Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Empirical Finance 12 (2005) 1–41
Journal of EMPIRICAL FINANCE
www.elsevier.com/locate/econbase

The econometrics of efficient portfolios

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Accepted 21 July 2003
Available online 26 April 2004

Abstract

It is well known that the standard mean variance approach can be inappropriate when return distributions feature skewness, fat tails or multimodes. This is typically the situation for portfolios including derivatives. In this case, it can be necessary to come back to the basic expected utility approach. In this paper, an efficient portfolio maximizes the expected utility of future wealth. This paper presents an analysis of the efficiency frontier, formed by a set of efficient portfolios corresponding to a parameterized class of utility functions. First, we discuss the estimation of an efficient portfolio and introduce several tests of the efficiency hypothesis, depending on what is known about the utility function and the budget level. Next we analyse the shape of the frontier and develop a procedure for testing the separability of the efficiency frontier into K independent funds. The inference is semi-nonparametric because the return distribution is left unspecified. We illustrate our approach by an application to portfolios including derivatives.

JEL classification: G13; C51
Keywords: Expected utility; Efficient portfolio; Efficiency test; Risk monitoring; Value at Risk; Fund separation theorem

1. Introduction

The definition of efficient portfolios and the analysis of efficient frontiers are the foundations of portfolio management theory[1]. In this domain, the mean variance approach

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[1] Roy, 1952; Arrow, 1953, 1965; Markowitz, 1952, 1976; Tobin, 1958; Feldstein, 1969; Sharpe, 1970; Merton, 1972; Huang and Litzenberger, 1988; Campbell, 2000; Gourieroux and Jasiak, 2002a.

0927-5398/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.jempfin.2003.07.001

and the assumption of gaussian returns play a central role. An alternative to the mean variance framework is based on the maximization of an expected utility of future wealth but, given the simplicity of the mean variance approach, efforts have been made in order to see in what circumstances it was equivalent to an expected utility approach. This equivalence can be reached by imposing severe restrictions either on the utility function (Samuelson, 1970; Levy and Markowitz, 1979; Epstein, 1985), or on the return distribution (Chamberlain, 1983). Another way of trying to keep the expected utility approach similar to the mean variance approach is to preserve a two-fund or a K-fund separation theorem. Again this condition can be obtained by imposing restrictions either on the utility function (Cass and Stiglitz, 1970), or on the return distribution [Ross, 1978; Chamber, 1983].

As far as the normality assumption is concerned, it is also often made for statistical inference. For instance, it is made in efficient portfolios estimation (Klein and Bawa, 1976; Jobson and Korkie, 1980), in efficiency test based on an asymptotic or a finite sample theory (Kroll and Levy, 1980; Jobson and Korkie, 1981, 1982, 1989; Kandel, 1984; Mc Kinlay, 1987; Gibbons et al., 1989; Huberman and Kandel, 1987) and in extensions taking into account liquidity constraints (Gourieroux and Jouneau, 1999).

However skewed, leptokurtic or multimodal distributions are frequently encountered in practice, especially when derivative assets are included in the portfolio (see, e.g., Clément et al., 2000). In this situation, the mean variance approach is not appropriate. For instance, the Basle Committee suggested a historical simulation of the Value at Risk to account for asymmetry and tails, instead of a mean variance approach, which would strongly underestimate the required capital.

In fact, the arguments for adopting the mean variance approach and the normality assumption for portfolio management and statistical inference are weak and mainly based on their simplicity of implementation. The aim of this paper is to propose econometric methods, which are valid for any kind of utility function and return distribution assuming that the returns are iid, but not necessarily normal. Many of the results can be extended to the case of heteroscedastic or autocorrelated returns, which is however left to the reader. These methods include the estimation of constrained and unconstrained efficient portfolios, the determination of implied stochastic discount factors (sdf's) and risk aversion coefficients, an efficiency test of a portfolio for known or parameterized utility function, and two- or K-fund separation tests. The paper is organised as follows. The efficient portfolios and the efficiency frontier are defined in Section 2. We discuss the unconstrained optimization of an expected utility function, and the optimization under a risk level constraint. As a byproduct, we introduce a utility-based Value at Risk. In Section 3 the allocation of an efficient portfolio is estimated by replacing the expected utility by its empirical counterpart, a procedure introduced by Gourieroux and Monfort (1998); Brandt (1999); Ait-Sahalia and Brandt (2001). The advantage of this approach is to bypass a preliminary estimation of the multivariate return distribution. We derive the asymptotic properties of the estimated efficient allocation with or without monitoring constraint, of the estimated performance and stochastic discount factor. Testing efficiency hypotheses is the aim of Section 4. Because the efficiency frontier is parameterized by the initial budget and the utility parameters, different hypotheses can be considered, depending whether the initial budget, or the utility parameters are assumed to be known or not. When they are not known, the procedure provides as a byproduct an estimation of the implied risk aversion parameter and of the individual budget level. In Section 5, we explain how to check if the efficiency frontier is generated by a finite number of portfolios, which is the K-fund separation hypothesis. Section 6 presents an application to portfolios containing both an underlying asset and a derivative. Section 7 concludes.

2. Efficient portfolios

The aim of this section is to define the structural parameters of interest, which will be estimated later on. We first review the standard myopic optimization of an expected utility function defining the efficient portfolio, the efficiency frontier, the implied performance and the implied stochastic discount factor. Then these results are extended to include a monitoring constraint, which leads to the definition of a utility-based Value at Risk (UVaR).

2.1. Definition

2.1.1. The optimization problem

Let us consider a market with a risk-free asset and n risky assets. At date $t-1$, all prices have been normalized to one, and at date t, they are equal to $1+R_t^{\mathrm{f}}$, $1+\tilde{R}_{1t},\ldots,1+\tilde{R}_{nt}$, respectively; R_t^{f} is the risk-free return and $\tilde{R}_{jt}$ is the return of asset j. A portfolio is defined by its allocation θ_{o}, $\theta_1,\ldots,\theta_n$ in the different assets. The portfolio value at date $t-1$ is $W_{t-1}=\theta_{\mathrm{o}}+\theta' e$, where $\theta=(\theta_1,\ldots,\theta_n)'$, $e=(1,\ldots,1)'$, whereas its future value is:

$$W_t = \theta_0(1+R_t^{\mathrm{f}}) + \theta'(e+\tilde{R}_t), \quad \text{where } \tilde{R}_t = \left(\tilde{R}_{1t},\ldots,\tilde{R}_{nt}\right).$$

An efficient portfolio maximizes the expected utility of the future wealth under the budget constraint. The introduction of an expected utility in the determination of efficient portfolios is useful in two different situations. First, the utility function may represent the preferences of the individual investors, and the comparison of a market portfolio with the set of efficient individual portfolios may be a basis for testing equilibrium models, whenever a K-fund separation theorem is satisfied. Second, a utility function (or a class of utility functions) may be introduced either by the fund manager in order to reveal new management strategies or pricing formulas by means of the associated stochastic discount factor (sdf), or by a monitoring agent in order to define an appropriate Value at Risk.

Let us introduce an (indirect) utility function $u(A;w)$, depending on some (multidimensional) parameter A, and satisfying the standard assumptions of increasingness and concavity with respect to w. The optimization problem is:

$$\max_{\theta_{\mathrm{o}},\theta}\ Eu(A;W_t)$$

$$\text{s.t.}: \theta_{\mathrm{o}} + \theta' e = W_{t-1},$$

where the expectation is evaluated at $t-1$. The quantity of risk-free asset can be eliminated by means of the budget constraint and the optimization problem becomes:

$$\max_{\theta} \; Eu\left(A;\theta'\left[(e+\tilde{R}_t)-\left(1+R_t^{\mathrm{f}}\right)e\right]+W_{t-1}\left(1+R_t^{\mathrm{f}}\right)\right).$$

Let us assume iid excess returns: $R_t=\tilde{R}_t-R_t^{\mathrm{f}}e$, and introduce the notation: $w=W_{t-1}(1+R_t^{\mathrm{f}})$. The optimization problem can be written as:

$$\max_{\theta} \; EU(R_t;\beta,\theta), \tag{2.1}$$

$$\text{where}: \beta=(A',w)', U(R_t;\beta,\theta)=u(A;\theta'R_t+w). \tag{2.2}$$

This problem is parameterized by the parameters A and the budget level w. The solution $\theta(\beta)=\theta(A,w)$ defines the efficient portfolio. The efficiency frontier corresponding to this class of utility function is made of the portfolios $\theta(\beta)$, β varying. In this framework, the efficient portfolios and the efficiency frontier are time and path independent due to the iid assumption on net excess returns, and the appropriate definition of the budget level w as $w=W_{t-1}(1+R_t^{\mathrm{f}})$. Note that in the mean variance approach the efficiency frontier is parameterized by A only, because the efficient risky allocation does not depend on w. This situation is very specific and generally an income effect has to be taken into account.

In practice, the class of utility functions can be chosen either to focus on some features of the portfolio return distribution such as skewness or tails[2], or to get an efficient allocation with some prior properties, such as to be not very diversified[3].

2.1.2. The efficient allocations

The efficient allocation and the efficiency frontier do not generally admit close form expressions. The efficient allocation satisfies the first-order conditions (foc):

$$E\left[(\tilde{R}_t-R_t^{\mathrm{f}}e)\frac{\partial u}{\partial W}(A;\theta'(\beta)R_t+w)\right]=0. \tag{2.3}$$

The lack of explicit expression of the efficient allocation is not a problem for portfolio management. As seen below the foc are moment conditions which can be used to derive a semiparametric estimation of the allocation of GMM type.

[2] For other choices of objective functions to solve the skewness problem see, e.g., Hanoch and Levy (1970), Jean (1971), Arditty and Levy (1975), Kraus and Litzenberger (1976), Lee (1977), Chunhachinda et al. (1997), Harvey and Siddique (2000), and to solve the tail problem see, e.g., Arzac and Bawa (1977), Leibowitz and Kogelman (1991), Campbell et al. (2001); Gourieroux and Monfort (2004).

[3] See, e.g., Sengupta and Sfeir (1985), Green and Hollifield (1992), Sankaran and Patil (1999) for discussion.

2.1.3. The implied stochastic discount factor

These conditions define the implied stochastic discount factor (also called pricing kernel) corresponding to the utility function (see Hansen and Richard, 1987). For period $(t-1, t)$, it is given by:

$$M_{t-1,t} = \frac{\partial u}{\partial W}[A;\theta'(\beta)R_t + w] \Big/ \left\{[1+R_t^{\mathrm{f}}]E[\frac{\partial u}{\partial W}[A;\theta'(\beta)R_t + w)]\right\}. \tag{2.4}$$

It is useful to estimate the stochastic discount factor. Indeed, it can be used to price derivatives written on the n basic assets.

2.1.4. The performance measures

The expected utility approach can also be used to define and compare portfolio performances. They are introduced for portfolios with the same initial value, $W_{t-1}=\bar{W}$ (say), or equivalently $w=\bar{w}$ (say), because the initial budget can have an effect on the structure of the efficient allocation. In the expected utility framework, the portfolios are usually ranked by means of the value of the objective function, i.e.:

$$\mathcal{P}(A;\theta) = Eu(A;\theta'R_t + \bar{w}). \tag{2.5}$$

The approach can be extended to define "the performance of a set of assets", by considering the maximal value of the performance over all portfolios based on these assets. For instance, the performance of the $n+1$ basic assets is measured by:

$$\mathcal{P}(A) = \max_{\theta} Eu(A;\theta'R_t + \bar{w}), \tag{2.6}$$

which is the value of the objective function at the optimum. For CARA utility function and gaussian returns, it is known that the utility-based definition of the performance is related to Sharpe performance.

More precisely we have:

$$\mathcal{P}(A) = -\exp(-A\bar{w})\exp - \frac{1}{2}S, \tag{2.7}$$

where $S=\mu'\Sigma^{-1}\mu$ is the Sharpe performance, $\mu=ER_t$, $\Sigma=V(R_t)$.

