Stock and Watson (1989) specification mentioned above. This becomes possible by estimating the dynamic factor model using the subspace algorithm methodology proposed by Kapetanios (2002) and discussed in Appendix B.

### 4.1. Dynamic factor models

In order to be able to formulate the dynamic CCI model in state space form, we rewrite the dynamics of $C_t$ as a first order $AR(1)$ model. Then, the state-space form of the dynamic factor models is[11]:

$$F_t = J_F C_t + D_F u_t \tag{22}$$

$$C_t = A_C C_{t-h} + B_C u_t \tag{23}$$

Here, $F_t$ is an $n$-dimensional vector of the observable, zero-mean, variables observed at time $t$. $C_t$ is an $L$-dimensional vector of unobserved state variables at time $t$, $u_t$ is a multivariate white noise sequence and $h$ is the time interval between to consecutive observations. We extract the common factors $C_t$ using the subspace algorithm described in Appendix B. Following that, we estimate the sensitivities (or loadings), $J_F$, of the observed variables to the common factors by regressing $C_t$ on $F_t$.[12] The estimation of the dynamics of $C_t$ is discussed in the following section along with the estimation of the rest of the model.

[10] The maximisation of the likelihood function of the state-space model is not feasible when the dimension of the model becomes too large due to computational problems. Stock and Watson (1989) identified a business cycle indicator using only up to 4 observable variables.

[11] Using the Euler discretisation scheme it is easy to show that Eq. (23) is a discrete time approximation of Eq. (8). Let $h$ be the time internal between two observations. Eq. (8) can be written as

$$C_t - C_{t-h} \simeq K_C \Theta_C h - K_C h C_{t-h} + \Sigma_C \sqrt{h u_t}$$

where $u_t$ is an iid error term. If $\Theta_C \equiv 0$, then

$$C_t \simeq (1 - K_C h) C_{t-h} + \Sigma_C \sqrt{h u_t}$$

which is equivalent to Eq. (23) by setting $A_C = (1 - K_C h)$ and $B_C = \Sigma_C \sqrt{h}$. We choose to estimate Eq. (8) instead of (23) simply because the bond pricing model is expressed in terms of the parameters in Eq. (8).

[12] As a robustness check of the subspace algorithm, we also estimate a static factor model using principal components analysis. The patterns of factor loadings of the two models are comparable. Examining the time series properties of the static and dynamic CCIs we find that the dynamic CCIs are, as expected, more responsive to changes to the underlying financial variables, but in broad terms the static and dynamic CCIs trace each other reasonably well. More detailed results of this comparison are available from the author upon request.

### 4.2. Variables included in the indicators

Perhaps the most important building block of our analysis is the choice of variables that we employ in order to estimate the common factor. After all, the estimated CCI will be a linear combination of these variables. The selection of these variables is guided by both theory and data availability. We examine variables that can be classified into 5 broad categories, all of which can be partially related to the credit conditions in the economy. These categories are: the government bond market, the short-term interbank market, the interest rate swap market, the corporate bond market and finally, the stock market.[13] More specifically we examine:

#### 4.2.1. Term structure of government bond yields

The first set of variables are designed to capture movements in the term structure of government bond yields. In particular, following work by Litterman and Scheinkman (1991), Lekkos (2000) and Lardic, Priaulet and Priaulet (2001) we examine three variables — the slope, level and curvature of the term structure.[14] The term structure of government bonds is the benchmark against which all corporate debt is priced; as such it does not contain information directly relevant for pricing corporate credit risk. However, as Friedman and Kuttner (1989), Estrella and Mishkin (1998) and Estrella and Hardouvelis (1991) have shown, the slope of the term structure does reflect expectations about future economic activity and can forecast changes in the business cycle. In addition, Campbell (1987) and Fama and French (1989) document that the slope of the term structure has predictive power for excess returns on a variety of assets. Hence, term structure variables can be viewed as a leading business cycle indicator capturing information about future changes in credit spreads. This is supported by the findings of Duffie and Singleton (1997) that shocks to the slope and the level of the term structure are transmitted to the interest rate swap spreads.

#### 4.2.2. Treasury eurodollar spread (TED)

This is the spread between 3-month LIBOR and the 3-month Treasury Bill (or T-Bill) rate. As such it should contain information about the short-term credit quality of the obligor. Nevertheless, as Grinblatt (1995) argues, TED spreads may also reflect changes in the liquidity of the interbank market in the form of a convenience yield that investors are willing to pay for the greater liquidity of the Treasury Bill market. To the extent that liquidity conditions will affect all spreads in the economy, they can become a source of systematic variation in corporate bonds, which the TED variable will be able to capture.

#### 4.2.3. Interest rate swap spreads

We also include the 3-year, 7-year and 10-year interest rate swap spreads. The presence of interest rate swap spreads serves two purposes. First, interest rate swaps are important instruments that financial and non-financial corporations use to hedge and transfer interest rate risks to and from their books. In addition, although swaps are affected by credit risks, they are not identical to corporate bond spreads, discussed later on in this section. One major difference is that swaps can

[13] Some overlap between the variables included in the CCIs and the term structures that we subsequently indent to model is unavoidable if we want to extract an indicator that genuinely reflects the credit conditions of the corporate sector. Specific econometric issues should not arise given that the estimated CCI is a linear combination of all variables used in its estimation and its correlation with any particular variable is limited.

[14] A detailed list of all variables included in the CCI models is provided at the end of the paper.

be defined as either assets or liabilities depending on the future evolution of interest rates. Hence, as discussed in Duffie and Huang (1996) and Li (1998) the spreads between swap counterparties with different credit ratings should be lower compared to their equivalent corporate bond spreads. Also, the presence of credit enhancements, i.e. collateral, guarantees etc., and credit rationing, differentiate the behaviour of swap spreads to those of corporate bond spreads.

*4.2.4. Absolute and relative corporate bond spreads*

Two sets of corporate bond spreads are examined. The first set, referred to as *absolute* corporate bond spreads, is the difference between AAA non-financial corporate bond yields and equivalent government bond yields across maturities. The second set, referred to as *relative* corporate bond spreads, is the spreads between BBB and AAA corporate bond yields. Both sets of spreads are examined because they reflect different risk factors. It is often argued that default risk in the highest quality bonds is so low that absolute bond spreads reflect a variety of other factors such as liquidity, taxes and also structural features of the underlying government bond market, such as the illiquidity of the long-term UK government bonds due to the MFR requirements. On the other hand relative bond spreads are mainly affected by changes in the credit conditions in the corporate sector of the economy.

*4.2.5. Stock returns*

To the extent that stock prices are forward looking, reflecting investors' expectations about future profitability and earnings, stock returns should also be included in the credit conditions indicator. A formal link between stock returns and the credit risk of a company is provided by Merton (1974) and subsequent extensions by CreditMetrics and KMV.

*4.3. Results*

Fig. 1 plots the estimated credit conditions indicators for the US and the UK. In addition, Table 1 reports the corresponding sensitivities of the financial variables to the CCIs. According to Table 1, both the US and UK indicators are negatively related to the slope of the term structure — a downward sloping term structure will increase the CCIs. This is in accordance with previous evidence, reported by Estrella and Hardouvelis (1991) and Estrella and Mishkin (1998), that a

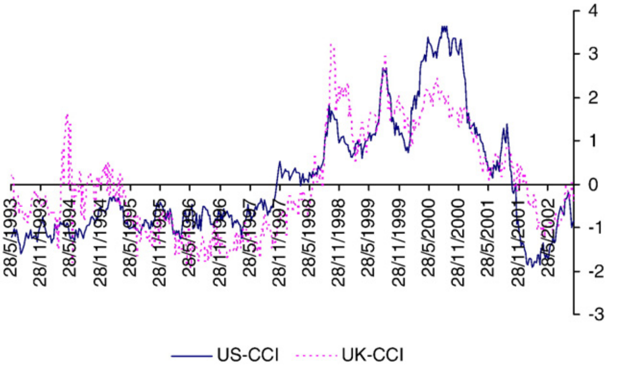


Fig. 1. US and UK credit cycles indicators. This figure plots the credit cycles indicators estimated using the dynamic factor model given by Eqs. (22) and (23). A detailed presentation of the subspace algorithm used to estimate the dynamic factor model is included in Appendix B. Our sample covers the period from March 28, 1993 to November 1, 2002.

Table 1
US and UK credit cycles indicators (CCI)

| | US | UK |
|---|---|---|
| SLOPE | −0.642** | −1.030** |
| CURVE | 0.288 | 0.105 |
| LEVEL | 1.791** | −0.188 |
| TED | 1.143** | 0.299** |
| 3Y-swap spread | 2.177** | 1.079** |
| 10Y-swap spread | 1.693** | 0.896** |
| 3M-BBBAAA | 1.054** | −0.494** |
| 3Y-BBBAAA | 1.366** | −0.016 |
| 10Y-BBBAAA | 1.554** | −0.069 |
| 30Y-BBBAAA | 1.647** | −0.220** |
| 3M-AAAGOV | 1.267** | 0.124** |
| 3Y-AAAGOV | 1.961** | 0.822** |
| 10Y-AAAGOV | 2.136** | 0.747** |
| 30Y-AAAGOV | 2.026** | 0.817** |
| FTSE | 0.11 | 0.068 |
| WRLDXUK | 0.12 | 0.061 |

This table reports the sensitivities (or factor loadings), $J_F$, of credit cycles indicators to the observed variables. The CCIs are estimated using the subspace algorithm discussed in Appendix B. The entries in the table have been estimated by regressing the estimated CCIs, $C_t$, on the observed variables, $F_t$. A detailed description of all financial variables included in the estimation of the CCIs is given in Appendix C. ** denotes significance at 1% confidence level.

downward slopping term structure is an indicator of a reduction in economic activity. Both CCIs are positively related to swap spreads and absolute corporate spreads. Relative corporate spreads have a small but negative impact on the CCIs. When stock returns are included they have a negligible effect on the indicator, because they receive almost zero loadings. Based on these factor loadings we can conclude that an increase in the indicators signals a deterioration of the credit conditions of the US and UK economies. This is consistent with the evidence reported in Fig. 1. The US-CCI captures the tightening of the monetary policy by the FED during 1994 as well as the Asian crisis at the end of 1997. US-CCI peaks in 1998 and 1999 and reaches its maximum during 2000. It, finally, points to an improvement of the credit conditions from October 2001 to July 2002, which is reversed during the last month in our sample. The UK-CCI is negative during the expansion period from 1993 to 1997 – with a brief exception during 1994 that reflects increased volatility in the UK bond market at the time – and it remains unaffected by the Asian crisis of 1997. It then rises rapidly during the Russian crisis of 1998 and the subsequent collapse of the LTCM fund and starts to unwind by the end of 2000.

## 5. Results for the default-free term structures

The estimated parameters for the 2-factor square-root process for the default-free short rate are reported in Table 2.[15] The estimated models share a number of features with previously estimated multifactor term structure models. Both factors have a significant impact on the instantaneous rate. In both the US and UK models, the first factor is strongly mean reverting. In the US model the second factor has a much slower rate of mean reversion and in the UK model the second factor is indistinguishable from a random walk. For that reason, the variation of the long-term yields for

[15] In the remainder of the paper we proceed without reporting the variances of the residuals, $[\Sigma_\varepsilon]_i$.