These standard notions of performance have been questioned in a series of papers by Chen and Knez (1996).[4] Their idea is to measure the portfolio performance by means of a "fair value" of the portfolio. In the utility framework, the "fair value" is computed with the pricing kernel given in Eq. (2.4). Due to the first-order conditions Eq. (2.3), the value given to portfolio $(\theta_{\mathrm{o}}, \theta)$ is necessarily $\theta_{\mathrm{o}}+\theta'e=\bar{w}$. Thus, by following this approach, all portfolios would admit the same fair value and would be considered as equivalent.[5]

[4] However, the Sharpe performance (or the Jensen performance) is largely used for comparison of initial funds (see, e.g., Gallo and Swanson, 1996; Detzler, 1999; Patro, 2001; Cesari and Panetta, 2002 for recent applications].

[5] Chen and Knez (1996) introduced four axioms to be satisfied by a performance measure. It seems that the axiom of linearity can be questioned.

2.2. Risk level constraint

The previous approach can be extended to derive efficient portfolios under a risk level constraint. The introduction of an additional risk level constraint appears in the risk management practices (Bodnar et al., 1998). Generally, the additional constraint is written on the Value at Risk.[6] However, the Value at Risk is based on a quantile function, which is the inverse of a survivor function. A survivor function can be written as $S(w_o)=E[\mathbb{1}_{W_t}>w_o]$; it is the expectation of the indicator function of a right infinite semiinterval. This indicator function is increasing with respect to W_t, but not concave, and cannot be interpreted as a utility function.[7]

In this section, we consider portfolio management under a risk level constraint represented by a minimal level for the expected utility function of the monitor. In this framework, the utility function of the monitor and the utility function of the fund manager have to be distinguished.

2.2.1. The monitoring constraint

Let us denote by $u_C(A_C;W)$ the monitor's utility function. For a given budget level w and portfolio allocation θ, the monitor's expected utility is $Eu_C(A_C;\theta' R_t+w)$. The monitoring constraint implies that this expected utility has to be sufficiently large:

$$Eu_C(A_C;\theta' R_t + w)\geq\gamma(w), \tag{2.8}$$

where $\gamma(w)$ is a threshold depending on w. For instance, it may be selected according to the maximal value of the monitor's expected utility as:

$$\gamma(w) = \alpha \max_\theta Eu_C[A_c;\theta' R_t + w],$$

where α is fixed between 0 and 1.

The monitoring constraint Eq. (2.8) restricts the set of admissible portfolios and may easily be written in monetary terms. Let us introduce the deterministic wealth w-UVaR yielding the same level of expected utility:

$$u_C[A_C; w - \mathrm{UVaR}(\theta, w)] = Eu_C(A_C; \theta' R_t + w).$$

The monitoring constraint is satisfied if and only if:

$$w - \mathrm{UVaR}(\theta, w)\geq u_C^{-1}[A_C;\gamma(w)],$$

$$\mathrm{UVaR}(\theta, w)\leq w - u_C^{-1}[A_C;\gamma(w)]. \tag{2.9}$$

The amount $\mathrm{UVaR}(\theta,w)$ defines the utility-based Value at Risk and the monitoring constraint is an inequality constraint on this Value at Risk.

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[6] See, e.g., Kluppelberg and Korn (1998), Kast et al. (1999), Basak and Shapiro (2001).

[7] This explains why the Value at Risk does not satisfy the minimal coherency conditions introduced in Artzner et al. (1999).

Example 1. For gaussian excess returns with mean μ and covariance matrix $\mathbf{\Omega}$ and an exponential (CARA) utility function, the constraint becomes:

$$w+\theta'\mu-\frac{A_C}{2}\theta'\mathbf{\Omega}\theta\geq-\frac{1}{A_C}\log[-\gamma(w)],$$

and the utility-based Value at Risk is given by:

$$\mathrm{UVaR}(\theta,w)=\frac{A_C}{2}\theta'\mathbf{\Omega}\theta-\theta'\mu.$$

It may be compared with the standard mean variance Value at Risk computed from the quantile function (Gourieroux and Jasiak, 2002b). The quantile constraint is:

$$P[\theta'R_t+w>a_C]\geq\alpha(w),$$

or:

$$w+\theta'\mu-a_C+(\theta'\mathbf{\Omega}\theta)^{1/2}\mathbf{\Phi}^{-1}(1-\alpha(w))\geq 0,$$

where $\alpha(w)$ is the selected loss probability.

The standard Value at Risk is given by:

$$\mathrm{VaR}=(\theta'\mathbf{\Omega}\theta)^{1/2}\mathbf{\Phi}^{-1}[\alpha(w)]-\theta'\mu,$$

and has to be smaller than: $w-a_C$.

The example shows that the main difference between the standard and utility-based Value at Risk comes from the loss function. Under the standard approach, the loss is assumed constant when $W>a_C$ or when $W<a_C$, creating a discontinuity in the criterion, whereas the loss is continuous when based on a concave utility function.[8] Thus, the VaR and UVaR depend differently on the expected return $\theta'\mu$ and volatility $\theta'\mathbf{\Omega}\theta$ of the portfolio. Indeed, the UVaR is linear in $\theta'\mathbf{\Omega}\theta$, whereas the VaR is linear in $(\theta'\mathbf{\Omega}\theta)^{1/2}$; thus, the UVaR is more sensitive to risk than the VaR.

2.2.2. The constrained optimization

If the monitoring constraint is imposed, the problem of the fund manager, with a utility function denoted by u_M, becomes.

$$\max_{\theta} Eu_M[A_M;\theta'R_t+w],\ \text{s.t.}\ Eu_C[A_C;\theta'R_t+w]\geq\gamma(w). \tag{2.10}$$

Then we can define as in the standard case the constrained efficient portfolio $\theta^c=\theta(A_M, A_C, w)$ and the constrained efficiency frontier, which is the set of constrained efficient portfolios, when A_M and w vary. Naturally, the efficient frontier will depend also on the monitor's preference u_C.

[8] The same argument is given to pass from the standard VaR to the TailVaR (see Artzner et al., 1999).

The first-order conditions become:

$$E\left[R_t\left(\frac{\partial u_M}{\partial w}[A_M;\theta^{c'}R_t+w]-\lambda^*\frac{\partial u_C}{\partial w}[A_C;\theta^{c\prime}R_t+w]\right)\right]=0, \tag{2.11}$$

where λ^* is the Kuhn–Tucker multiplier associated with the risk level constraint. Thus, the implied stochastic discount factor involves a combination of both marginal utility functions:

$$M^*_{t-1,t}=\left\{\frac{\partial u_M}{\partial w}[A_M;\theta^{c'}R_t+w]-\lambda^*\frac{\partial u_C}{\partial w}[A_C;\theta^{c'}R_t+w]\right\} \Big/ \left\{(1+R_t^{\mathrm{f}})E\left[\frac{\partial u_M}{\partial w}(A_M;\theta^{c'}R_t+w)-\lambda^*\frac{\partial u_C}{\partial w}(A_C;\theta^{c'}R_t+w)\right]\right\}. \tag{2.12}$$

Note that $M^*_{t-1,t}$ is always nonnegative, due to the inequality restrictions satisfied by the Kuhn–Tucker multiplier.

3. Estimation of efficient portfolios

The aim of this section is to derive estimators of the constrained or unconstrained efficient portfolios and efficiency frontiers, without specifying a parametric return distribution. The natural idea is to replace the theoretical optimization problem by its sample counterpart. Thus, the model is semiparametric, because the utility functions are parameterized and the return distribution is left unspecified. Nevertheless, the estimator of the functional parameter of interest $\theta(\beta)$, β varying, will converge at the parametric rate $1/\sqrt{T}$. This is a consequence of the preliminary smoothing involved when computing an expected utility. To simplify the presentation of the statistical results, we consider a framework of iid returns. The properties are easily extended to a markovian framework with unspecified transition by using kernel M-estimators (Gourieroux et al., 2000) (see Brandt, 1999 for an application of this technique to an intertemporal framework and Ait-Sahalia and Brandt, 2001 for variable selection).[9] The proof of the asymptotic properties are available on the website http://www.crest.fr/pageperso/lfa/monfort/monfort.htm.

3.1. Unconstrained efficient portfolio

Let us denote by $R_1,\ldots, R_T$ iid observations on the excess return vectors. The efficient allocation $\theta(\beta)$ can be estimated by replacing in the optimization problem the expected utility by its sample counterpart; the estimator is defined by:

$$\hat{\theta}_T(\beta)=\arg\max_\theta \frac{1}{T}\sum_{t=1}^{T}U(R_t;\beta,\theta). \tag{3.1}$$

[9] See Pastor (2000) for an alternative Bayesian approach, which assumes gaussian likelihood, but could be extended to a nonparametric framework.

This estimator converges to the efficient allocation $\theta(\beta)$, and admits the asymptotic expansion:

$$\sqrt{T}[\hat{\theta}_T(\beta)-\theta(\beta)] = -\left[E\left(\frac{\partial^2 U}{\partial\theta\partial\theta'}[R;\beta,\theta(\beta)]\right)\right]^{-1} \times \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial U}{\partial\theta}(R_t;\beta,\theta(\beta))+o_P(1), \tag{3.2}$$

where $o_P(1)$ is negligible in probability.

We deduce the following property, which extends the properties of the estimated efficient portfolios in the gaussian mean variance framework (Jobson and Korkie, 1980).

Proposition 1. *Under standard regularity conditions:*

(i) The estimator $\hat{\theta}_T(\beta)$ converges to $\theta(\beta)$.

(ii) The asymptotic distribution of $\sqrt{T}[\hat{\theta}_T(\beta)-\theta(\beta)]$ is multivariate normal, with zero mean and covariance matrix:

$$V_{\text{as}}\left[\sqrt{T}\left[\hat{\theta}_T(\beta)-\theta(\beta)\right]\right] = J(\beta)^{-1}C(\beta)J(\beta)^{-1},$$

$$\textit{where}: C(\beta) = \text{var}\left[\frac{\partial U}{\partial\theta}(R;\beta,\theta(\beta))\right] = E\left[\frac{\partial U}{\partial\theta}(R;\beta,\theta(\beta))\frac{\partial U}{\partial\theta'}(R;\beta,\theta(\beta))\right],$$

$$= E\left[RR'\left[\frac{\partial u}{\partial w}[A;\theta'(\beta)R+w]\right]^2\right],$$

$$J(\beta) = -E\left(\frac{\partial^2 U}{\partial\theta\partial\theta'}(R;\beta,\theta(\beta))\right)$$

$$= -E\left\{RR'\frac{\partial^2 u}{\partial w^2}[A;\theta'(\beta)R+w]\right\}.$$

(iii) The asymptotic normality is also satisfied when we consider two efficient portfolios on the efficiency frontier associated with parameters β_0 and β_1, say. Their asymptotic covariance is:

$$\text{Cov}_{\text{as}}\left[\sqrt{T}[\hat{\theta}_T(\beta_o)-\theta(\beta_o)],\sqrt{T}[\hat{\theta}_T(\beta_1)-\theta(\beta_1)]\right]$$

$$= J(\beta_o)^{-1}C(\beta_0,\beta_1)J(\beta_1)^{-1},$$

where:

$$C(\beta_0,\beta_1)=\mathrm{Cov}\left[\frac{\partial U}{\partial\theta}(R,\beta_0,\theta(\beta_0)),\frac{\partial U}{\partial\theta}(R,\beta_1,\theta(\beta_1))\right]$$

$$=E\left(RR'\frac{\partial u}{\partial w}[A;\theta'(\beta_0)R+w]\frac{\partial u}{\partial w}[A;\theta'(\beta_1)R+w]\right).$$

Note that the estimator of the efficient allocation can be considered as a moment estimator associated with the moment restrictions Eq. (2.3). Because the number of moment restrictions is equal to the parameter size, θ is just identified, and the estimator is automatically GMM.

Let us illustrate Proposition 1 for standard families of utility functions.

Example 2. The exponential (or CARA) utility function. In this case, the efficient allocation in the risky assets does not depend on the budget level w. We may choose:

$$\beta=A, U(R;A;\theta)=-\exp(-A\theta'R),$$

and deduce:

$$J(A)=A^2E[RR'\exp[-A\theta'(A)R]],$$

$$C(A)=A^2E\{RR'\exp[-2A\theta'(A)R]\},$$
$$V_{\mathrm{as}}(\sqrt{T}[\hat\theta_T(A)-\theta(A)])=\frac{1}{A^2}E[RR'\exp(-A\theta'(A)R)]^{-1}E\{RR'\exp[-2A\theta'(A)R]\}$$
$$\times E[RR'\exp(-A\theta'(A)R)]^{-1}.$$

The form of the asymptotic variance can easily be compared with the formula derived by Jobson–Korkie in the gaussian framework, which is $(1+S)[\frac{\Sigma^{-1}}{A^2}+\theta(A)\theta'(A)]$. If the excess return is gaussian, we get:

$$J(A)=A^2E[RR'\exp(-\Sigma^{-1}\mu R)],$$

$$C(A)=A^2E[RR'\exp(-2\Sigma^{-1}\mu R)].$$

Since $E[RR'\exp-\alpha'R]$

$$=\frac{\partial^2}{\partial\alpha\partial\alpha'}E[\exp-\alpha'R]$$

$$=[\Sigma+(-\mu+\Sigma\alpha)(-\mu+\Sigma\alpha)']\exp\left[-\alpha'\mu+\frac{\alpha'\Sigma\alpha}{2}\right],$$

we deduce: $J(A) = \Sigma \exp - \frac{1}{2}S, C(A) = \Sigma + \mu\mu'$, and

$$V_{\mathrm{as}}(\sqrt{T}[\hat{\theta}_T(A) - \theta(A)]) = \exp S\left[\frac{\Sigma^{-1}}{A^2} + \theta(A)\theta'(A)\right].$$

By convexity inequality $\exp S \geq 1 + S$, and the asymptotic variance is larger than the one derived by Jobson–Korkie. This loss of information is due to the nonparametric feature of our approach, which does not assume normality, when deriving the estimated efficient portfolio. This loss of information is the counterpart of the greater robustness of our estimator, which is consistent even if the return distribution is not gaussian.

Example 3. The power (or CCRA) utility function

$$\text{We have : } U(R;\beta,\theta) = \frac{(\theta' R + w)^{1-\gamma}}{1-\gamma}, \gamma > 0, \beta = (\gamma, w),$$

$$J(\beta) = \gamma E\{RR'[(\theta(\beta)'R + w)^{-\gamma-1}\},$$

$$C(\beta) = E\{RR'(\theta'(\beta)R + w)^{-2\gamma}\}.$$

In this case,

$$\begin{aligned}\theta(\beta) &= \mathrm{argmax}_\theta E(\theta' R + w)^{1-\gamma} \\ &= w \ \mathrm{argmax}_\varphi E(\varphi' R + 1)^{1-\gamma}, \text{ if } w > 0, \\ &= w\theta(\gamma, 1).\end{aligned}$$

Therefore, the allocation in the risky assets is proportional to a fixed allocation, when w varies.

In particular, when $\gamma = 1$, we get the limiting case of the logarithmic utility function for which:

$$\beta = w, U(R; w, \theta) = \log(\theta' R + w),$$

$$J(w) = C(w) = E(RR'(\theta' R + w)^{-2})$$

3.2. Performance and stochastic discount factor

It is also possible to deduce approximations of the performance and implied stochastic discount factor associated with the efficient portfolio.

The performance $\mathcal{P}(A;\ \theta(\beta))$ is consistently estimated by:

$$\hat{\mathcal{P}}(A,\beta) = \frac{1}{T}\sum_{t=1}^{T} u[A; \hat{\theta}_T(\beta)' R_t + w], \tag{3.3}$$

which is the optimal value of the empirical objective function. The asymptotic properties of the estimated performance are given in the proposition below.

Proposition 2. *We get:*

$$\sqrt{T}\left(\hat{\mathcal{P}}(A;\beta)-\mathcal{P}[A;\theta(\beta)]\right)\xrightarrow{d}N[0,VU(R;\beta;\theta(\beta))].$$

It is interesting to compare this result with the mean variance gaussian framework considered in Jobson and Korkie (1981). Under the restrictive gaussian assumption, the natural estimator of the Sharpe performance is:

$$\hat{S}=\hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu}, \tag{3.4}$$

where μ, Σ are the two first sample moments of the excess return. It is known (Jobson and Korkie, 1981), that:

$$V_{\mathrm{as}}[\sqrt{T}(\hat{S}-S)]=2S(2+S). \tag{3.5}$$

In our framework, the estimator of the Sharpe performance $\tilde{S}$ is defined by: $\hat{\mathcal{P}}(A;\beta)=-\exp(-A\bar{w})\exp-(1/2)\tilde{S}$. It differs from $\hat{S}$ because the expected utility is approximated by its sample counterpart, whereas the analytical computation is performed in the gaussian case. By applying Proposition 2 and the delta method, it is easily checked that:

$$V_{\mathrm{as}}[\sqrt{T}(\tilde{S}-S)]=4(\exp S-1). \tag{3.6}$$

The asymptotic variance depends on S only as in the mean variance framework. Moreover, by the convexity inequality, we get:

$$4[\exp S-1]\geq 2S(2+S).$$

Thus, the standard gaussian formula Eq. (3.5) is simply the second-order expansion of the most general formula Eq. (3.6). Moreover, the comparison between both formulas provides immediately the loss of information due to the nonparametric approach. This loss of information is rather small.

The estimation of the efficient portfolio can also be considered as a first step before pricing derivatives. The stochastic discount factor associated with the efficient portfolio is consistently estimated by:

$$\hat{M}_{t-1,t}(R)=\frac{\partial u}{\partial W}[A;\hat{\theta}'_T(\beta)R+w]/[(1+R_t^{\mathrm{f}})\frac{1}{T}\sum_{t=1}^{T}\frac{\partial u}{\partial W}(A;\hat{\theta}'_T(\beta)R_t+w)]. \tag{3.7}$$

The estimated sdf can be used to price a derivative asset with maturity 1 and payoff $g(R)$. The estimated price at τ will be:

$$\hat{C}_\tau(g) = \frac{1}{(1+R^{\mathrm{f}}_\tau)}\left[\frac{1}{T}\sum_{t=1}^{T} g(R_t)\frac{\partial u}{\partial w}[A;\hat{\theta}'_T(\beta)R_t + w]\right] \Big/ \left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial u}{\partial w}[A;\hat{\theta}'_T(\beta)R_t + w]\right]. \tag{3.8}$$

The estimated derivative prices are jointly asymptotically normal. Their joint asymptotic distribution will be used to construct confidence bands for derivative prices, which can be the basis for arbitragist strategies.

3.3. Constrained efficient portfolio

Under a monitoring constraint, the efficient portfolio is defined by the optimization problem Eq. (2.10) and the estimator of this portfolio is the solution of:

$$\max_\theta \frac{1}{T}\sum_{t=1}^{T} u_M[A_M;\theta' R_t + w] \tag{3.9}$$

$$\text{s.t.}\ \frac{1}{T}\sum_{t=1}^{T} u_C[A_C;\theta' R_t + w] \geq \gamma(w).$$

When the number of observations T tends to infinity, the estimated allocation, denoted by $\hat{\theta}^c_T(\delta)$ with $\delta=(A_M, A_C, w)$, tends to its theoretical counterpart solution of Eq. (2.10) and the estimator is consistent.

There exist two regimes depending whether the monitoring constraint is binding or not. The estimated allocation may correspond to the unconstrained one, if the constraint is not binding. Otherwise, it is the solution of the optimization problem:

$$\max_\theta \frac{1}{T}\sum_{t=1}^{T}[u_M[A_M;\theta' R_t + w] - \lambda u_C[A_C;\theta' R_t + w]],$$

where λ is the Lagrange multiplier associated with the equality constraint. The introduction of the monitoring constraint implies a change in the utility function of the fund manager.

The asymptotic distribution of $\sqrt{T}[\hat{\theta}^c_T(\delta) - \theta^c(\delta)]$ is not the same if hypothesis H_c defined by $Eu_C[A_C;\theta^c(\delta)R_t + w] = \gamma(w)$ is satisfied or not.

(i) If H_c is not satisfied, that is if $Eu_C[A_C;\theta^c(\delta)R_t + w] > \gamma(w)$, the asymptotic behavior is the same for the constrained estimator $\hat{\theta}^c_T(\delta)$ and for the unconstrained one $\hat{\theta}_T(\beta)$.

(ii) If hypothesis H_c is satisfied, that is, if the (theoretical) monitoring constraint is binding, $\sqrt{T}[\hat{\theta}_T^c(\delta) - \theta^c(\delta)]$ is asymptotically distributed as a mixture of normal distributions, with equal weights 1/2 (see Appendix A).

(iii) Finally hypothesis H_c can be tested by a Wald procedure based on the statistic

$$\xi_T^W = \frac{\sum_{t=1}^{T} [u_C(A_C; \hat{\theta}_T^{c\prime} R_t + w) - \gamma(w)]^2}{T\hat{\sigma}^2},$$

where the standardization $\hat{\sigma}^2$ is given in Appendix A. Under the null hypothesis H_c, the test statistic ξ_T^W is asymptotically distributed as an equally weighted mixture of the point mass at 0 and a chi-square with one degree of freedom. Therefore the critical region of a test at asymptotic level 5% is $\chi^2_{90\%}$, where $\chi^2_{90\%}$ denotes the quantile at 90% of $\chi^2(1)$.[10]

4. Tests of portfolio efficiency

In this section we introduce a given portfolio θ^o and consider different efficiency hypotheses:

$$H_{o,\beta} = \{\theta^o \text{ is efficient for } \beta = (A, w)\} = \{\theta^o = \theta(\beta)\}, \text{ where } \beta = (A, w) \text{ is given;}$$

$$H_{o,w} = \{\theta^o \text{ is efficient for } w\} = \{\exists A : \theta^o = \theta(A, w)\}, \text{ where } w \text{ is given;}$$

$$H_o = \{\theta^o \text{ is efficient}\} = \{\exists A, w : \theta^o = \theta(A, w)\}.$$

It is important to note that, in the standard mean variance framework, the efficiency frontier is characterized either by means of the subspace of efficient portfolios, or by their mean variance representation. Both characterizations are in a one to one relationship due to the two-fund separation theorem. In our general framework, the appropriate representation is by means of portfolios. However, some hypotheses have still standard interpretations. For instance, when $H_{o,w}$ is satisfied, the efficiency frontier constructed from the risk-free asset and portfolio θ_o intersects the whole efficiency frontier. This is the intersection condition introduced by Huberman and Kandel (1987). Thus, it could be possible to

[10] Asymptotic distributions corresponding to mixture of chi-square have also been obtained by De Roon et al. (2001) in a framework of short sale constraints. Our framework is more complicated since the monitoring constraint has to be estimated by the fund manager.

extend the approach by considering a subset of K portfolios $\theta_1^{\mathrm{o}},\ldots,\theta_K^{\mathrm{o}}$ and testing the hypothesis:

$$\bar{H}_{\mathrm{o},w} = \{\exists A, \alpha : \alpha_1\theta_1^{\mathrm{o}} + \ldots + \alpha_K\theta_K^{\mathrm{o}} = \theta(A, w)\}.$$

In the rest of the section, we derive the corresponding test statistics, and, if $H_{\mathrm{o},w}$ or H_{o} is satisfied, we obtain estimators of the implied risk aversion parameter A or budget level w, for which the given portfolio is efficient.[11] Moreover, the test statistics provide natural measures for the lack of efficiency.