Table 2
Two-factor default-free term structure model

| Coefficients | Estimate (st. error) | |
|---|---|---|
| | US | UK |
| $[\delta_X]_1$ | 0.031 (0.0001) | 0.00419 (0.0003) |
| $[\delta_X]_2$ | 0.0061 (0.0007) | 0.00436 (0.0005) |
| $[K_X]_{11}$ | 0.0337 (0.0034) | 1.00E−05 (constrained) |
| $[K_X]_{22}$ | 0.6648 (0.0685) | 0.43620 (0.0343) |
| $[\Theta_X]_1$ | 6.2554 (1.0121) | 0.00432 (0.0005) |
| $[\Theta_X]_2$ | 5.4845 (0.5481) | 7.7463 (0.7418) |
| $[\Xi_X]_1$ | −0.0567 (0.005) | −0.0773 (0.0109) |
| $[\Xi_X]_2$ | −0.1937 (0.0665) | −0.1040 (0.0268) |

This table reports the estimated coefficients for the 2-factor specification of the default-free term structure model, given by Eqs. (2)–(5). The coefficients are estimated using the Kalman filter methodology described in Section 3, along with the heteroskedasticity and serial correlation robust standard errors, calculated according to Hamilton (1994). The model is estimated using AAA zero-coupon corporate bond yields for the US and the UK. Our sample covers the period from March 23, 1993 to November 1, 2002. $[\bullet]_{ij}$ denotes the $ij$-th element of the matrix in square brackets.

both countries will be dominated by the second factor. Furthermore, our factors have a straightforward economic interpretation. In both the US and the UK, the first factor is highly positively correlated with the 30-year yield. The second factor is highly negatively correlated with the difference between the 30-year and the 1-year yields. In the case of the US, the correlation between the estimated factor and the slope of the term structure is −99%, while in the case of the UK the correlation is −97%.

This structure of correlations between the default-free yield curve and the underlying state variables is also reflected in the patterns of $B(\tau)$s, as reported in panel A, Figs. 2–5. $B(\tau)$s capture the contemporaneous effect that shocks to factors have on the default-free zero-coupon yields across maturities.[16] The first factor has roughly equal weights, $B_2(\tau)$, across maturities. As a result, a shock to this factor will affect equally all maturities, resulting in a parallel shift of the yield curve. $B_2(\tau)$ has a strong positive impact on the short-end of the yield curve, which tends to zero for longer maturities. Hence, a shock to the second factor will increase the short-term rates while leaving the long-term rates unaffected resulting in an inversion of the yield curve. Given this structure of correlations and in order to facilitate the interpretation of our results, we will refer to the first factor as the LEVEL factor and to the second as the SLOPE factor.

While this economic interpretation of our factors gives more credibility to our results, it also signals some model misspecification. According to our model specification the two unobserved state variables should be independent of each other. Nevertheless, our empirical results indicate that the LEVEL and SLOPE factors are negatively correlated. This is not an uncommon feature of multivariate square-root affine models (see also Duffee, 1999). The trade-off between a more flexible factor correlation structure and the ability of the square-root process to fit the volatility dynamics and exclude negative rates is discussed in Dai and Singleton (2000) and Duffie and Singleton (1999). Despite this apparent misspecification, the overall fit of our model is quite good. The mean absolute difference between the actual and

[16] The $B(\tau)$s reported in Figs. 2–5 have been scaled by the standard deviation of the corresponding factor. In addition, they have been multiplied by 100 to improve the readability of the figures.

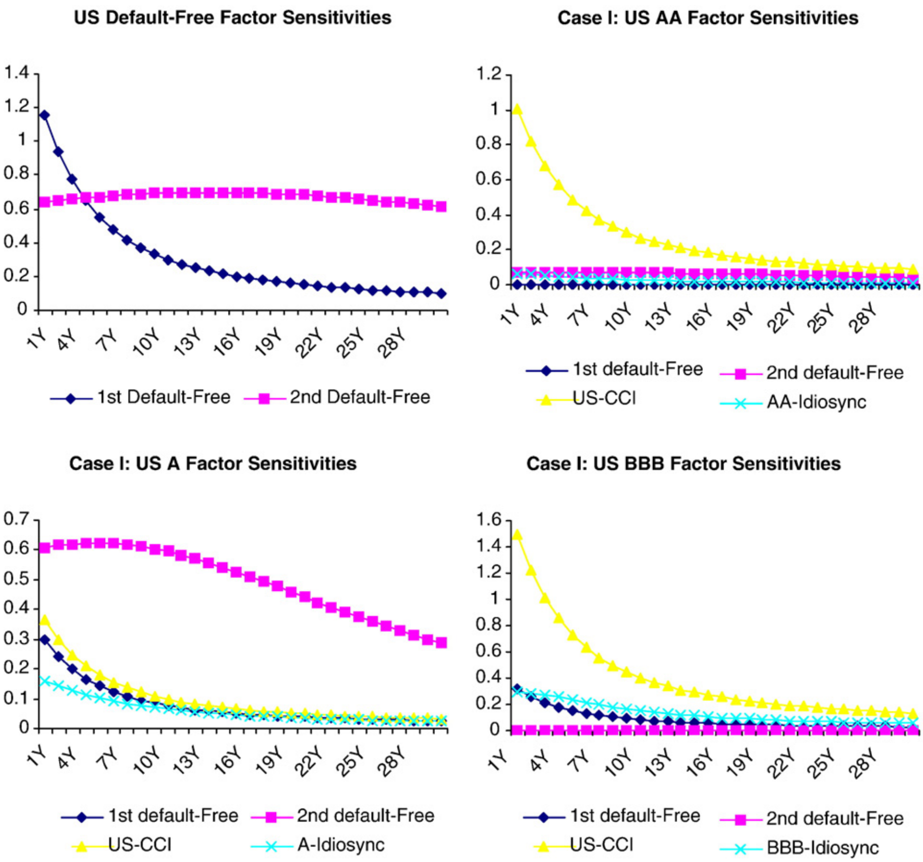


Fig. 2. Case I — Term structure of US factor sensitivities. Panel A plots the sensitivities of the default-free term structure (in levels) to shocks to the two default-free state variables. Panels B, C and D plot the sensitivities of the AA, A and BBB corporate bond spreads to shocks to the two default-free, the US-CCI and the 3 idiosyncratic state variables. All shocks are equal to 1 standard deviation of the corresponding state variable.

model yields ranges from 9 bps to 15 bps for the US rates and from 8 bps to 17 bps for the UK rates, depending on maturity.[17] Finally, the estimated market prices of risk, $[\Xi_X]_1$ and $[\Xi_X]_2$, are negative and statistically significant. As a result, investors receive a positive risk premium for both systematic factors.

## 6. Results of the term structure of defaultable bonds

In this section we present our estimates for Case I and Case II models for defaultable bonds. We concentrate our analysis on the properties of the default intensities and the factor dynamics because these define the credit spreads and default probabilities. In the last part of this section, we examine the ability of our model to fit corporate bond yields.

[17] Graphs of both actual and fitted zero-coupon yields for all maturities examined, are available from the author upon request.

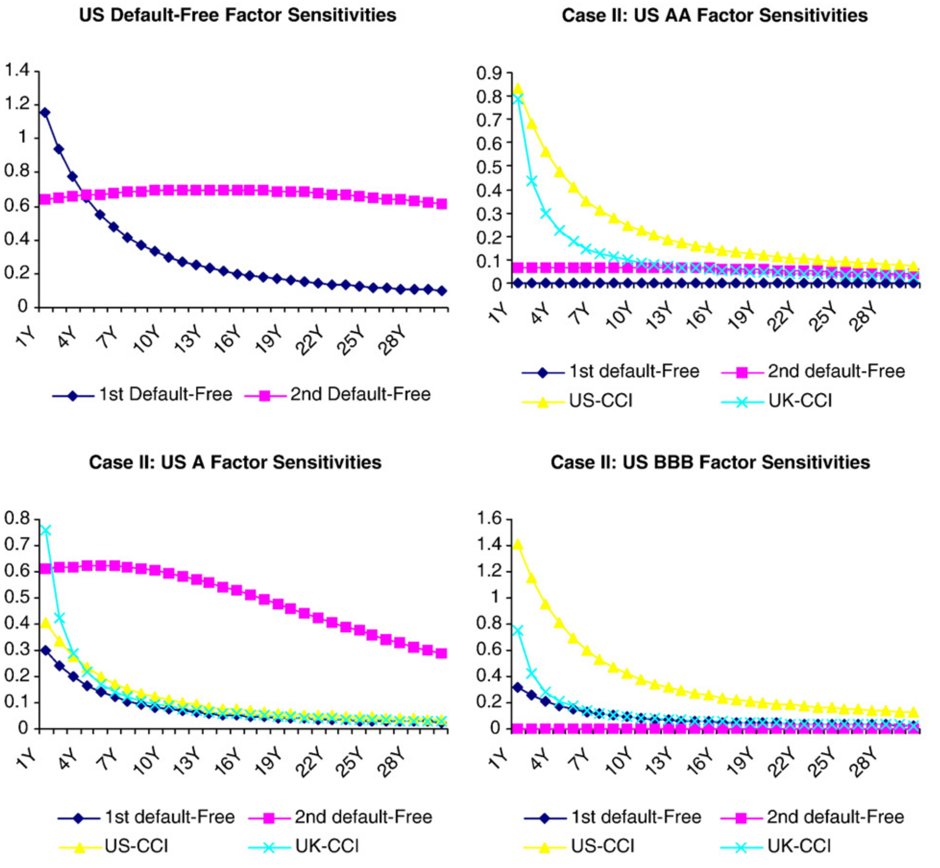


Fig. 3. Case II — Term structure of US factor sensitivities. Panel A plots the sensitivities of the default-free term structure (in levels) to shocks to the two default-free state variables (Panel 1 is the same as in Fig. 2. It is repeated to facilitate presentation.). Panels B, C and D plot the sensitivities of the AA, A and BBB corporate bond spreads to shocks to the two default-free, the US-CCI and the UK-CCI state variables. All shocks are equal to 1 standard deviation of the corresponding state variable.

### *6.1. Default-intensities: Case I*

Case I corresponds to the specification of the model where the default intensities for the AA, A and BBB corporate bonds depend upon the two default-free factors, the domestic credit cycles indicator and three additional factors, each idiosyncratic to one of the three rating classes. Table 3, Case I, reports the parameters of the continuous-time specification of the CCI dynamics. The significance of the estimated coefficients of mean reversion differs between the two countries. The UK-CCI exhibits strong mean reversion (towards zero), while the US-CCI is not statistically significant from a random walk process.[18]

[18] The parameters (and std. errors) $A_C$, of the equivalent $AR(1)$ specification (23) are 0.9912 (0.0059) for the US-CCI and 0.962 (0.012) for the UK-CCI. An array of unit-root tests verified our initial conclusion that the US-CCI follows a random walk process.