4.1. The efficient allocation hypothesis for a given β

Because the estimation of efficient portfolio considered in Section 3 is an M-estimation procedure, it is natural to use the associated testing theory based on general Wald, or score principle (see Gourieroux and Monfort, 1989; 1995). In the special case of efficiency analysis, for a given β, they are respectively given by:

$$\hat{\xi}_{T,W}(\beta) = T\left[\hat{\theta}_T(\beta) - \theta^{\mathrm{o}}\right]'\hat{J}(\beta)\hat{C}(\beta)^{-1}\hat{J}(\beta)\left[\hat{\theta}_T(\beta) - \theta^{\mathrm{o}}\right], \tag{4.1}$$

$$\hat{\xi}_{T,S}(\beta) = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial U}{\partial\theta'}(R_t;\beta,\theta^{\mathrm{o}})\hat{C}(\beta)^{-1}\sum_{t=1}^{T}\frac{\partial U}{\partial\theta}(R_t;\beta,\theta^{\mathrm{o}}), \tag{4.2}$$

where $\hat{J}$, $\hat{C}$ are consistent estimators of J and C under the null hypothesis. From expansion (3.2), these statistics are asymptotically equivalent under the null hypothesis, with identical chi-square $\chi^2(n)$ distribution. Therefore, the efficiency hypothesis $H_{\mathrm{o},\beta}$ is rejected at the asymptotic level α, if $\hat{\xi}_{T,W}(\beta)$ [or $\hat{\xi}_{T,S}(\beta)$] is larger than $\chi^2_{1-\alpha}(n)$, is accepted otherwise.

It has to be noted that twice the difference between the unconstrained and constrained objective functions (which is the analogous of the likelihood ratio statistic) is:

$$\hat{\xi}(\beta) = 2\left\{\sum_{t=1}^{T}U[R_t;\beta,\hat{\theta}_T(\beta)] - \sum_{t=1}^{T}U(R_t;\beta,\theta^{\mathrm{o}})\right\}$$

$$\simeq \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial U}{\partial\theta}(R_t;\beta,\theta^{\mathrm{o}})\right]'(J(\beta))^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial U}{\partial\theta}(R_t;\beta,\theta^{\mathrm{o}})\right].$$

In general, under the null hypothesis $H_{\mathrm{o},\beta}$, this test statistic is asymptotically distributed as a linear combination of independent χ^2 distributed variables (Gourieroux and Monfort, 1989, 1995). Nevertheless, it is equivalent to the previous test statistics if

[11] In the extension, it will be also possible to derive the allocation α leading to the intersection.

$C(\beta)=J(\beta)$; this condition is for instance satisfied for the logarithmic utility function (see Example 3).

4.2. The efficient allocation hypothesis for a fixed budget level

4.2.1. The test statistics

The general Wald and score testing principle can also be developed when parameter A is not a priori given.

The test of the hypothesis $H_{o,w}=\{\exists\, A\colon \theta^o=\theta(A,w)\}$, where w is given, can be based on one of the statistics below, defined as the minimum with respect to the preference parameter A of the functional statistics $\xi_{T,W}(A,w)$ and $\xi_{T,S}(A,w)$:

$$\hat{\hat{\xi}}_{T,W}(w)=\min_A \hat{\xi}_{T,W}(A,w), \tag{4.3}$$

$$\hat{\hat{\xi}}_{T,S}(w)=\min_A \hat{\xi}_{T,S}(A,w). \tag{4.4}$$

The minimal values are obtained for:

$$\hat{A}_{T,W}(w)=\arg\min_A \hat{\xi}_{T,W}(A,w), \text{ and } \hat{A}_{T,S}(w)=\arg\min_A \hat{\xi}_{T,S}(A,w), \tag{4.5}$$

which indicate empirically the implied risk aversion parameters that are the most favorable for efficiency. Under the hypothesis that the portfolio is efficiently managed[12], the estimator $\hat{A}$ provides a consistent estimator of the utility function of the manager. The estimator of the individual risk aversion derived under the assumption of optimal allocation of financial assets might be compared with the estimator derived from consumption based model and computed by GMM from the Euler conditions (Hansen, 1982).

4.2.2. The asymptotic properties

Let us assume that the null hypothesis is satisfied and that there exists a unique value A_o, say, such that $\theta^o=\theta(A_o,w)$. $\hat{A}_{T,S}(w)$ will tend to A_o, and:

$$\sqrt{T}\left(\hat{A}_{T,W}(w)-A_o\right) \quad \text{and} \quad \sqrt{T}\left(\hat{A}_{T,S}(w)-A_o\right),$$

are asymptotically equivalent and asymptotically normal under the null hypothesis (see the website). Their asymptotic distribution is

$$N\left\{0,\left[\frac{\partial\theta'}{\partial A}(A_o,w)\Sigma^{-1}(A_o,w)\frac{\partial\theta}{\partial A'}(A_o,w)\right]^{-1}\right\},$$
$$\text{where } \Sigma(A_o,w)=J^{-1}(A_o,w)C(A_o,w)J^{-1}(A_o,w)$$

[12] It is also possible to construct a specification test of this hypothesis.

For instance, for CARA utility function and gaussian excess returns, we get:

$$V_{\mathrm{as}}\left[\sqrt{T}(\hat{A}_{T,W}(w) - A_{\mathrm{o}})\right]$$

$$= A^2 \exp S\left\{\theta'(A)\left[\frac{\Sigma^{-1}}{A^2} + \theta(A)\theta(A)'\right]^{-1}\theta(A)\right\}^{-1}$$

$$= A^2 \exp S\left\{\mu'\Sigma^{-1}\left[\Sigma^{-1} + \Sigma^{-1}\mu\mu'\Sigma^{-1}\right]^{-1}\Sigma^{-1}\mu\right\}^{-1},$$

which is an increasing function of A. The larger the risk aversion, the less accurate the estimator.

The expression of the asymptotic covariance matrix is easily understood. Let us assume for a while that the unknown preference parameters are close to a given value A_{o}. Then we can approximate the efficient allocation by:

$$\theta[A,w] \simeq \theta[A_{\mathrm{o}},w] + \frac{\partial\theta}{\partial A'}(A_{\mathrm{o}},w)(A - A_{\mathrm{o}}).$$

After replacing the underlying efficient allocation by its estimator, we get an approximated linear model:

$$\hat{\theta} \simeq \theta(A_{\mathrm{o}},w) + \frac{\partial\theta}{\partial A'},(A_{\mathrm{o}},w)(A - A_{\mathrm{o}}) + \text{error term},$$

where the error term is heteroscedastic with variance $V_{as}\hat{\theta}$. The asymptotic variance is simply the variance of the GLS estimator of preference parameter A in this linear model. It involves the sensitivity of the allocation with respect to the preference parameter.

Moreover, $\hat{\xi}_{T,W}(w)$ and $\hat{\xi}_{T,S}(w)$ are asymptotically equivalent and asymptotically chi-square distributed with $n-p$ degrees of freedom, n being the dimension of θ and p the dimension of A. Therefore, the hypothesis $H_{\mathrm{o},w}$ is rejected at asymptotic level α, if $\hat{\xi}_{T,W}(w)$ [or $\hat{\xi}_{T,S}(w)$] is larger than $\chi^2_{1-\alpha}(n-p)$, and is accepted, otherwise.

4.3. The efficient allocation hypothesis

4.3.1. The test statistics

We can now consider the null hypothesis:

$$H_{\mathrm{o}} = \{\exists A, w : \theta^{\mathrm{o}} = \theta(A,w)\}.$$

The test statistics are $\hat{\xi}_{T,W} = \min_{\beta}\hat{\xi}_{T,W}(\beta)$ and $\hat{\xi}_{T,S} = \min_{\beta}\hat{\xi}_{T,S}(\beta)$. It can be shown that these statistics are asymptotically equivalent under the hypothesis H_{o}, with a chi-square distribution with $n-p-1$ degrees of freedom (see the web site). Therefore, the efficiency hypothesis is rejected at asymptotic level α, if $\hat{\hat{\xi}}_{T,W}$ (or $\hat{\hat{\xi}}_{T,S}$) is larger than $\chi^2_{1-\alpha}(n-p-1)$, is accepted otherwise.

As a byproduct, we derive the values of A, w providing the optimum of the objective function, and therefore estimate the part of the portfolio invested in the risk-free asset. Moreover, if we observe a sequence of managed portfolios at different dates θ^j, $j=1,\ldots,J$, and if they are efficient, we can reconstitute the sequence (A_j, w_j), $j=1,\ldots,J$ and then determine how the manager is fixing his/her risk aversion as a function of his/her budget level.

4.3.2. Decomposition of the test statistics

Using general results on the tests of a sequence of nested hypotheses (see Gourieroux and Monfort, 1995, Chapter 19), a decomposition of the test statistics is easily derived. More precisely, let A_o, w_o be the parameter values for which the null hypothesis H_o is satisfied. By definition of the test statistics, we have:

$$\hat{\hat{\xi}}_{T,W} \leq \hat{\hat{\xi}}_{T,W}(w_o) \leq \hat{\hat{\xi}}_{T,W}(A_o, w_o).$$

The different efficiency hypotheses $H_{o,Ao,wo}$, $H_{o,wo}$ and H_o may be analysed jointly, since under the smallest null hypothesis $H_{o,Ao,wo}$ the statistics:

$$\hat{\hat{\xi}}_{T,W},\ \hat{\hat{\xi}}_{T,W}(w_o) - \hat{\hat{\xi}}_{T,W},\ \hat{\hat{\xi}}_{T,W}(A_o, w_o) - \hat{\hat{\xi}}_{T,W}(w_o)$$

are asymptotically independent with chi-square distributions. Their degrees of freedom are $n-p-1$, 1, and p, respectively.

5. Multifund separation hypothesis

In the mean variance framework, the efficiency frontier is generated by two basic portfolios, that are the portfolio entirely invested in the risk-free asset and the portfolio, whose allocations in the risky assets are $\eta=(VR)^{-1}ER$. This is the well-known two-fund separation theorem. The possibility for the efficiency frontier to be included in a low dimensional linear subspace of portfolios depends on the type of utility functions and return distributions (see, e.g., Cass and Stiglitz, 1970; Ross, 1978, Litzenberger and Ramaswamy, 1979; Chamberlain, 1983). The conditions to impose on the excess return distribution in order to ensure that the portfolio choice may be based on a limited number of mutual funds for any utility function are rather restrictive and not satisfied in practice. However, the multifund separation condition is of great practical relevance, because it allows for a simplified management of portfolio positions or for a pricing of financial instruments by comparing with some benchmark portfolios.

The test of multifund separation hypothesis can be seen as a first step before considering the spanning properties of a given set of portfolio.[13] Indeed, it is natural to follow the general to specific approach below:

first step: Test for the K-fund separation hypothesis.
second step: If the hypothesis is accepted, determine a generating set of portfolios.
third step: Analyse how the set of portfolios, which candidate for spanning is related to the generating set estimated in the second step.

In this section, we introduce testing procedures on the shape of the efficiency frontier without imposing a parametric form of the return distribution. We first describe the two-fund separation hypothesis before explaining how to proceed empirically with the K-fund separation hypothesis. We also explain how to estimate a generating set of portfolios when a fund separation hypothesis is accepted.

5.1. Two-fund separation hypothesis

5.1.1. The null hypothesis

If there is a risk-free asset, we have to check if the set of efficient portfolios is such that the allocation in risky assets $\theta(\beta)$ is included in a one-dimensional subspace. The null hypothesis is:

$$H_{0,2} = \{\exists\theta^1 \in \mathbb{R}^n, \forall\beta, \exists\mu(\beta) \in \mathbb{R} : \theta(\beta) = \mu(\beta)\theta^1\}. \tag{5.1}$$

The null hypothesis is in a mixed form (Gourieroux and Monfort, 1989), which involves two auxiliary parameters $\mu(\beta)$, θ^1, the first one being functional. θ^1 is a generating risky allocation, whereas $\mu(\beta)$ gives the coordinate of the risky part of the efficient portfolio according to the preference parameters and the initial wealth. Under the null, these parameters are defined up to a multiplicative scalar, and therefore, it is possible to introduce an identifying constraint, usually on the length of the generating portfolio θ^1: $(\theta^1)'Q\theta^1 = 1$ (say).