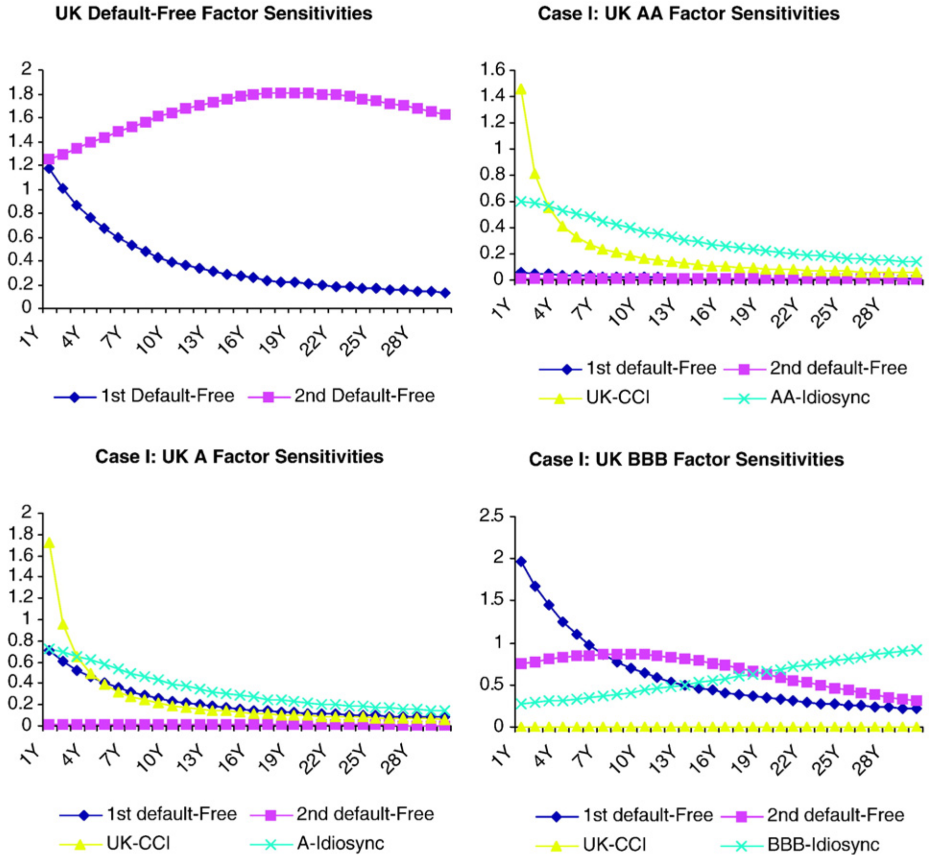


Fig. 4. Case I — Term structure of UK factor sensitivities. Panel A plots the sensitivities of the default-free term structure (in levels) to shocks to the two default-free state variables. Panels B, C and D plot the sensitivities of the AA, A and BBB corporate bond spreads to shocks to the two default-free, the UK-CCI and the 3 idiosyncratic state variables. All shocks are equal to 1 standard deviation of the corresponding state variable.

The sensitivity of the vector of default intensities, $\Lambda_t$, to these factors, in conjunction with the factor dynamics is reported in Tables 4 and 5. First, we turn our attention to the factor loadings on the vector of default intensities (given by matrices $\Gamma$, $H$ and $\Pi$). According to the estimated loadings, the US AA-rated term structure does not differ significantly from the underlying default-free rates. The AA default intensity has zero loading on the SLOPE factor and small, though statistically significant, loadings on the LEVEL and AA-idiosyncratic factors.[19] The factor with the highest loading on AA default intensity is the US credit cycles indicator. The US A default intensity depends upon both the SLOPE and LEVEL default-free factors. In addition, it is

[19] The fact that the default intensity does not depend on the default-free factors does not mean that the AA and default-free term structures are uncorrelated. The term structures of defaultable bonds are linked to the default-free interest rates through the dependence of the vector of default intensities, $Y_t$, on the default-free state variables, $X_t$, given by Eq. (10).

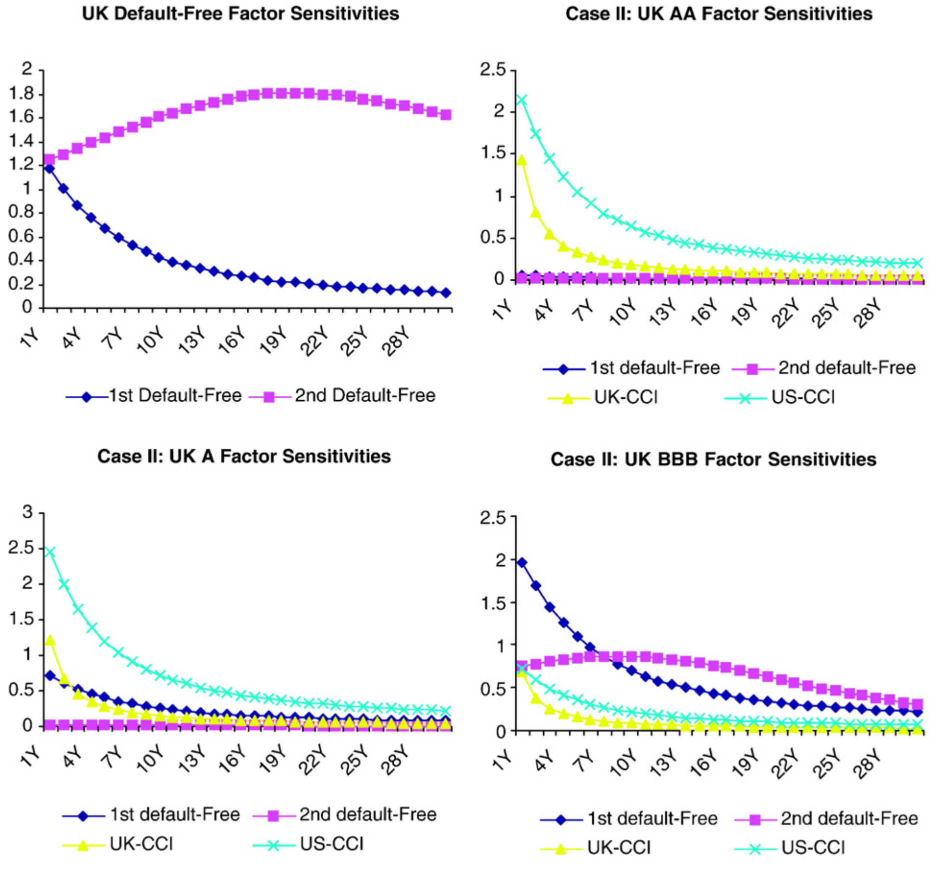


Fig. 5. Case II — Term structure of UK factor sensitivities. Panel A plots the sensitivities of the default-free term structure (in levels) to shocks to the two default-free state variables (Panel 1 is the same as in Fig. 2. It is repeated to facilitate presentation.). Panels B, C and D plot the sensitivities of the AA, A and BBB corporate bond spreads to shocks to the two default-free, the UK-CCI and the US-CCI state variables. All shocks are equal to 1 standard deviation of the corresponding state variable.

also affected by shocks to both the US-CCI and the A-idiosyncratic factor. Finally, the BBB default intensity depends upon the SLOPE default-free factor, and exhibits a strong positive dependence upon the US credit cycles and the BBB idiosyncratic factor.

The UK factor loadings on default intensities do not always follow the same pattern as in the US. The AA default intensity is not affected by any of the default-free factors but is affected by the UK credit cycles indicator and the AA-idiosyncratic factor. The SLOPE factor does affect the A and BBB intensities and the LEVEL factor is significant for the BBB intensity. Similarly to the US, the magnitude of the CCI factor loadings increases as the credit quality of the bonds deteriorates. In contrast, the loadings of the idiosyncratic factors exhibit the opposite pattern to that of the US, with the importance of the idiosyncratic factors decreasing with credit quality.

Additional information about the significance of the underlying factors, in determining the shape of the term structure of corporate bond yields is provided by examining the patterns of the

Table 3
Credit cycles indicators

| Coefficients | Estimate (st. error) | |
|---|---|---|
| | US | UK |
| *Case I* | | |
| $[K_C]_1$ | 0.4607 (0.3351) | 2.1246 (0.6797) |
| $[\Theta_C]_1$ | 0.00 (fixed) | 0.00 (fixed) |
| $[\Sigma_C]_1$ | 1.2541 (0.0618) | 2.4661 (0.1365) |
| *Case II* | | |
| $[K_C]_{11}$ | 0.4607 (0.3351) | 2.1246 (0.6797) |
| $[K_C]_{22}$ | 7.2680 (1.6554) | 5.4374 (1.3221) |
| $[\Theta_C]_1$ | 0.00 (fixed) | 0.00 (fixed) |
| $[\Theta_C]_2$ | 0.00 (fixed) | 0.00 (fixed) |
| $[\Sigma_C]_{11}$ | 1.2541 (0.0618) | 2.4661 (0.1365) |
| $[\Sigma_C]_{22}$ | 2.6242 (0.1442) | 2.5445 (0.1252) |

This table reports the estimated dynamics of the credit cycles indicators (CCIs) for the US and the UK. The parameters are estimated by maximising the log-likelihood function given by Eq. (20). Because the CCIs are, by construction, zero mean variables we have restricted the long-term mean of the processes to zero. In Case II, we orthogonalise the indicators, by regressing the UK-CCI (US-CCI) on the US-CCI (UK-CCI) before estimating their dynamics. The dynamics of the local indicator remain the same as in Case I. Our sample covers the period from March 23, 1993 to November 1, 2002. $[\bullet]_{ij}$ denotes the $ij$-th element of the matrix in square brackets.

responses that shocks to the underlying state variables have on the term structure of credit spreads. We have chosen to show the effect of shocks to credit spreads rather than to corporate bond yields for two reasons. First, the term structure of corporate bond yields is dominated by the default-free

Table 4
Default intensities sensitivities: Case I

| Coefficients | Estimate (st. error) | |
|---|---|---|
| | US | UK |
| $[\Lambda_0]_1$ | 9.16E−05 (0.0001) | 3.25E−05 (4.6E−05) |
| $[\Lambda_0]_2$ | 8.79E−05 (0.0003) | 0.0001 (0.0003) |
| $[\Lambda_0]_3$ | 0.0002 (3.2E−05) | 0.0002 (2.7E−05) |
| $[\Gamma]_{11}$ | 1.04E−05 (0.0001) | 0.0002 (0.0008) |
| $[\Gamma]_{12}$ | 0.00032 (0.0009) | 3.45E−05 (0.0002) |
| $[\Gamma]_{21}$ | 0.0016 (0.0001) | 0.0027 (0.0006) |
| $[\Gamma]_{22}$ | 0.0029 (0.0003) | 3.85E−05 (0.0002) |
| $[\Gamma]_{31}$ | 0.0017 (0.0006) | 0.0079 (0.0002) |
| $[\Gamma]_{32}$ | 2.52E−05 (na) | 0.0025 (0.0003) |
| $[H]_1$ | 0.0047 (0.028) | 1.00E−07 (constrained) |
| $[H]_2$ | 0.0104 (0.0013) | 0.0559 (0.0018) |
| $[H]_3$ | 0.0333 (0.0014) | 0.0344 (0.0021) |
| $[\Pi]_{11}$ | 0.0008 (0.0003) | 0.0322 (0.0075) |
| $[\Pi]_{22}$ | 0.0049 (0.0017) | 0.0043 (0.0052) |
| $[\Pi]_{33}$ | 0.0705 (0.0063) | 0.0011 (6.85E−0.5) |

This table reports the estimated sensitivities of the state variables on the vector of default intensities as given by Eq. (7). The specification of the model corresponds to Case I, that includes the domestic cycles indicator and 3 idiosyncratic factors. The coefficients are estimated using the Kalman filter methodology described in Section 3, along with the heteroskedasticity and serial correlation robust standard errors, calculated according to Hamilton (1994). The model is estimated using AA, A and BBB zero-coupon corporate bond yields for the US and the UK. Our sample covers the period from March 23, 1993 to November 1, 2002. $[\bullet]_{ij}$ denotes the $ij$-th element of the matrix in square brackets.

Table 5
Default intensities dynamics: Case I

| Coefficients | Estimate (st. error) | |
|---|---|---|
| | US | UK |
| $[\Xi_C]_1$ | 0.0082 (0.028) | 0.0313 (0.0030) |
| $[\Xi_C]_2$ | −0.019 (0.003) | 0.0431 (0.0023) |
| $[\Xi_C]_3$ | −0.0216 (0.0027) | 0.0468 (0.0001) |
| $[K_Z]_{11}$ | 0.2544 (0.0588) | 0.5031 (0.3584) |
| $[K_Z]_{22}$ | 0.3236 (0.1637) | 0.0445 (0.0112) |
| $[K_Z]_{33}$ | 0.0029 (na) | 1.23E−05 (na) |
| $[\Theta_Z]_1$ | 4.2958 (0.9181) | 0.2426 (0.1723) |
| $[\Theta_Z]_2$ | 0.06385 (0.3118) | 0.0345 (0.0360) |
| $[\Theta_Z]_3$ | 1.419 (na) | 9.47E−05 (na) |
| $[\Xi_Z]_1$ | −1.00E−07 (constrained) | −0.4676 (0.3619) |
| $[\Xi_Z]_2$ | −0.0383 (0.15) | −1.00E−07 (constrained) |
| $[\Xi_Z]_3$ | −0.010 (0.0125) | −0.0969 (0.0026) |

This table reports the dynamics of the state variables given by Eqs. (8) and (9) along with the corresponding prices of risk. The specification of the model corresponds to Case I, that includes the domestic cycles indicator and 3 idiosyncratic factors. The coefficients are estimated using the Kalman filter methodology described in Section 3, along with the heteroskedasticity and serial correlation robust standard errors, calculated according to Hamilton (1994). The model is estimated using AA, A and BBB zero-coupon corporate bond yields for the US and the UK. Our sample covers the period from March 23, 1993 to November 1, 2002. $[\bullet]_{ij}$ denotes the $ij$-th element of the matrix in square brackets.

factors; hence the impact of the additional factors could not be properly appreciated. Second, because the credit spreads reflect changes in credit risk and default probabilities more clearly than the levels of corporate bond yields.

These responses, reported in panels B, C and D, in Figs. 2 and 4, are estimated as the difference between $B_j(\tau)$ and $B(\tau)$, for all credit classes and for maturities from 1 to 30 years.[20] According to Fig. 2, a shock to the LEVEL, SLOPE and AA-idiosyncratic factor will have a negligible impact on AA corporate bond spreads. A positive shock to the US-CCI factor, which corresponds to deterioration of the credit conditions in the US, will increase the short-maturity spreads but will leave the long-term spreads unaffected. Hence, in times of financial crises the term structure of AA credit spreads will have a tendency to flatten or even to invert.[21] The same holds for the term structure of BBB corporate spreads, where the US-CCI is again the dominant factor. The term structure of A credit spreads, especially the short-end, is affected by all four factors. Increases in the level of the term structure will have a positive impact across maturities — although the effect will be stronger at the short-end. The short-maturity A-rated spreads are also positively related to the US-CCI, A-idiosyncratic and SLOPE factors. The impact of the SLOPE factor requires some further interpretation due to its negative correlation with the actual slope of the default-free yield curve (defined as the difference between the 30-year and the 1-year zero-coupon rates). Positive shocks to the slope factor, which correspond to flattening of the actual default-free yield curve,

[20] Again, the responses have been scaled to reflect a shock equal to one standard deviation of the underlying factor and have been multiplied by 100.

[21] The final shape of the term structure of credit spreads will depend upon the shape of the term structure before the shock.

will have a significant positive effect on the short-maturity credit spreads. Put another way, a flattening of the actual yield curve will result in a flat or an inverted term structure of credit spreads. This means that the SLOPE and CCI factors have very similar effects. This is not surprising, given the ample empirical evidence on the ability of the slope of the term structure to anticipate changes in economic conditions (see Estrella and Hardouvelis, 1991; Estrella and Mishkin, 1998).

The pattern of the sensitivities for the UK credit spreads are reported in Fig. 4. For AA and A credit spreads, the UK-CCI has a significant impact, especially at the short-end. Similar to the US results, the importance of the UK idiosyncratic factors increases as the credit quality of the bonds deteriorates. One difference relative to the US results is that the AA- and A-idiosyncratic factors have a more significant role — especially for long maturities. Only for the BBB spreads the patterns of sensitivities are totally different compared to all other results. Nevertheless, as will be discussed in the following section, this reflects the inability of our model to provide an adequate fit to the UK BBB bond yields rather than a different structural relationship between the state variables and the BBB corporate bond spreads.

### 6.2. Default-intensities: Case II

Case II corresponds to the specification of the model where the default intensities depend upon the two default-free factors and the US and UK credit conditions indicators. In the modelling stages of our research we have assumed that the state variables are orthogonal to each other.

Table 6
Default intensities sensitivities: Case II

| Coefficients | Estimate (st. error) | |
|---|---|---|
| | US | UK |
| $[\Lambda_0]_1$ | 9.78E−05 (0.0002) | 0.00013 (0.0002) |
| $[\Lambda_0]_2$ | 3.08E−05 (2.1E−05) | 0.00017 (1.02E−05) |
| $[\Lambda_0]_3$ | 0.0002 (0.0001) | 0.0002 (2.5E−05) |
| $[\Gamma]_{11}$ | 1.04E−05 (0.0009) | 0.0002 (0.0008) |
| $[\Gamma]_{12}$ | 0.00032 (0.0001) | 3.45E−05 (0.0002) |
| $[\Gamma]_{21}$ | 0.0016 (0.0001) | 0.0027 (0.0006) |
| $[\Gamma]_{22}$ | 0.0029 (0.0003) | 3.85E−05 (0.0002) |
| $[\Gamma]_{31}$ | 0.0017 (0.0005) | 0.0074 (0.0002) |
| $[\Gamma]_{32}$ | 2.5E−05 (na) | 0.0025 (0.0003) |
| $[H]_{11}$ | 0.0321 (0.0013) | 0.0226 (0.0011) |
| $[H]_{12}$ | 0.0171 (0.0010) | 0.0333 (0.0019) |
| $[H]_{21}$ | 0.0210 (0.0011) | 0.0286 (0.0016) |
| $[H]_{22}$ | 0.0173 (0.0009) | 0.0576 (0.0022) |
| $[H]_{31}$ | 0.0463 (0.0029) | 0.0195 (0.0020) |
| $[H]_{32}$ | 0.0204 (0.0017) | 0.0236 (0.0020) |

This table reports the estimated sensitivities of the state variables on the vector of default intensities as given by Eq. (7). The specification of the model corresponds to Case II, that includes the US and UK cycles indicators. The coefficients are estimated using the Kalman filter methodology described in Section 3, along with the heteroskedasticity and serial correlation robust standard errors, calculated according to Hamilton (1994). The model is estimated using AA, A and BBB zero-coupon corporate bond yields for the US and the UK. Our sample covers the period from March 23, 1993 to November 1, 2002. $[\bullet]_{ij}$ denotes the $ij$-th element of the matrix in square brackets.

Nevertheless, Fig. 1 shows that the US and UK CCIs are highly correlated. We orthogonalise the two indicators by regressing the foreign CCI on the domestic one and using the residuals as our new proxy for the foreign CCI.

The sensitivities of the default intensities, $\Lambda_t$, to these factors are reported in Table 6 and the effects of shocks to the factors on the credit spreads are reported in Figs. 3 and 5. The loadings of the default-free factors are identical to the loadings in Case I. The new feature introduced in Case II is the sensitivity of the default intensities to the two CCI factors. In the US model, the default intensities of all rating classes have positive and significant loadings on the US-CCI. The loadings of the UK-CCI on all US default intensities are smaller compared to the loadings of the US-CCI but again they are positive and statistically significant. This result provides clear evidence that credit conditions outside the US play a significant role in determining the dynamics of the US corporate spreads. Equivalently, in the UK model both CCIs have a positive and statistically significant effect. An additional feature of the UK results is the dominant role of the US credit cycles in the determination of the UK default intensities and credit spreads. The loadings of the US-CCI are larger than those of the UK-CCI across credit classes. This result is corroborated by the effects of the two CCI factors on the term structures of UK credit spreads, reported in Fig. 5. A one standard deviation shock to the US-CCI will have a larger impact on the term structure of UK credit spreads across maturities and rating classes.

Finally, Table 7 reports the remaining parameters for the CCI processes. Because the CCIs are observable factors and their dynamics are estimated in a separate stage (reported in Table 3) the only parameters left for estimation are the market prices of risk for the two factors. Given that the CCIs follow "Vasicek-type" Gaussian processes, positive market prices are required in order for investors to gain excess return for bearing the systematic risk of the US and UK CCI factors. In the US model, most factors have positive and significant prices of risk — the only negative parameter is the price of risk for the US-CCI for A-rated corporate bonds. In the UK model, we are unable to accurately estimate the prices of risk for A and BBB-rated bonds. The only prices of risk that are statistically significant are those of the AA corporate bonds. The price of risk for the US-CCI factor is negative and statistically significant.

Table 7
Default intensities dynamics: Case II

| Coefficients | Estimate (st. error) | |
|---|---|---|
| | US | UK |
| $[\Xi_C]_{11}$ | −0.0929 (0.385) | 0.4967 (0.1495) |
| $[\Xi_C]_{12}$ | 0.3839 (0.1615) | −0.1672 (0.0443) |
| $[\Xi_C]_{21}$ | −0.3570 (0.0515) | −0.0554 (0.2172) |
| $[\Xi_C]_{22}$ | 0.9647 (0.1444) | 0.0356 (0.1663) |
| $[\Xi_C]_{31}$ | 0.01777 (0.006) | −0.1167 (0.1663) |
| $[\Xi_C]_{32}$ | 0.00527 (0.0005) | 0.0576 (0.056) |

This table reports the dynamics of the state variables given by Eqs. (8) and (9) along with the corresponding prices of risk. The specification of the model corresponds to Case II, that includes the US and UK credit cycles indicators. The coefficients are estimated using the Kalman filter methodology described in Section 3, along with the heteroskedasticity and serial correlation robust standard errors, calculated according to Hamilton (1994). The model is estimated using AA, A and BBB zero-coupon corporate bond yields for the US and the UK. Our sample covers the period from March 23, 1993 to November 1, 2002. $[\bullet]_{ij}$ denotes the $ij$-th element of the matrix in square brackets.