The hypothesis $H_{0,2}$ can be considered as a spanning hypothesis, in which the set of generating portfolios is not a priori given. This problem is standard in a mean variance framework. It is considered here in the general framework of expected utility maximization.

5.1.2. The test statistic

For instance, the test of the fund separation hypothesis $H_{0,2}$ can be based on a Wald-type statistic:

$$\xi_T^*(\beta,\theta) = T(\hat{\theta}_T(\beta) - \theta)'\,\mathbf{\Omega}(\beta)(\hat{\theta}_T(\beta) - \theta),\ \ \beta \text{ varying},$$

[13] For test of spanning in a mean variance framework see, e.g., Huberman and Kandel (1987), Jobson and Korkie (1989), Ferson et al. (1993), De Roon et al. (2001).

where $\mathbf{\Omega}(\beta)$ is a given positive definite matrix. More precisely, let us consider the optimization problem:

$$\min_{\theta^1} \max_{\beta} \min_{\mu(\beta)} \xi_T^*(\beta, \mu(\beta)\theta^1) = T(\hat{\theta}_T(\beta) - \mu(\beta)\theta^1)' \, \Omega(\beta)(\hat{\theta}_T(\beta) - \mu(\beta)\theta^1),$$
$$\text{s.t.}: \ \theta^{1\prime} Q\theta^1 = 1, \tag{5.2}$$

which provides an estimator $\hat{\theta}_T^1$ of the generating risky allocation, an estimator $\hat{\mu}_T(.)$ of the size function $\mu(.)$, and the optimal value $\hat{\xi}_T^*$ of the objective function.

The first optimization (minimization) provides the size $\mu(\beta)$ for β and θ^1 given. The second optimization (maximization) provides the goodness of fit of the efficiency frontier by a space generated by θ^1. Finally, the last optimization (minimization) look for the most favorable direction. Also note that the metric $\mathbf{\Omega}(\beta)$ depends on β. Indeed, the initial wealth is one component of the β parameter. Thus, we can expect that the investment in risky assets will increase with the amount to be invested, implying an heteroscedastic effect.[14]

The first optimization can be performed analytically to get the coordinate of the efficient risky allocations in terms the generating risky allocation:

$$\hat{\mu}_T(\beta) = \frac{(\theta^1)'\mathbf{\Omega}(\beta)\hat{\theta}_T(\beta)}{(\theta^1)'\mathbf{\Omega}(\beta)\theta^1}. \tag{5.3}$$

By replacing in the initial problem, we have:

$$\min_{\theta^1} \max_{\beta} \left\{ T\hat{\theta}_T(\beta)'\mathbf{\Omega}(\beta)\hat{\theta}_T(\beta) - T\frac{[\hat{\theta}_T(\beta)'\mathbf{\Omega}(\beta)\theta^1]^2}{\theta^{1\prime}\mathbf{\Omega}(\beta)\theta^1} \right\}, \text{s.t. } \theta^{1\prime} Q\theta^1 = 1.$$

Let us denote by $\tilde{\beta}_T(\theta^1)$ the solution in β of the first maximization, by $\hat{\theta}_T^1$ the solution of the final minimisation in θ^1, by $\hat{\beta}_T$ the value $\hat{\beta}_T = \tilde{\beta}_T(\hat{\theta}_T^1)$, and by $\hat{\alpha}_T$ the Lagrange multiplier associated with the constraint. Let us assume that the corresponding limit problem:

$$\min_{\theta_1} \max_{\beta} \left\{ \theta(\beta)'\mathbf{\Omega}(\beta)\theta(\beta) - \frac{[\theta(\beta)'\mathbf{\Omega}(\beta)\theta^1]^2}{\theta^{1\prime}\mathbf{\Omega}(\beta)\theta^1} \right\}, \ \text{s.t.}: \theta^{1\prime} Q\theta^1 = 1,$$

is well defined and yields unique solutions, denoted by $\beta(\theta^1)$, θ_o^1 and $\beta_o = \beta(\theta_o^1)$, respectively.

[14] Even if in the standard mean variance framework the risky allocation is independent of *w*.

5.1.3. Asymptotic expansions

The asymptotic expansion below can be derived (see the website):

$$\begin{bmatrix} \frac{\partial^2\psi}{\partial\beta\partial\beta'}+\frac{\partial\theta'}{\partial\beta}\frac{\partial^2\psi}{\partial\theta\partial\theta'}\frac{\partial\theta}{\partial\beta} & \frac{\partial\theta'}{\partial\beta}\frac{\partial^2\psi}{\partial\theta\partial\theta^{1\prime}} & 0 \\ \frac{\partial^2\psi}{\partial\theta^1\partial\theta'}\frac{\partial\theta}{\partial\beta'} & \frac{\partial^2\psi}{\partial\theta^1\partial\theta^{1\prime}}, & -2Q\theta_0^1 \\ 0 & \theta_0^{1\prime}Q & 0 \end{bmatrix}\begin{bmatrix}\sqrt{T}(\hat{\beta}_T-\beta_0)\\ \sqrt{T}(\hat{\theta}_T^1-\theta_0^1)\\ \sqrt{T}\hat{\alpha}_T\end{bmatrix}$$

$$=\begin{bmatrix} \frac{-\partial\theta'}{\partial\beta'}\frac{\partial^2\psi}{\partial\theta\partial\theta'} \\ -\frac{\partial^2\psi}{\partial\theta^1\partial\theta'} \\ 0\end{bmatrix}\sqrt{T}[\hat{\theta}_T(\beta_o)-\theta(\beta_o)]+o_p(1),$$

$$\text{where : }\ \psi(\beta,\theta^1,\theta)=\theta'\mathbf{\Omega}(\beta)\theta-\frac{[\theta'\mathbf{\Omega}(\beta)\theta^1]^2}{\theta^{1\prime}\mathbf{\Omega}(\beta)\theta^1}.$$

The asymptotic normality of the estimator follows, as well as the asymptotic distribution of the test statistics $\xi_T = T\psi[\hat{\beta}_T,\hat{\theta}_T^1,\hat{\theta}_T(\hat{\beta}_T)]$ (see the website).

The test statistic ξ_T and the estimators $\hat{\beta}_T$, $\hat{\theta}_T^1$ can be used in a general to specific approach.

First step: Compare the statistic ξ_T with the asymptotic critical value to test the separation hypothesis.
Second step 1: If the null hypothesis is accepted, $\hat{\theta}_T^1$ provides an estimated generating risky allocation.
Second step 2: If the null hypothesis is rejected, $\hat{\beta}_T$ gives the risk aversion and initial wealth which are the least favorable to the separation hypothesis.

5.2. K-fund separation hypothesis

The null hypothesis becomes:

$$H_{o,K}=\{\exists K\in\mathbb{N}^*,\exists\theta^1,\ldots,\theta^{K-1}\in\mathbb{R}^*,\forall\beta,\exists\mu_1(\beta),\ldots,\mu_{K-1}(\beta)\in\mathbb{R}:\theta(\beta)$$
$$=\mu_1(\beta)\theta^1+\ldots+\mu_{K-1}(\beta)\theta^{K-1}\}.$$

Even if it is still possible to follow a two-step approach by first estimating K, θ^1,. . ., θ^{K-1}, $\mu_1(.)$,. . ., $\mu_{K-1}(.)$ and then computing a test statistic, the asymptotic properties of the corresponding estimators and test statistics are difficult to derive. However, we can introduce a principal component approach, whose statistical properties may be analysed by simulation. This approach is the analogous of the singular value decomposition of the volatility–covolatility matrix usually performed in a mean variance framework. Like in the standard approach, it will be based on a spectral analysis of a well-chosen matrix. This approach may be applied along the following lines.

(i) Select a set of values $\beta_1,\ldots,\beta_J$ and compute the corresponding estimated efficient portfolios $\hat{\theta}_T(\beta_1),\ldots,\hat{\theta}_T(\beta_J)$, $(J>n>K)$.
(ii) Consider the symmetric (n,n) matrix:

$$\Lambda_T = [\hat{\theta}_T(\beta_1),\ldots,,\hat{\theta}_T(\beta_J)][\hat{\theta}_T(\beta_1),\ldots,\hat{\theta}_T(\beta_J)]' .$$

(iii) Perform a spectral decomposition of this matrix, that is compute the eigenvalues $\hat{\lambda}_{1T}\geq\hat{\lambda}_{2T}\geq\ldots\geq\hat{\lambda}_{nT}$, and an associated basis of eigenvectors.
(iv) Then estimate the number of mutual funds by the index $\hat{K}-1$, for which the eigenvalues become nonsignificant, and estimate the basic mutual funds by the eigenvectors associated with $\hat{\lambda}_{1T},\ldots,\hat{\lambda}_{\hat{K}-1,T}$.

6. Application

6.1. The data

We consider the daily Hang Sen Index (HSI) over the period January 2nd, 1997, to January 29th, 1999, corresponding to 501 observations. The analysis of this series is illustrative for the analysis of many other series; that is, the problems faced and the way to deal with them will be similar for other series. Note that the period of analysis covers approximately 2 years. This period is rather short compared to the whole set of available data on the index. However, a similar period (1 year) is usually considered by practitioners for rolling portfolio management and is suggested by the regulator (Basle

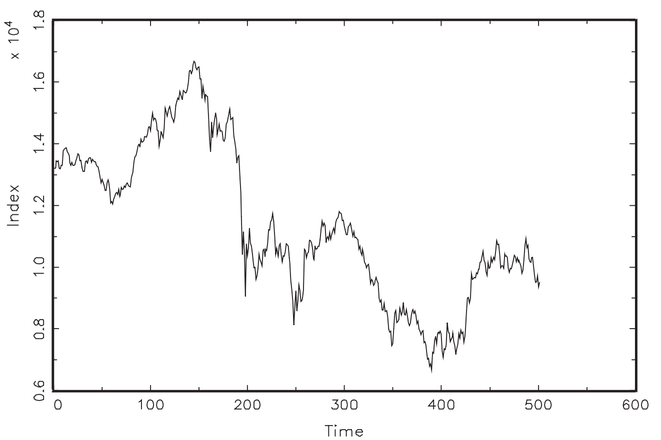

Fig. 1. Index level.

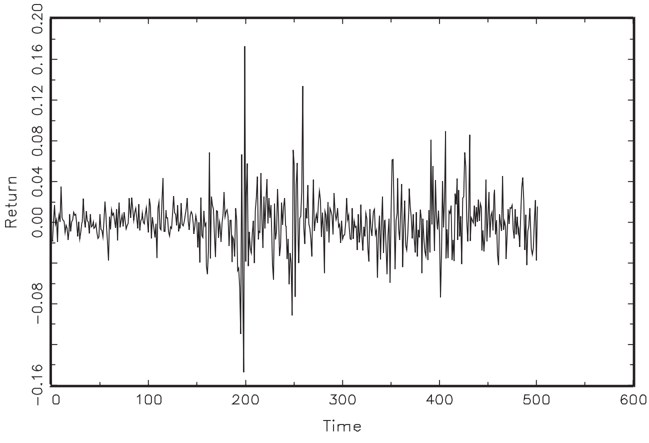

Fig. 2. Daily index return.

Committee) for risk supervision. Clearly, a 1- or 2-year period is smaller than the periodicity of business cycles, and with no surprise we can observe a period of strongly monotonic prices either increasing, or decreasing. A rolling portfolio management on such a 1- or 2-year periods has the advantage of fitting the cycle without being obliged to model the regime switching of expansion–recession. The HSI is a value-weighted index of 33 stocks highly traded on the Hong Kong Stock Exchange, which accounts for about 70% of total market capitalization. The Hong Kong Stock Exchange is the second largest Asian market after Tokyo.