### 6.3. Model performance

#### 6.3.1. In-sample performance

The ability of the two specifications of our model to describe the data is examined in Tables 8 and 9. These tables report the mean error (ME), mean absolute error (MAE), the root mean squared error (RMSE) and the serial correlation coefficients of the difference between "actual" and "model" corporate bond yields for the US and the UK. ME is used to examine whether the models systematically over (under)-estimate the actual bond yields. MAE assesses the fit of the models and the RMSE captures the variation of the absolute errors. Finally, if the models are well specified then their errors should be uncorrelated.

In the US, there is very little difference in the ability of the two specifications to fit the data. Both the MAE and the RMSE for Case I and Case II specifications are very similar for all rating classes. Case II produces smaller MAE and RMSE for 1-year and 5-year AA bond yields and 5-year and 30-year BBB bond yields. In addition, Case I systematically overestimates AA bond yields. For all other maturities and rating classes both models produce similar mean absolute errors ranging between 13 and 26 basis points.

Table 8
In-sample performance of the US corporate bond models

| | Case I: domestic | | | | Case II: dom. & int. | | | |
|---|---|---|---|---|---|---|---|---|
| | 1Y | 5Y | 10Y | 30Y | 1Y | 5Y | 10Y | 30Y |
| *AA* | | | | | | | | |
| *ME* | 0.0011 | 0.0005 | 7.1E−05 | 0.0004 | 8.8E−05 | 1.9E−05 | −0.0006 | 0.0002 |
| *MAE* | 0.0051 | 0.0022 | 0.001725 | 0.0013 | 0.0026 | 0.0018 | 0.0017 | 0.0012 |
| *RMSE* | 0.0065 | 0.002867 | 0.0022 | 0.0016 | 0.0034 | 0.0023 | 0.0021 | 0.0015 |
| $\rho_1$ | 0.36 | −0.29 | −0.17 | 0.09 | −0.23 | −0.17 | 0.01 | 0.11 |
| $\rho_5$ | 0.33 | −0.04 | −0.01 | 0.17 | −0.06 | −0.04 | 0.02 | 0.15 |
| $\rho_{12}$ | 0.25 | −0.01 | 0.02 | 0.21 | −0.01 | −0.02 | 0.05 | 0.19 |
| *A* | | | | | | | | |
| *ME* | −0.0004 | 2.6E−05 | 1.6E−05 | −0.0008 | −7.0E−05 | −0.0002 | 4.9E−05 | −0.0003 |
| *MAE* | 0.002 | 0.0015 | 0.0013 | 0.0018 | 0.0019 | 0.0015 | 0.0013 | 0.0017 |
| *RMSE* | 0.0027 | 0.0021 | 0.0018 | 0.0024 | 0.0027 | 0.0021 | 0.0019 | 0.0022 |
| $\rho_1$ | 0.16 | −0.24 | −0.23 | 0.58 | −0.43 | −0.32 | −0.23 | 0.55 |
| $\rho_5$ | 0.10 | −0.02 | −0.07 | 0.34 | −0.01 | −0.05 | −0.07 | 0.31 |
| $\rho_{12}$ | 0.05 | −0.04 | −0.04 | 0.28 | −0.05 | −0.05 | −0.04 | 0.28 |
| *BBB* | | | | | | | | |
| *ME* | 0.0004 | 9.2E−05 | 5.5E−05 | 0.0016 | 0.0002 | 8.5E−05 | −0.001 | 0.0006 |
| *MAE* | 0.0037 | 0.0031 | 0.0022 | 0.0020 | 0.0035 | 0.0023 | 0.0021 | 0.0016 |
| *RMSE* | 0.0048 | 0.0041 | 0.0029 | 0.0025 | 0.0049 | 0.0030 | 0.0027 | 0.0021 |
| $\rho_1$ | 0.06 | −0.27 | −0.44 | 0.29 | −0.12 | −0.09 | −0.02 | 0.24 |
| $\rho_5$ | −0.08 | −0.05 | −0.08 | 0.36 | −0.08 | −0.05 | −0.02 | 0.30 |
| $\rho_{12}$ | 0.02 | 0.01 | 0.01 | 0.35 | 0.01 | −0.02 | 0.03 | 0.27 |

The table reports measures of in-sample yield errors for the AA, A and BBB US corporate bonds. *ME* denotes Mean Error, *MAE* denotes Mean Absolute Error, *RMSE* denotes Root Mean Squared Error and $\rho_1$, $\rho_5$ and $\rho_{12}$ denote the 1st, 5th and 12th order autocorrelation of the yield errors. Case I refers to the specification of the model that includes 2 default-free factors, the US CCI and 3 idiosyncratic factors. Case II refers to the specification that includes 2 default-free factors and the US and UK CCIs. Our sample covers the period from March 23, 1993 to November 1, 2002.

Table 9
In-sample performance of the UK corporate bond models

| | Case I: domestic | | | | Case II: dom. & int. | | | |
|---|---|---|---|---|---|---|---|---|
| | 1Y | 5Y | 10Y | 30Y | 1Y | 5Y | 10Y | 30Y |
| *AA* | | | | | | | | |
| *ME* | 0.0006 | 0.0001 | 0.0001 | −0.0003 | 0.0001 | 0.0002 | −0.0016 | 0.0004 |
| *MAE* | 0.0025 | 0.0023 | 0.0024 | 0.0014 | 0.0025 | 0.0020 | 0.0028 | 0.0019 |
| *RMSE* | 0.0034 | 0.0031 | 0.0032 | 0.0022 | 0.0035 | 0.0027 | 0.0038 | 0.0026 |
| $\rho_1$ | −0.16 | −0.12 | 0.05 | −0.17 | −0.12 | 0.01 | 0.50 | 0.13 |
| $\rho_5$ | 0.01 | 0.05 | 0.22 | −0.04 | 0.02 | 0.02 | 0.44 | 0.27 |
| $\rho_{12}$ | 0.02 | 0.03 | 0.13 | 0.10 | 0.02 | −0.03 | 0.28 | 0.35 |
| *A* | | | | | | | | |
| *ME* | 0.0005 | 0.0002 | 0.0001 | 0.0002 | 0.0003 | 0.0002 | −0.0024 | 0.001 |
| *MAE* | 0.0038 | 0.0032 | 0.0026 | 0.0018 | 0.0039 | 0.0027 | 0.0035 | 0.0024 |
| *RMSE* | 0.0055 | 0.0048 | 0.0039 | 0.0027 | 0.0060 | 0.0040 | 0.0046 | 0.0033 |
| $\rho_1$ | −0.12 | −0.15 | −0.15 | −0.23 | −0.10 | −0.02 | 0.33 | 0.04 |
| $\rho_5$ | 0.05 | 0.06 | 0.09 | 0.01 | 0.03 | 0.02 | 0.28 | 0.27 |
| $\rho_{12}$ | −0.01 | −0.01 | −0.01 | −0.02 | −0.01 | −0.05 | 0.14 | 0.17 |
| *BBB* | | | | | | | | |
| *ME* | −0.0002 | −0.0010 | 0.0022 | −0.0005 | 0.0002 | −0.0002 | −0.0002 | 0.0012 |
| *MAE* | 0.0021 | 0.0025 | 0.0046 | 0.0084 | 0.0015 | 0.0018 | 0.0019 | 0.0037 |
| *RMSE* | 0.0031 | 0.0036 | 0.0057 | 0.0112 | 0.0026 | 0.0029 | 0.0033 | 0.0054 |
| $\rho_1$ | 0.07 | 0.56 | 0.76 | 0.60 | −0.39 | 0.06 | 0.24 | 0.91 |
| $\rho_5$ | −0.03 | 0.40 | 0.71 | 0.04 | −0.01 | 0.02 | 0.20 | 0.75 |
| $\rho_{12}$ | 0.00 | 0.29 | 0.58 | 0.01 | 0.01 | 0.00 | 0.14 | 0.61 |

The table reports measures of in-sample yield errors for the AA, A and BBB UK corporate bonds. *ME* denotes Mean Error, *MAE* denotes Mean Absolute Error, *RMSE* denotes Root Mean Squared Error and $\rho_1$, $\rho_5$ and $\rho_{12}$ denote the 1st, 5th and 12th order autocorrelation of the yield errors. Case I refers to the specification of the model that includes 2 default-free factors, the UK CCI and 3 idiosyncratic factors. Case II refers to the specification that includes 2 default-free factors and the UK and US CCIs. Our sample covers the period from March 23, 1993 to November 1, 2002.

In the UK Case I model, despite systematically overestimating AA and A bond yields produces a smaller MAE for the 1-year, 10-year and 30-year maturities. In contrast, Case II outperforms Case I specification across all BBB maturities. Serial correlation in yield errors in small with the exception of the 30-year BBB bond yields.

### *6.3.2. Out-of-sample performance*

The biggest advantage of interest rate models over statistical models, that employ Vector Autoregressive (VAR) methodologies to describe the dynamics of interest rates, is their ability to characterise the evolution of the entire term structure of interest rates and not just the dynamics of the bond yields used in their estimation. A useful test for the ability of the models to fit the data is to employ Eq. (11) to generate "cross-sectional" out-of-sample forecasts for the yields not included in the estimation of the model parameters. We use the estimated models to generate cross-sectional forecasts for the 6-month, 3-year, 7-year and 20-year AA, A and BBB zero-coupon yields that were not used in the estimation stage of our research. Tables 10 and 11 report the out-of-sample performance of the US and UK corporate bond models respectively.