The index level is displayed in Fig. 1 and its return in Fig. 2.

In addition to this index, we consider a series of daily implied volatilities (see Fig. 3), which are derived by averaging daily the Black–Scholes implied volatilities of the six

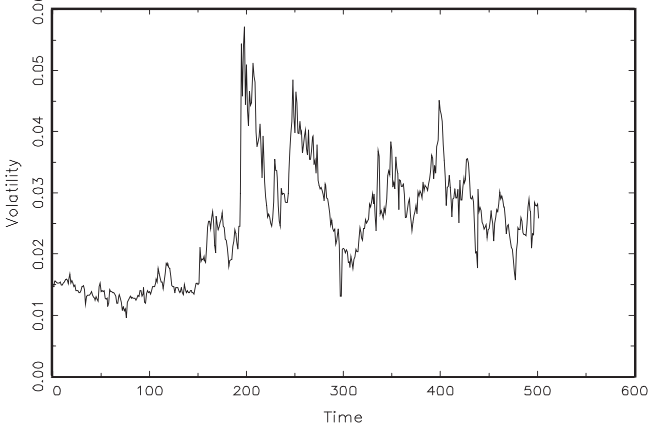

Fig. 3. Implied volatility.

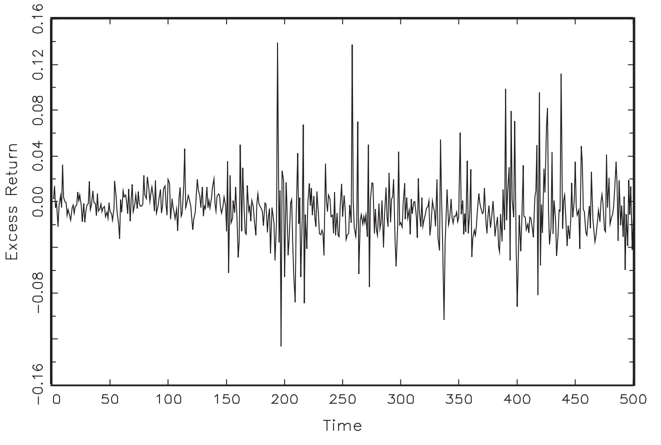

Fig. 4. Daily call excess return: maturity 1 month.

most frequently traded call and put options written on the index. The option data have been obtained from the Hong Kong Futures Exchange. As usual, the options are not written on the index itself, which is not traded, but on index futures. Nevertheless, the put-call parity with the index itself seems to hold extremely well (see the discussion in Duan and Zhang, 2001). The other bias due to averaging over different strikes and maturities can also be neglected in a first step, especially since the liquid options are close to at the money options. This is the reason why we prefer to keep the average itself which can be computed (and is reported) daily, instead of focusing on a given moneyness strike and maturity, while diminishing the observation frequency (see Duan and Zhang, 2001 for the building of a weekly data base of this type).

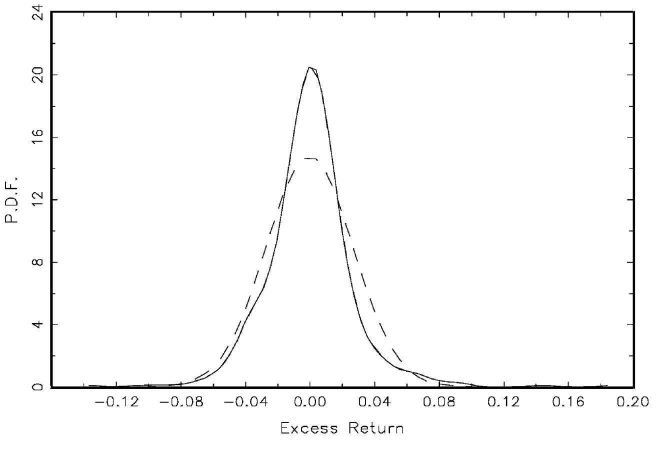

Fig. 5. P.D.F. of the index excess return (daily).

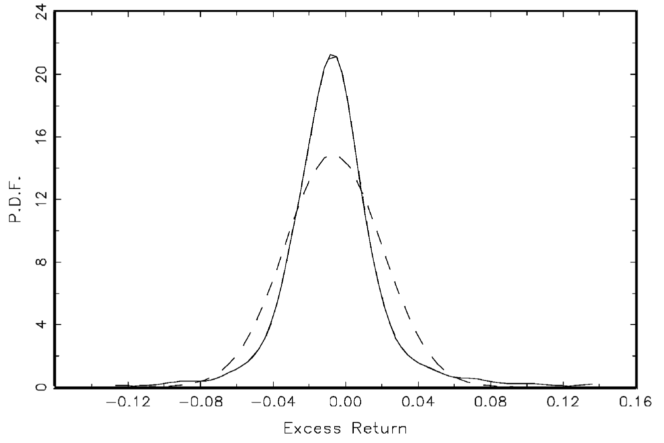

Fig. 6. P.D.F. of the call excess return (daily, $t = 1$ month).

From these series we can compute the index returns at different horizons, as well as the returns of calls with various maturities and strike moneyness, by reapplying the Black and Scholes (1973) formula. Assuming an annual risk-free rate of 3%, we construct two pairs of excess returns:

pair A: daily excess returns of the index itself and of a call with maturity 1 month and strike moneyness 0.9; the excess return of the call is displayed in Fig. 4.

pair B: 3-month excess returns of the index and of a call with maturity 6 months and strike moneyness 0.9; in this case, the daily series of returns deal with overlapping periods.

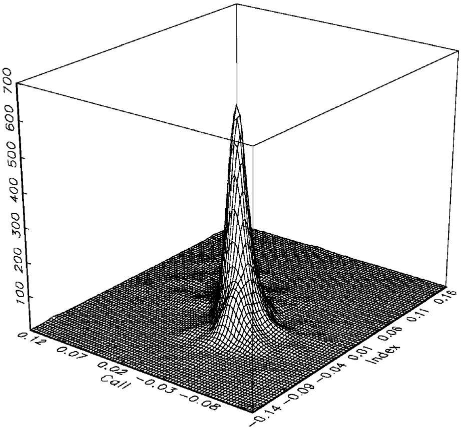

Fig. 7. Joint P.D.F.: index and call (daily, $t = 1$ month).

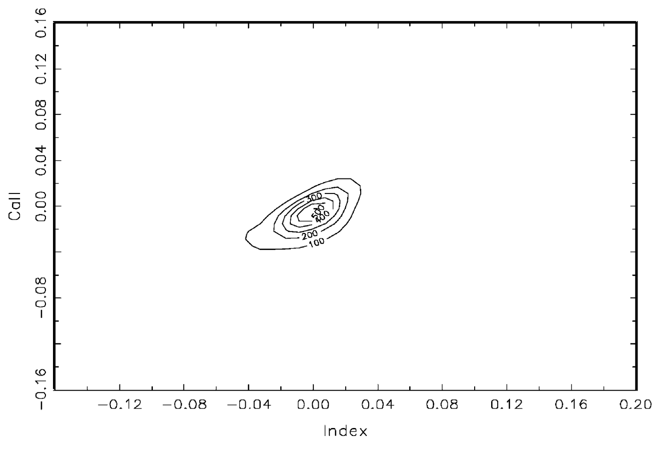

Fig. 8. Equidensity curves.

For each pair we provide the nonparametric estimates of the P.D.F. of each excess return, as well as their joint P.D.F. For daily returns, the univariate P.D.F.'s feature no skewness, but their tails are much thicker than gaussian tails (see Figs. 5–8); in Figs. 5 and 6, the dotted curves are the P.D.F.'s of the gaussian distribution with the same mean and variance as the other P.D.F. plotted in the same figure. The kurtosis are 10.9 for the index return and 9.1 for the call return, respectively.

The excess return for the index is close to zero, even slightly negative due to the observation period within the cycle. However, the expected return of the call is much more negative, especially for 3-month horizon. This fact may seem surprising at first

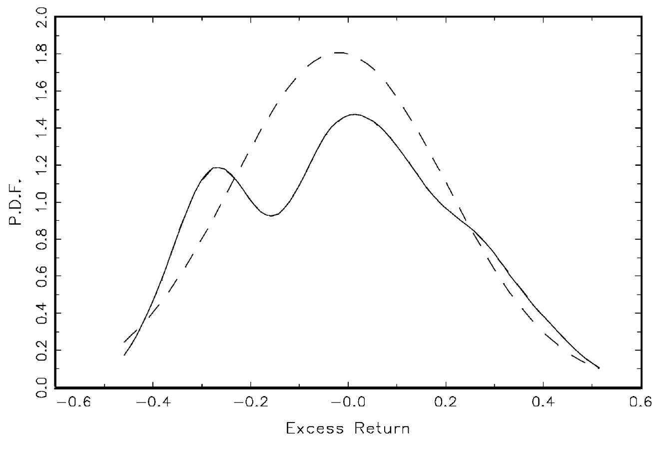

Fig. 9. P.D.F. of the index excess return (3 months).

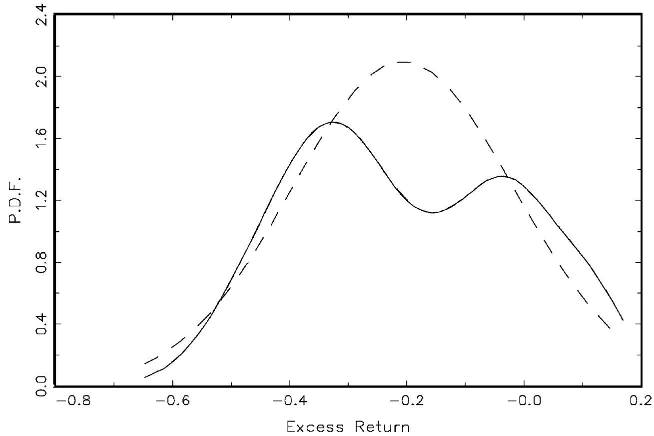

Fig. 10. P.D.F. of the call excess return (3 months, $t = 6$ months).

sight. Indeed, at equilibrium in the usual mean variance framework, the expected excess return is generally positive due to the risk premium effect. However, this result assumes gaussianity or at least a mean variance framework. However, an asset with negative expected return can be demanded by investors if its distribution is skewed especially for the extreme risk. Our example is typically of this kind because an option is introduced to hedge a risk of a given sign; an option is an "insurance" product which can have a cost in terms of expected return. More practically, the investors have to manage portfolios in various situations, including expansion and recession periods. In such periods, it is important to compare the usual mean variance approach with a more appropriate one.

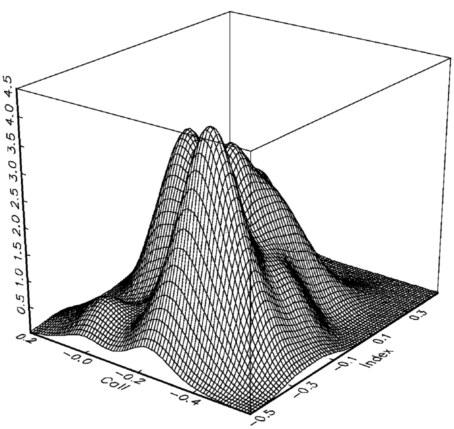

Fig. 11. Joint P.D.F.: index and call (3 months, $t = 6$ months).