In the US, Case I produces very well balanced forecasts (ME close to zero) compared to Case II that underestimates bond yields for most maturities and rating classes. Case I fails to fit the

Table 10
Out-of-sample performance of the US corporate bond models

| | Case I: domestic | | | | Case II: dom. and int. | | | |
|---|---|---|---|---|---|---|---|---|
| | 6M | 3Y | 7Y | 20Y | 6M | 3Y | 7Y | 20Y |
| *AA* | | | | | | | | |
| *ME* | 4.3E−18 | −1.2E−17 | −3.0E−18 | 2.2E−18 | −0.0003 | −0.0021 | −0.0031 | −0.0042 |
| *MAE* | 0.0062 | 0.0029 | 0.0018 | 0.0015 | 0.0031 | 0.0029 | 0.0032 | 0.0042 |
| *RMSE* | 0.0082 | 0.0038 | 0.0023 | 0.0021 | 0.0043 | 0.0037 | 0.0038 | 0.0047 |
| $\rho_1$ | 0.50 | 0.12 | 0.03 | 0.26 | 0.17 | 0.13 | 0.09 | 0.34 |
| $\rho_5$ | 0.43 | 0.08 | 0.03 | 0.23 | 0.16 | 0.07 | 0.00 | 0.23 |
| $\rho_{12}$ | 0.29 | 0.11 | 0.02 | 0.17 | 0.15 | 0.08 | 0.03 | 0.17 |
| *A* | | | | | | | | |
| *ME* | 1.4E−18 | −1.0E−18 | 3.3E−18 | 1.6E−19 | −0.0022 | −0.0007 | −0.0026 | 0.0042 |
| *MAE* | 0.0036 | 0.0017 | 0.0015 | 0.0017 | 0.0032 | 0.0018 | 0.0027 | 0.0043 |
| *RMSE* | 0.0047 | 0.0022 | 0.0020 | 0.0022 | 0.0043 | 0.0024 | 0.0033 | 0.0048 |
| $\rho_1$ | 0.67 | 0.20 | 0.26 | 0.53 | 0.27 | 0.24 | 0.23 | 0.51 |
| $\rho_5$ | 0.52 | 0.12 | 0.17 | 0.26 | 0.28 | 0.09 | 0.06 | 0.24 |
| $\rho_{12}$ | 0.32 | 0.00 | 0.13 | 0.21 | 0.15 | 0.04 | 0.01 | 0.18 |
| *BBB* | | | | | | | | |
| *ME* | 8.1E−18 | −8.3E−18 | −5.0E−18 | −4.2E−18 | 0.0014 | −0.0035 | −0.0043 | −0.0061 |
| *MAE* | 0.0062 | 0.0037 | 0.0025 | 0.0024 | 0.0043 | 0.0043 | 0.0046 | 0.0061 |
| *RMSE* | 0.0078 | 0.0048 | 0.0033 | 0.0030 | 0.0063 | 0.0058 | 0.0053 | 0.0068 |
| $\rho_1$ | 0.22 | 0.13 | 0.18 | 0.46 | 0.14 | 0.11 | 0.13 | 0.56 |
| $\rho_5$ | 0.19 | 0.02 | 0.05 | 0.51 | 0.01 | 0.04 | 0.09 | 0.53 |
| $\rho_{12}$ | 0.020 | 0.00 | 0.10 | 0.44 | 0.03 | 0.01 | 0.10 | 0.44 |

The table reports measures of out-of-sample yield errors for the AA, A and BBB US corporate bonds. The model is estimated using data on 1-, 5-, 10-, and 30-year bonds and then Eq. (11) is used to generate "theoretical" 6-month, 3-, 7- and 20-year yields. The yield errors are the difference between actual yields and "theoretical" yields. *ME* denotes Mean Error, *MAE* denotes Mean Absolute Error, *RMSE* denotes Root Mean Squared Error and $\rho_1$, $\rho_5$ and $\rho_{12}$ denote the 1st, 5th and 12th order autocorrelation of the yield errors. Case I refers to the specification of the model that includes 2 default-free factors, the US CCI and 3 idiosyncratic factors. Case II refers to the specification that includes 2 default-free factors and the US and UK CCIs. Our sample covers the period from March 23, 1993 to November 1, 2002.

short-end of the AA and A yield curves, but dominates Case II forecasts for all other maturities. The average fit error ranges between 15 and 37 basis points. In contrast, the performance of the two models is mixed in the UK data. Case I performs better at the long-end of the AA and A yields, while Case II errors are smaller at the short-end of A and across all BBB yields. Hence, both in-sample and out-of-sample statistics verify our conclusion that Case I specification fails to provide an adequate fit to the UK BBB yields. Finally, strong serial correlation is estimated for most yield errors.

## 7. Conclusions

In this paper we develop a framework that allows us to model the joint evolution of multiple term structures of defaultable bond yields and at the same time to capture several "stylised" features of corporate bond spreads, documented in previous econometric studies. In our most general specification, we identify the following key variables as the main risk-drivers for the default intensities of corporate bonds with different credit ratings: (i) a set of default-free state-

Table 11
Out-of-sample performance of the UK corporate bond models

| | Case I: domestic | | | | Case II: dom. & int. | | | |
|---|---|---|---|---|---|---|---|---|
| | 6M | 3Y | 7Y | 20Y | 6M | 3Y | 7Y | 20Y |
| *AA* | | | | | | | | |
| *ME* | −3.2E−18 | 3.4E−18 | −4.5E−18 | 1.9E−18 | −0.0012 | −0.0002 | 0.0012 | 0.0042 |
| *MAE* | 0.0103 | 0.0027 | 0.0022 | 0.0025 | 0.0042 | 0.0026 | 0.0024 | 0.0044 |
| *RMSE* | 0.0127 | 0.0037 | 0.0030 | 0.0032 | 0.0055 | 0.0034 | 0.0031 | 0.0054 |
| $\rho_1$ | 0.79 | 0.10 | 0.11 | 0.42 | 0.44 | 0.14 | 0.22 | 0.54 |
| $\rho_5$ | 0.74 | 0.17 | 0.18 | 0.46 | 0.41 | 0.14 | 0.18 | 0.53 |
| $\rho_{12}$ | 0.66 | .13 | 0.10 | 0.37 | 0.23 | 0.12 | 0.06 | 0.42 |
| *A* | | | | | | | | |
| *ME* | −9.2E−18 | 8.6E−18 | 1.2E−17 | −1.0E−17 | 0.0012 | −0.0007 | 0.0014 | 0.0036 |
| *MAE* | 0.0130 | 0.0040 | 0.0027 | 0.0032 | 0.0054 | 0.0037 | 0.0032 | 0.0043 |
| *RMSE* | 0.0162 | 0.0056 | 0.0039 | 0.0045 | 0.0078 | 0.0053 | 0.0042 | 0.0059 |
| $\rho_1$ | 0.65 | 0.22 | 0.11 | 0.52 | 0.27 | 0.12 | 0.15 | 0.60 |
| $\rho_5$ | 0.58 | 0.14 | 0.10 | 0.49 | 0.20 | 0.03 | 0.12 | 0.51 |
| $\rho_{12}$ | 0.45 | 0.10 | −0.04 | 0.34 | 0.08 | 0.01 | −0.01 | 0.37 |
| *BBB* | | | | | | | | |
| *ME* | −1.2E−19 | 1.3E−18 | −7.3E−18 | −8.2E−18 | 0.0001 | −0.0006 | 0.0008 | −0.0016 |
| *MAE* | 0.0043 | 0.0039 | 0.0036 | 0.0051 | 0.0032 | 0.0026 | 0.0021 | 0.0017 |
| *RMSE* | 0.0057 | 0.0051 | 0.0045 | 0.0064 | 0.0042 | 0.0034 | 0.0029 | 0.0068 |
| $\rho_1$ | 0.72 | 0.88 | 0.79 | 0.75 | 0.49 | 0.37 | 0.27 | 0.80 |
| $\rho_5$ | 0.52 | 0.77 | 0.70 | 0.41 | 0.40 | 0.29 | 0.25 | 0.55 |
| $\rho_{12}$ | 0.34 | 0.64 | 0.56 | 0.28 | 0.29 | 0.24 | 0.09 | 0.40 |

The table reports measures of out-of-sample yield errors for the AA, A and BBB UK corporate bonds. The model is estimated using data on 1-, 5-, 10-, and 30-year bonds and then Eq. (11) is used to generate "theoretical" 6-month, 3-, 7- and 20-year yields. The yield errors are the difference between actual yields and "theoretical" yields. *ME* denotes Mean Error, *MAE* denotes Mean Absolute Error, *RMSE* denotes Root Mean Squared Error and $\rho_1$, $\rho_5$ and $\rho_{12}$ denote the 1st, 5th and 12th order autocorrelation of the yield errors. Case I refers to the specification of the model that includes 2 default-free factors, the UK CCI and 3 idiosyncratic factors. Case II refers to the specification that includes 2 default-free factors and the UK and US CCIs. Our sample covers the period from March 23, 1993 to November 1, 2002.

variables, (ii) a set of observable factors that capture changes in the credit conditions — both domestic and international, and (iii) a set of factors each idiosyncratic to each credit class under consideration.

The main contribution of our framework, in the literature of credit risk modelling is its ability to analyze, in a systematic and consistent manner, the relative significance and the magnitude of the impact of these risk-drivers on the US and UK, AA, A and BBB corporate bond yields. The main conclusions that can be drawn from our results are that in both the US and UK, the risk-free factors, although significant, are not the main driving factors of corporate bond spreads. Instead, the main sources of risk are variations in the domestic and international credit conditions. In both the US and UK, the significance of the domestic credit cycles increases as the credit quality deteriorates and have a much larger impact on the short-end of the credit curves. A direct implication of our results is that, contrary to conventional wisdom, in times of adverse credit conditions we should expect a flattening or even an inversion of the term structure of credit spreads. Furthermore, we are able to demonstrate that in an international setting, global risk factors are significant and that the dominant role of the US financial markets is highlighted by their leading role in the evolution of the UK term structure of corporate bond spreads. Finally, credit rating idiosyncratic factors, when included, are also significant in determining the dynamics of term structures of corporate bond yields. Again, the significance of idiosyncratic factors increases as the credit quality deteriorates, but the AA and A-idiosyncratic factors are more important in the UK corporate bond market.

In addition to providing these insights into the empirical attributes of the corporate bond yields, our model can be employed in a wide variety of applications. The ability of the model to capture the correlations of corporate bond yields and credit spreads between obligors of multiple credit classes or from multiple countries can be employed to develop portfolio credit risk models. The existence of common factors between the default-free and corporate term structures allows for integrated market and credit risk modelling, suggested by both practitioners (see Jarrow and Turnbull, 2000) and regulators. The flexibility of the model to jointly model the default-free and the defaultable term structures of interest rates can also be employed for improving the valuation and hedging of derivatives whose payoff depends on the spread between a corporate and a government bond at some future date, i.e. spreads options. Finally, the model can be applied, in a way similar to Duffie and Garleanu (2001), to price credit derivatives, such as Collateralised Debt Obligations (CDOs), where the main risk factor is the correlation between the default intensities of the obligors in the underlying pool of assets. We intend to investigate these in future research.

## Appendix A. Estimation of zero-coupon yields

Let $V_j(t, \tau, c_\tau)$ denote the price, at time $t$, of a $j$-th class defaultable coupon paying bond that matures at time $t+\tau$, where $\tau$=6 months to 30 years, that pays a semi-annual coupon of $c_\tau$. The coupon rates for each of these bonds are

$$C_\tau = \left\{ \begin{array}{lll} 0 & \text{if} & \tau \leq 1 \text{ year} \\ c_\tau & \text{if} & \tau > 1 \; year \end{array} \right\}$$

In order to estimate a set of zero-coupon bonds, $V_j(t, \tau)$ we express each coupon paying bond as a portfolio of zero-coupon bonds plus a pricing error

$$V_j(t,\tau,c_\tau) = \sum_{k=1}^{\tau} c_\tau V_j(t,k;\beta) + V_j(t,\tau;\beta) + \varepsilon_{j,\tau} \tag{A1}$$

where $V_j(t, k; \beta)$ is a parametric discount function with parameter vector $\beta=(\beta_0, \beta_1, \beta_2, \beta_3, l_1, l_2)'$.

The functional form we use is based on the Svensson (1994, 1995) extension of the Nelson and Siegel (1987) model. According to Svensson, the term structure of zero coupon yields is given by

$$y_j(t,\tau;\beta) = \beta_0 + \beta_1 \left( \frac{1-e^{-\frac{\tau}{l_1}}}{\frac{\tau}{l_1}} \right) + \beta_2 \left( \frac{1-e^{-\frac{\tau}{l_1}}}{\frac{\tau}{l_1}} - e^{-\frac{\tau}{l_1}} \right) + \beta_3 \left( \frac{1-e^{-\frac{\tau}{l_2}}}{\frac{\tau}{l_2}} - e^{-\frac{\tau}{l_2}} \right) \tag{A2}$$

For each date and for each term structure in our sample, we estimate a parameter vector $\beta$, by minimising the sum of squared differences between actual and theoretical (model) yields. This results in more balanced errors across maturities compared to minimising bond pricing errors.

Bond pricing errors are often larger for short maturities given that the sensitivity of bond prices to interest rates increases with maturity (or duration).

## Appendix B. Dynamic factor estimation

Kapetanios (2002) describes an algorithm that allows the estimation of the above model without the need to maximise the log-likelihood function of the state-space model. Let

$$F_t^f = \mathcal{O}\mathcal{K}F_t^p + \mathcal{E}E_t^f$$

where $F_t^f=(F_t', F_{t+1}', F_{t+2}', \ldots)'$, $F_t^p=(F_{t-1}', F_{t-2}', \ldots)'$, $E_t^f=(u_t', u_{t+1}', \ldots)'$, $\mathcal{O}=[J_F', A_C'J_F', (A_C^2)'J_F', \ldots]'$, $\mathcal{K} = [\overline{B}_C, (A_C-\overline{B}_C J_F)\overline{B}_C, (A_C-\overline{B}_C J_F)^2\overline{B}_C, \ldots]$, $\overline{B}_C = B_C D_F^{-1}$ and

$$\mathcal{E} = \begin{pmatrix} D_F & 0 & \cdots & 0 \\ J_F B_C & D_F & \ddots & \vdots \\ J_F A_C B_C & \ddots & \ddots & 0 \\ \vdots & \cdots & J_F B_C & D_F \end{pmatrix} \tag{B3}$$

From the above it is easy to prove that (i) $F_t^f=\mathcal{O}C_t+\mathcal{E}E_t^f$ and (ii) $C_t = \mathcal{K}F_t^p$. Given that, the task is to estimate $\mathcal{K}$. In practice, we truncate the leads and lags in $F_t^f$ and $F_t^p$ and work with their sample approximations $F_{s,t}^f=(F_t', F_{t+1}', F_{t+2}', \ldots, F_{t+s-1}')'$ and $F_{q,t}^p=(F_{t-1}', F_{t-2}', \ldots, F_{t-q}')'$. Let $\mathcal{F}$ denote the product of $\mathcal{O}$ times $\mathcal{K}$, i.e. $\mathcal{F} = \mathcal{O}\mathcal{K}$. An estimate of $\mathcal{F}$, denoted $\widehat{\mathcal{F}}$, can be obtained by regressing $F_{s,t}^f$ on $F_{q,t}^p$. Kapetanios (2002) suggests an estimate of $\mathcal{K}, \widehat{\mathcal{K}}$, which is based on a singular value decomposition of $\widehat{\Gamma}^f \widehat{\mathcal{F}}\widehat{\Gamma}^p$ where $\widehat{\Gamma}^f$ and $\widehat{\Gamma}^p$ are the sample autocovariances of $F_{s,t}^f$ and $F_{q,t}^p$ respectively.[22] Then $\widehat{\mathcal{K}}$ is given by

$$\widehat{\mathcal{K}} = \widehat{\mathcal{S}}_m^{1/2}\widehat{\mathcal{V}}_m\widehat{\Gamma}^{p^{-1}} \tag{B4}$$

where $\widehat{\mathcal{U}}\widehat{\mathcal{S}}\widehat{\mathcal{V}}$ represents the singular value decomposition of $\widehat{\Gamma}^f \widehat{\mathcal{F}}\widehat{\Gamma}^p, \widehat{\mathcal{S}}_m$ denotes the matrix containing the first $L$ columns of $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{V}}_m$ denotes the heading $L\times L$ submatrix of $\widehat{\mathcal{V}}$. $\widehat{\mathcal{S}}$ contains the singular values of $\widehat{\Gamma}^f \widehat{\mathcal{F}}\widehat{\Gamma}^p$ in decreasing order. Having estimated $\widehat{\mathcal{K}}$, the unobserved common factor $C_t$, is given by $\widehat{\mathcal{K}}F_t^p$.

## Appendix C. Data Definitions

The following variables have been included in our models:

- Slope of the term structure: 10-year − 3-month government bond yields.
- Level of the term structure: Changes in the 3-month T-Bill rate.
- Curvature: (30-year rate + 3-month rate)/2 − 10-year rate.
- TED Spread: 3-month LIBOR − 3-month T-Bill rate.
- 3-year Swap Spread: 3-year swap rate − 3-year government bond rate.

[22] The singular value decomposition algorithm allows us to decompose any $M\times N$ matrix $A$, with $M \geq N$ as

$$A = UWV'$$

where $U$ is an $M\times N$ column-orthogonal matrix, $W$ is an $N\times N$ diagonal matrix and $V'$ is the transpose of an $M\times N$ orthogonal matrix $V$. For more information on singular value decomposition algorithms see Golub and Van Loan (1983) and Press, Flannery, Teukolsky and Vetterling (1992).

- 7-year Swap Spread: 7-year swap rate − 7-year government bond rate.
- 10-year Swap Spread: 10-year swap rate − 10-year government bond rate.
- 3-month Absolute Corporate Bond Spread: 3-month AAA corporate bond yield − 3-month government bond yield.
- 3-year Absolute Corporate Bond Spread: 3-year AAA corporate bond yield − 3-year government bond yield.
- 10-year Absolute Corporate Bond Spread: 10-year AAA corporate bond yield − 10-year government bond yield.
- 30-year Absolute Corporate Bond Spread: 30-year AAA corporate bond yield − 30-year government bond yield.
- 3-month Relative Corporate Bond Spread: 3-month BBB corporate bond yield − 3-month AAA corporate bond yield.
- 3-year Relative Corporate Bond Spread: 3-year BBB corporate bond yield − 3-year AAA corporate bond yield.
- 10-year Relative Corporate Bond Spread: 10-year BBB corporate bond yield − 10-year AAA corporate bond yield.
- 30-year Relative Corporate Bond Spread: 30-year BBB corporate bond yield − 30-year AAA corporate bond yield.
- UK Stock Returns: Log-differences of the FTSE-100.
- WRLDXUK: Log-differences of the Morgan Stanley World Index excluding the UK.
- US Stock Returns: Log-differences of the S&P-500.
- WRLDXUS: Log-differences of the Morgan Stanley World Index excluding the US.

## Appendix D. Kalman Filter Algorithm

The Kalman filter is an algorithm for calculating an optimal forecast of $X_t^*$, on the basis of information observed up to time $t-h$. Let $F_{t-h}$ denote the information set (filtration) generated by all observations up to time $t-h$.

First the prediction set is given by

$$\widehat{X}^*_{t|t-h} = E[X_t^* | \mathcal{F}_{t-h}] = \Phi_1 + \Phi_2 X^*_{t-h}, \tag{D5}$$

with a mean root squared error of

$$\Sigma_{t|t-h} = E[(X_t^* - \widehat{X}^*_{t|t-h})(X_t^* - \widehat{X}^*_{t|t-h})' | \mathcal{F}_{t-h}] = \Phi_2 \Sigma_{t-h} \Phi_2' + \Sigma_{\eta,t} \tag{D6}$$

Second in the update step, a more precise information is obtained by using the new information provided by $\mathcal{F}_t$:

$$\widehat{X}_t^* = E[X_t^* | \mathcal{F}_t] = \widehat{X}^*_{t|t-h} + \Sigma_{t|t-h} \Phi_2' \mathcal{P}_t^{-1} v_t \tag{D7}$$

with mean squared error of

$$\Sigma_t = E[(X_t^* - \widehat{X}_t^*)(X_t^* - \widehat{X}_t^*)' | \mathcal{F}_t] \tag{D8}$$

$$= \Sigma_{t|t-h} - \Sigma_{t|t-h} \Phi_2' \mathcal{P}_t^{-1} \Phi_2 \Sigma_{t|t-h} \tag{D9}$$

where $v_t$ and $\mathcal{P}_t$ are the prediction errors and their covariance matrix, given by:

$$v_t = R_t - A_{X^*}(\tau) - B_{X^*}(\tau)X_t^* \tag{D10}$$

$$\mathcal{P}_t = \Phi_2 \Sigma_{t|t-h}\Phi_2' + \Sigma_{\eta,t} \tag{D11}$$

Finally, the Gaussian log-likelihood is given by:

$$\log L(R_1, \ldots, R_T) = \sum_{t=1}^{T} -\frac{T}{2}\log(2\pi) - \frac{1}{2}\log|\mathcal{P}_t| - \frac{1}{2}v_t'\mathcal{P}_t^{-1}v_t \tag{D12}$$

## Appendix E. Monte Carlo simulation

As it was discussed in Section 3, in the case of the square-root interest rate models, the error term in the transition Eq. (16) is not normally distributed. Moreover, the conditional variance of that error, Eq. (17), is a function of the unobserved factor $X_{it}^*$. The application of Kalman filter estimation in that case introduces a certain degree of bias in our estimators. In order to investigate the extent of this bias we conducted a Monte Carlo simulation, which is a two-factor version of the simulations reported by de Jong (2000), Duan and Simonato (1999) and Lund (1997). The Monte Carlo experiment proceeds as follows. First we assume that the default-free parameters reported in Table 2 are the "true" parameter values. We then use these parameters and a discretised