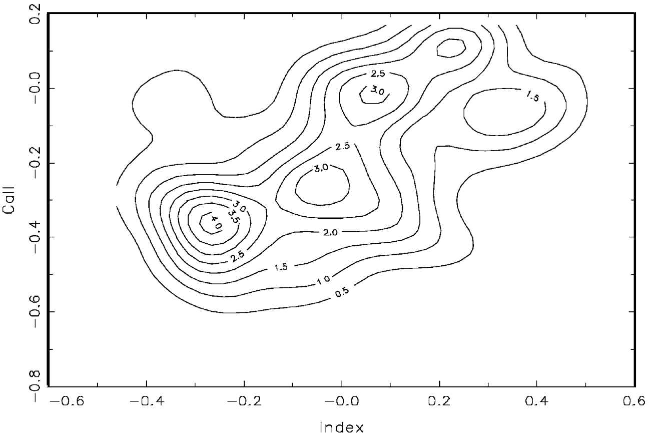

Fig. 12. Equidensity curves.

For 3-month returns, the univariate distributions feature two modes and are skewed (see Figs. 9, 10, 11 and 12), whereas the kurtosis are 2.2 for the index return and 1.9 for the call return, respectively. We also note the complicated joint density, which features several modes. The presence of several modes is not surprising. Indeed, let us consider the standard Black and Scholes (1973) model; the price of a call at maturity is equal to $(S_T\text{--}K)^{+}$, where S_T is the price of the underlying asset and K the strike. Therefore the distribution of the call price at maturity is the mixture of a point mass at 0 and a truncated log-normal distribution. It admits two modes (resp. one mode) if the strike is smaller [resp. larger] than the mode of the log normal. As a consequence, we expect four modes [resp. two modes] for the distribution of a call return (see also Clément et al., 2000).

6.2. Estimation of optimal portfolios

For each investment horizon, we consider the optimal allocations for a portfolio including the riskless asset, the basic risky asset (i.e., the index) and the call written on the index. The mean variance optimal allocation and the optimal allocation based on a CARA utility function are both displayed as function of the risk aversion parameter.

In Figs. 13 and 14, the dotted lines correspond to the mean variance allocations and the solid line to the CARA utility based allocations; the allocations in the basic asset and in the call are positive and negative, respectively. If the returns were gaussian, the mean variance and the CARA utility-based allocations would be identical. Figs. 13 and 14 show that this is not the case. Moreover, different departures from gaussianity, that are thick tails for case A and bimodality for case B, have different consequences on the allocations. At short horizon, the mean variance allocations are both larger in absolute value, whereas at 3-month horizon, the mean

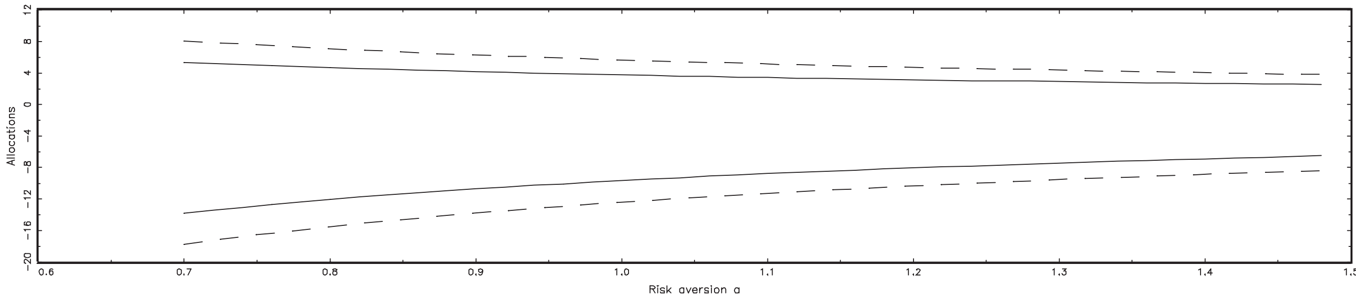

Fig. 13. Optimal portfolios: expected utility and mean variance (daily).

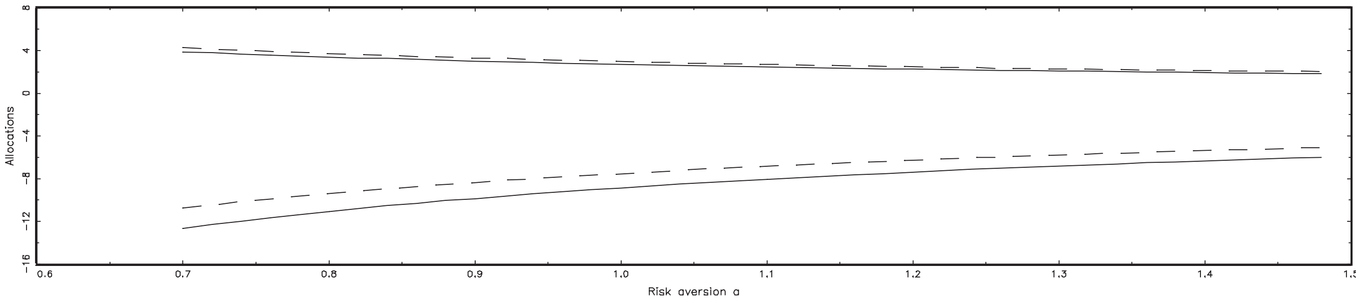

Fig. 14. Optimal portfolios: expected utility and mean variance (3 months).

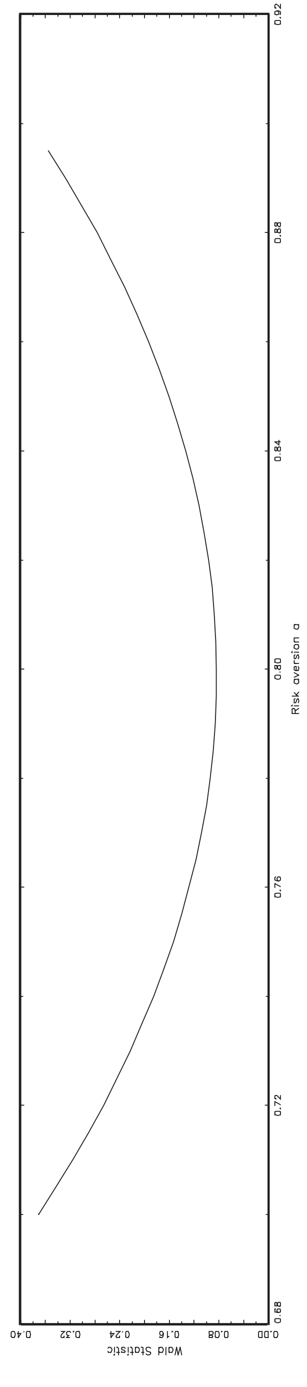

Fig. 15. Functional Wald statistic for an MV portforlio (daily).

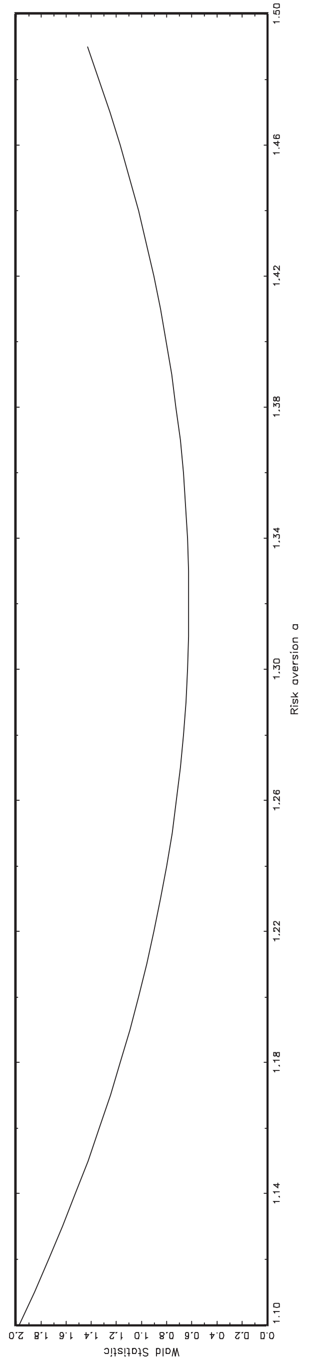

Fig. 16. Functional Wald statistic for an MV portforlio (3 months).

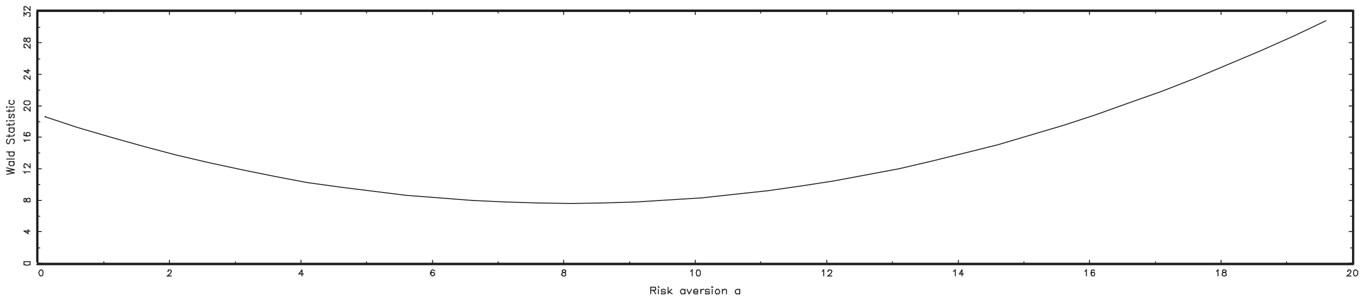

Fig. 17. Functional Wald statistic for an arbitrage portforlio in risky assets (daily).

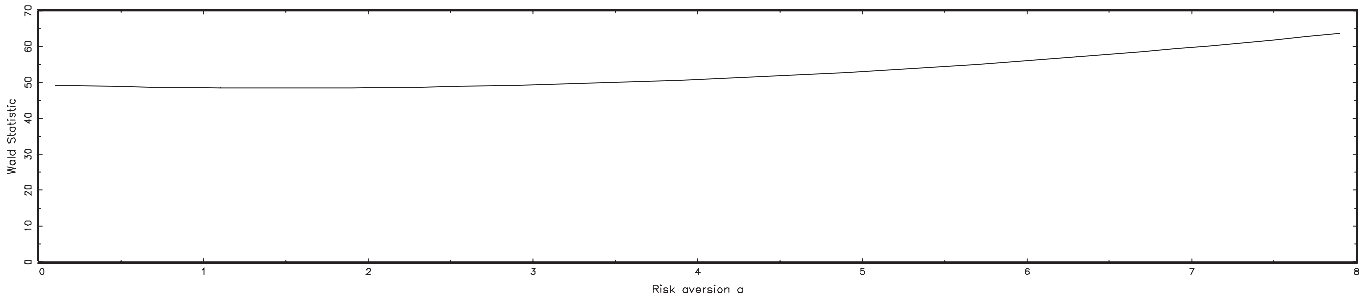

Fig. 18. Functional Wald statistic for an arbitrage portforlio in risky assets (3 months).

variance allocation corresponding to the call is smaller in absolute value than the one corresponding to the CARA utility function.

6.3. Efficiency tests and estimation of the risk aversion parameter

As an illustration for each pair of risky assets, we use as a benchmark the optimal mean variance portfolio corresponding to the risk aversion parameter $A=1$. Then we examine the efficiency of this benchmark portfolio under the CARA utility assumption. This is done by considering the functional Wald statistic $\hat{\hat{\xi}}_{T,W}(A)$ and its minimum $\hat{\hat{\xi}}_{T,W}=\min_A \hat{\hat{\xi}}_{T,W}(A)$, which allows to test the null hypothesis under which the portfolio considered is CARA efficient for some risk aversion parameter A not necessarily equal to 1. The critical region of the test at asymptotic level α is $\{\hat{\hat{\xi}}_{T,W} \leq \chi^2_{1-\alpha}(1)\}$.

At 1-day horizon, $\hat{\hat{\xi}}_{T,W}$ is equal to 0.08 (see Fig. 15) and the null hypothesis is accepted. The estimated risk aversion parameter under the null hypothesis is $\hat{A}=0.8$, and is smaller than the risk aversion parameter corresponding to the mean variance approach, i.e., $A=1$.