Table 12
Monte Carlo simulations

| Coefficients | True value | Estimate (st. error) | | | |
|---|---|---|---|---|---|
| | | Median | Mean | St. dev. | *t*-statistic |
| *US* | | | | | |
| $[\delta_X]_1$ | 0.031 | 0.030 | 0.029 | 0.09 | −0.35 |
| $[\delta_X]_2$ | 0.0061 | 0.006 | 0.0063 | 1.56 | 0.006 |
| $[K_X]_{11}$ | 0.0337 | 0.0335 | 0.029 | 0.08 | −1.77 |
| $[K_X]_{22}$ | 0.6648 | 0.664 | 0.6642 | 0.89 | 0.007 |
| $[\Theta_X]_1$ | 6.2554 | 6.255 | 6.2551 | 0.29 | 0.01 |
| $[\Theta_X]_2$ | 5.4845 | 5.4845 | 5.4847 | 0.008 | 0.79 |
| $[\Xi_X]_1$ | −0.0567 | −0.057 | −0.0569 | 0.02 | 0.15 |
| $[\Xi_X]_2$ | −0.1937 | −0.1938 | −0.1939 | 0.01 | −0.31 |
| *UK* | | | | | |
| $[\delta_X]_1$ | 0.00419 | 0.0042 | 0.00421 | 0.0047 | 0.067 |
| $[\delta_X]_2$ | 0.00436 | 0.00433 | 0.044 | 0.0025 | 0.77 |
| $[K_X]_{11}$ | 1.00E−05 (constr.) | n.a. | n.a. | n.a. | n.a. |
| $[K_X]_{22}$ | 0.43620 | 0.435 | 0.432 | 0.08 | −1.18 |
| $[\Theta_X]_1$ | 0.00432 | 0.00432 | 0.0043 | 0.009 | −0.067 |
| $[\Theta_X]_2$ | 7.7463 | 7.7464 | 7.7467 | 0.05 | 0.19 |
| $[\Xi_X]_1$ | −0.0773 | −0.0773 | −0.0775 | 0.015 | −0.42 |
| $[\Xi_X]_2$ | −0.1040 | −0.106 | −0.108 | 0.09 | −0.70 |

This table reports the results from the Monte Carlo simulations for the estimation of the bias in the QML Kalman filter estimation of the two-factor square-root interest rate models. For each parameter, we report its "true " value (from Table 2) together with the median, mean and standard deviation of the estimates from 1000 simulation runs. The *t*-statistic corresponds to the test for the difference between the true and mean parameters.

version of Eq. (3), with a time step equal to 1 week, to simulate 500 observations of $X_{1t}^*$ and $X_{2t}^*$ (both the time step and the length of the time series are chosen to match the properties of our data sample). Once the values of the factors are simulated we use the term structure Eq. (5) to generate 1-, 5-, 10- and 30-year interest rates, with independent normally distributed measurement errors. We then estimate the parameters of the models using the QML Kalman filtering approach discussed in Section 3. This simulation run is repeated for 1000 times and the results are reported in Table 12 together with the *t*-statistic for the difference between the "true" and the mean of the simulated parameters. Our results are consistent with the previous evidence that the bias in the QML estimators is by and large limited and only the cases of borderline non-stationarity are affected.

## References


Altman, E.I., 1990. Corporate Financial Distress. Wiley, New York.

Altman, E.I., Kishore, V.M., 1998. Default and returns on high yield bonds: analysis through 1997. Working Paper. NYU Solomon Center.

Arvanitis, A., Gregory, J., Laurent, J., 1999. Building models for credit spreads. Journal of Derivatives 6, 27–43.

Bakshi, G., Madan, D., Zhang, F., 2001. Investigating the sources of default risk: lessons from empirically evaluating credit risk models. Working Paper. University of Maryland.

Ball, C.A., Torous, W.N., 1996. Unit roots and the estimation of interest rate dynamics. Journal of Empirical Finance 3, 215–238.

Black, F., Cox, J., 1976. Valuing corporate securities: some effects on bond indenture provisions. Journal of Finance 31, 351–367.

Campbell, J.Y., 1987. Stock returns and the term structure. Journal of Financial Economics 18, 373–399.

Chen, R.R., Scott, L., 1995. Multi-factor Cox–Ingersoll and Ross models of the term structure: estimates and tests from a Kalman filter model. Working Paper. University of Georgia.

Collin-Dufresne, P., Goldstein, R., 2001. Do credit spreads reflect stationary leverage ratios? Journal of Finance 56, 1929–1957.

Collin-Dufresne, P., Goldstein, R., Martin, S., 2001. The determinants of credit spread changes. Journal of Finance 56, 2177–2208.

Cox, J., Ingersoll, C.J., Ross, S., 1985. A theory of the term structure of interest rates. Econometrica 53, 385–407.

Dai, Q., Singleton, K., 2000. Specification analysis of affine term structure models. Journal of Finance 55, 1943–1978.

de Jong, F., 2000. Time series and cross-section information in affine term structure models. Journal of Business and Economic Statistics 18, 300–314.

Duan, J.C., Simonato, J.G., 1999. Estimating exponentially-affine term structure models by Kalman filter. Review of Quantitative Finance and Accounting 13, 111–135.

Duffie, D., 1992. Dynamic Asset Pricing. Princeton University Press, Princeton, New Jersey.

Duffee, G., 1998. The relation between treasury yields and corporate bond spreads. Journal of Finance 53, 2225–2242.

Duffee, G., 1999. Estimating the price of default risk. Review of Financial Studies 12, 197–226.

Duffie, D., Garleanu, N., 2001. Risk and valuation of collateralised debt obligation. Financial Analysts Journal 57, 41–59.

Duffie, D., Huang, M., 1996. Swap rates and credit quality. Journal of Finance 51, 921–949.

Duffie, D., Kan, R., 1996. A yield factor model of interest rates. Mathematical Finance 6, 921–950.

Duffie, D., Singleton, K., 1997. An econometric model of the term structure of interest-rate swap yields. Journal of Finance 52, 1287–1321.

Duffie, D., Singleton, K., 1999. Modelling term structures of defaultable bonds. Review of Financial Studies 12, 687–720.

Duffie, D., Singleton, K., 2003. Credit Risk: Pricing, Measurement and Management. Princeton University Press, Princeton, New Jersey.

Elton, E.J., Gruber, M.J., Agrawal, D., Mann, C., 2001. Explaining the rate spread on corporate bonds. Journal of Finance 56, 247–277.

Eom, Y.H., Helwege, J., Huang, J., 2002. Structural models of corporate bond pricing: an empirical analysis. Working Paper. Penn State University.

Estrella, A., Hardouvelis, G.A., 1991. The term structure as a predictor of real economic activity. Journal of Finance 46, 555–576.

Estrella, A., Mishkin, F.S., 1998. Predicting US recession: financial variables as leading indicators. Review of Financial Studies 80, 45–61.

Fama, E.F., French, K.R., 1989. Business conditions and expected returns in stocks and bonds. Journal of Financial Economics 18, 373–399.

Fons, J.S., Carty, L., Kaufman, J., 1994. Corporate Bond Defaults and Default Rates: 1970–1993. Moody's Special Report.

Friedman, B.M., Kuttner, K.N., 1989. Why does the Paper-Bill spread predict real economic activity? In: Stock, J., Watson, M. (Eds.), Business Cycles, Indicators and Forecasting. University of Chicago Press, Chicago.

Geske, R., 1977. The valuation of corporate liabilities as compound options. Journal of Financial and Quantitative Analysis 12, 541–552.

Golub, G.A., Van Loan, C., 1983. Matrix Computations. John Hopkins University Press, Baltimore.

Gong, G.H., Remolona, E.M., 1996a. A Three-factor econometric model of the US term structure. Working Paper. Capital Markets Function, Federal Reserve Bank of New York.

Gong, G.H., Remolona, E.M., 1996b. Two factors along the yield curve. Working Paper. Capital Markets Function, Federal Reserve Bank of New York.

Grinblatt, M., 1995. An Analytic Solution for Interest-rate Swap Spreads. Working Paper. Anderson Graduate School of Management, University of California, Los Angeles.

Hamilton, J.D., 1994. State-space models. In: Engle, R.F., McFadden, D.L. (Eds.), Handbook of Econometrics, vol. IV. Elsevier Science B.V., Holland.

Houweling, P., Vorst, T., 2001. An empirical comparison of default swap pricing models. Working Paper. Erasmus University, Rotterdam.

Huge, B., Lando, D., 1999. Swap pricing with two-sided default risk in a rating-based model. European Finance Review 3, 239–268.

Jarrow, R.A., Turnbull, S.M., 1995. Pricing derivatives on financial securities subject to credit risk. Journal of Finance 50, 53–86.

Jarrow, R.A., Turnbull, S.M., 2000. The intersection of market and credit risk. Journal of Banking and Finance 24, 271–299.

Jarrow, R.A., Lando, D., Turnbull, S.M., 1997. A Markov model for the term structure of credit risk spreads. Review of Financial Studies 10, 481–523.

Jarrow, R.A., Lando, D., Yu, F., 2002. Default risk and diversification: theory and applications. Working Paper. Cornell University.

Jones, E.P., Mason, S., Rosenfeld, E., 1984. Contingent claims analysis of corporate capital structures: an empirical investigation. Journal of Finance 39, 611–625.

Kapetanios, G., 2002. A new dynamic factor estimation method for large datasets. Bank of England, Mimeo.

Lando, D., 1998. Cox processes and credit risky securities. Review of Derivatives Research 2, 99–120.

Lardic, S., Priaulet, P., Priaulet, S., 2001. PCA of the yield curve dynamics: questions of methodologies. Working Paper. HSBC-CCF.

Lekkos, I., 2000. A critique of factor analysis of interest rates. Journal of Derivatives 1, 72–83.

Leland, H., Toft, K., 1996. Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads. Journal of Finance 51, 987–1019.

Li, H., 1998. Pricing swaps with default risk. Review of Derivatives Research 2, 231–250.

Li, H., 2000. A model of pricing defaultable bonds and credit ratings. Working Paper. John M. Olin School of Business, Washington University, St. Louis.

Litterman, R., Scheinkman, J., 1991. Common factors affecting bond returns. Journal of Fixed Income 1, 54–61.

Longstaff, F., Schwartz, E., 1995. Valuing risk debt: a new approach. Journal of Finance 50, 789–821.

Lund, J., 1997. Econometric analysis of continuous-time arbitrage-free models of the term structure of interest rates. Working Paper. Department of Finance, Aarhus School of Business, Denmark.

Merton, R., 1974. On the pricing of corporate debt: the risk structure of interest rates. Journal of Finance 29, 449–470.

Nelson, C.R., Siegel, A.F., 1987. Parsimonious modelling of the yield curves. Journal of Business 60, 473–489.

Pedrosa, M., Roll, R., 1998. Systematic risk in corporate bond credit spreads. Journal of Fixed Income 3, 7–26.

Pennacchi, G.G., 1991. Identifying the dynamics of real interest rates and inflation: evidence using survey data. Review of Financial Studies 4, 53–86.

Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T., 1992. Numerical Recipes in C++: The Art of Scientific Computing. Cambridge University Press, Cambridge.

Stock, J.H., Watson, M., 1989. New indices of coincident and leading indicators. NBER Macroeconomics Annual 4, 351–394.

Stock, J.H., Watson, M., 2002. Macroeconomic forecasting using diffusion indices. Journal of Business and Economic Statistics 20, 147–162.

Svensson, L., 1994. Estimating and interpreting forward rates: Sweden 1992–1994. Working paper. IMF.
Svensson, L., 1995. Estimating forward rates with the extended Nelson and Siegel method. Quarterly Review. Sveriges Riksbank.
Wilson, T., 1997a. Portfolio credit risk (I). Risk 10, 111–116.
Wilson, T., 1997b. Portfolio credit risk (II). Risk 11, 56–61.
Yu, F., 2002. Modelling expected returns on defaultable bonds. Journal of Fixed Income 12, 69–81.