At 3-month horizon, $\hat{\hat{\xi}}_{T,W}$ is equal to 0.7 (see Fig. 16), the null is still accepted (when a correction of the asymptotic behavior for overlapping rolling periods is introduced), and the estimated risk aversion parameter $\hat{A}$ is larger than 1 and approximately equal to 1.3.

Let us now provide an example in which the efficiency hypothesis will be rejected. The benchmark portfolio is the arbitrage portfolio in the risky assets corresponding to the weights $(1,-1)$ for the asset and the call, respectively. In both cases, i.e., for daily and 3-month horizons, efficiency of the arbitrage portfolio is strongly rejected using the Wald statistic (see Figs. 17 and 18).

7. Concluding remarks

The aim of this paper was to develop an approach of efficiency analysis in a general framework of expected utility maximization, without imposing arbitrary parametric restrictions on the return distribution. We have seen that the standard approaches proposed in the gaussian mean variance framework can be extended to the general case. They include the estimation of efficient portfolios, of performances, of implied risk aversion and implied stochastic discount factors as well as tests of various efficiency or fund separation hypothesis, or estimation of spanning sets of portfolios. Statistical inference is based on standard nonlinear semiparametric approaches, such as M-estimation. The procedures have been illustrated by an application to a portfolio including both basic assets and derivatives. The application has shown that the semiparametric approach is easy to implement and provides reasonable results for the rather small number of observations (1 or 2 years) used in practice for rolling portfolio management or computation of Value at Risk. These approaches are easily extended to conditional dynamic framework, and to other types of criteria describing for instance ambiguity.

Appendix A. Asymptotic distribution of the constrained estimator $\sqrt{T}[\hat{\boldsymbol{\theta}}_T^c(\boldsymbol{\delta}) - \boldsymbol{\theta}^c(\boldsymbol{\delta})]$ under $\boldsymbol{H_c}$

For sake of simplicity, $\hat{\theta}_T(\beta)$, $\hat{\theta}_T^c(\delta)$ and $\theta^c(\delta)$ will be denoted by $\hat{\theta}_T$, $\hat{\theta}_T^c$ and θ^c, respectively; $\frac{1}{T}\sum_{t=1}^{T} u_C(A_C;\theta' R_t + w) - \gamma(w)$ will be denoted by $g_T(\theta)$ and $Eu_C(A_C;\ \theta' R_t+w)-\gamma(w)$ will be denoted by $g_\infty(\theta)$.

If the monitoring constraint of problem (2.10) is binding (hypothesis H_c) we have $g_\infty(\theta^c)=0$. The inequality constrained estimator $\hat{\theta}_T^c$ is defined by:

$$\begin{cases} \max_\theta \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} u_M(A_M;\theta' R_t + w) = \dfrac{1}{T} L_T(\theta), \\ \text{s.t. } g_T(\theta) \geq 0. \end{cases}$$

The equality constrained estimator $\hat{\theta}_T^0$ is defined by:

$$\begin{cases} \max_\theta \dfrac{1}{T} L_T(\theta), \\ \text{s.t. } g_T(\theta) = 0. \end{cases}$$

Therefore, the estimator is:

$$\hat{\theta}_T^c = \hat{\theta}_T \mathbb{1}_{\{g_T(\hat{\theta}_T)\geq 0\}} + \hat{\theta}_T^0 \mathbb{1}_{\{g_T(\hat{\theta}_T)<0\}},$$

and admits two different expressions according to the sign of the estimated monitoring constraint. We deduce:

$$\sqrt{T}(\hat{\theta}_T^c - \theta^c) = \sqrt{T}(\hat{\theta}_T - \theta^c)\mathbb{1}_{\{g_T(\hat{\theta}_T)\geq 0\}} + \sqrt{T}(\hat{\theta}_T^0 - \theta^c)\mathbb{1}_{\{g_T(\hat{\theta}_T)<0\}}.$$

(a) Asymptotic behavior of $\sqrt{T}(\hat{\theta}_T^0 - \theta^c)$ under H_c.
The first-order conditions are:

$$\begin{cases} \dfrac{\partial L_T}{\partial \theta}(\hat{\theta}_T^0) - \dfrac{\partial g_T(\hat{\theta}_T^0)}{\partial \theta}\hat{\lambda}_T = 0 \\ g_T(\hat{\theta}_T^0) = 0. \end{cases}$$

Asymptotic expansions around the limiting value θ^c give:

$$\begin{cases} \dfrac{1}{\sqrt{T}}\dfrac{\partial L_T}{\partial\theta}(\theta^c)+\dfrac{1}{T}\dfrac{\partial^2 L_T}{\partial\theta\partial\theta'}(\theta^c)\sqrt{T}(\hat{\theta}^0_T-\theta^c)-\dfrac{\partial g_\infty}{\partial\theta}(\theta^c)\dfrac{\hat{\lambda}_T}{\sqrt{T}}=o_P(1),\\ \sqrt{T}[g_T(\hat{\theta}^0_T)-g_T(\theta^c)]+\sqrt{T}g_T(\theta^c)=0,\end{cases}$$

and by using $\frac{1}{\sqrt{T}}\frac{\partial L_T}{\partial\theta}(\theta^c)=J(\theta^c)\sqrt{T}(\hat{\theta}_T-\theta^c)$, we deduce:

$$\begin{cases} J(\theta^c)\sqrt{T}(\hat{\theta}_T-\theta^c)-J(\theta^c)\sqrt{T}(\hat{\theta}^0_T-\theta^c)-\dfrac{\partial g_\infty}{\partial\theta}(\theta^c)\dfrac{\hat{\lambda}}{\sqrt{T}}=o_P(1),\\ \dfrac{\partial g_\infty}{\partial\theta}\sqrt{T}(\hat{\theta}^0_T-\theta^c)+\sqrt{T}g_T(\theta^c)=o_P(1).\end{cases}$$

Moreover, we have:

$$\sqrt{T}g_T(\theta)=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(u_{C,t}-\gamma)$$

where $u_{C,t}$ stands for $u_C(A_C;\ \theta^{c\prime}R_t+w)$ and γ for $\gamma(w)$.

Finally, we get the system:

$$\begin{pmatrix} J & \dfrac{\partial g_\infty}{\partial\theta}\\ \dfrac{\partial g_\infty}{\partial\theta'} & 0\end{pmatrix}\begin{pmatrix}\sqrt{T}(\hat{\theta}^0_T-\theta^c)\\ \dfrac{\hat{\lambda}_T}{\sqrt{T}}\end{pmatrix}=\begin{pmatrix} J\sqrt{T}(\hat{\theta}_T-\theta^c)\\ -\dfrac{1}{\sqrt{T}}\displaystyle\sum_{t=1}^{T}(u_{C,t}-\gamma)\end{pmatrix}.$$

This system gives:

$$\hat{\lambda}_T=\frac{1}{\dfrac{\partial g_\infty}{\partial\theta'}J^{-1}\dfrac{\partial g_\infty}{\partial\theta}}\left[\frac{\partial g_\infty}{\partial\theta'}\sqrt{T}(\hat{\theta}_T-\theta^c)+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(u_{C,t}-\gamma)\right],$$

and:

$$\sqrt{T}(\hat{\theta}^0_T-\theta^c)=(I-P)\sqrt{T}(\hat{\theta}_T-\theta^c)-p\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(u_{C,t}-\alpha),$$

$$\text{with}:\ P=\frac{1}{\dfrac{\partial g_\infty}{\partial\theta'}J^{-1}\dfrac{\partial g_\infty}{\partial\theta}}J^{-1}\frac{\partial g_\infty}{\partial\theta}\frac{\partial g_\infty}{\partial\theta'},$$

$$p=\frac{1}{\dfrac{\partial g_\infty}{\partial\theta'}J^{-1}\dfrac{\partial g_\infty}{\partial\theta}}J^{-1}\frac{\partial g_\infty}{\partial\theta}.$$

Or, noting $u_{M,t}=u_M(A_M;\ \theta^{c'}R_t+w)$ and assuming that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\begin{pmatrix} u_{C,t}-\gamma \\ \dfrac{\partial u_{M,t}}{\partial \theta}\end{pmatrix} \xrightarrow{d} N(0,Q),$$

where Q is block decomposed as: $Q=\begin{pmatrix}\sigma_c^2 & q' \\ q & C\end{pmatrix}$, we get:

$$\sqrt{T}(\hat{\theta}_T^0-\theta^c)=[-p,(I-P)J^{-1}]\begin{bmatrix}\dfrac{1}{\sqrt{T}}\sum_{t=1}^{T}(u_{C,t}-\gamma)\\ \dfrac{1}{\sqrt{T}}\sum_{t=1}^{T}\dfrac{\partial u_{M,t}}{\partial \theta}\end{bmatrix}.$$

Therefore:

$$\sqrt{T}(\hat{\theta}_T^c-\theta^c)\xrightarrow{d} N(0,\Sigma^c),$$

with: $\Sigma^c=\sigma_c^2pp'+(I-P)\Sigma(I-P)'-(I-P)J^{-1}qp'-pq'J^{-1}(I-P)',\Sigma=J^{-1}CJ^{-1}$.

(b) Asymptotic distribution of $\sqrt{T}(\hat{\theta}_T^c-\theta^c)$ and test of H_c.

We have:

$$\sqrt{T}(\hat{\theta}_T^c-\theta^c)=\sqrt{T}(\hat{\theta}_T-\theta^c)\mathbb{1}_{\{g_T(\hat{\theta}_T)\geq 0\}}+\sqrt{T}(\hat{\theta}_T^0-\theta^c)\mathbb{1}_{\{g_T(\hat{\theta}_T)<0\}}.$$

Under H_c,

$$\sqrt{T}g_T(\hat{\theta}_T)=\sqrt{T}\hat{g}_T(\theta^c)+\frac{\partial g_\infty}{\partial \theta}\sqrt{T}(\hat{\theta}_T-\theta^c)+o_p(1)$$

$$=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(u_{C,t}-\gamma)+\frac{\partial g_\infty}{\partial \theta}\sqrt{T}(\hat{\theta}_T-\theta^c)+o_p(1),$$

is asymptotically distributed as $N(0,\sigma_g^2)$, (say); therefore, $P[g_T(\hat{\theta}_T)\geq 0]\rightarrow 1/2$ and the asymptotic distribution of $\sqrt{T}(\hat{\theta}_T^c-\theta^c)$ is $\frac{1}{2}N(0,\Sigma)+\frac{1}{2}N(0,\Sigma^c)$.

Moreover

$$\sqrt{T}g_T(\hat{\theta}_T^c)=\begin{cases}0, \text{ if } g_T(\hat{\theta}_T)<0,\\ \sqrt{T}g_T(\hat{\theta}_T), \text{if } g_T(\hat{\theta}_T)\geq 0\end{cases},$$

and, under H_c:

$$\sqrt{T}g_T(\hat{\theta}_T)\xrightarrow[T\to\infty]{d}N(0,\sigma_g^2),$$

with $\sigma_g^2=\sigma_c^2+\frac{\partial g_\infty}{\partial\theta'}\Sigma\frac{\partial g_\infty}{\partial\theta}+2\frac{\partial g_\infty}{\partial\theta'}J^{-1}q.$

Therefore:

$$\frac{Tg_T^2(\hat{\theta}_T^c)}{\hat{\sigma}_g^2}\xrightarrow{d}\frac{1}{2}\varepsilon_0+\frac{1}{2}\chi^2(1),$$

where ε_0 is the point mass at zero and $\hat{\sigma}_g^2$ a consistent estimator of σ_g^2.

Thus, the critical region of the Wald test of H_c at asymptotic level α is:

$$\frac{Tg_T^2(\hat{\theta}_T^c)}{\hat{\sigma}_g^2}\geq\chi_{1-2\alpha}^2(1).$$

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