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# Cross-sectional tests of deterministic volatility functions

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**Abstract**

We study the cross-sectional performance of option pricing models in which the volatility of the underlying stock is a deterministic function of the stock price and time. For each date in our sample of FTSE 100 index option prices, we fit an implied binomial tree to the panel of all European style options with different strike prices and maturities and then examine how well this model prices a corresponding panel of American style options. We find that the implied binomial tree model performs no better than an ad-hoc procedure of smoothing Black–Scholes implied volatilities across strike prices and maturities. Our cross-sectional results complement the time-series findings of Dumas et al. [J. Finance 53 (1998) 2059].


*JEL classification:* G12; G13
*Keywords:* Deterministic volatility functions; Option pricing

## 1. Introduction

Dumas et al. (1998, DFW) show that option pricing models based on deterministic volatility functions perform no better in out-of-sample pricing and hedging than an ad-hoc procedure of smoothing Black–Scholes implied volatilities across strike prices and maturities. In practice, however, deterministic volatility functions are used more often for cross-sectional pricing than for out-of-sample pricing and hedging. Typically, an implied binomial tree is fitted to the cross-section of all liquid options to then contemporaneously price illiquid and exotic derivatives (a procedure also referred to as no-arbitrage pricing). The relative failure of deterministic volatility functions along the time-series dimension, as

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documented by DFW, does not automatically imply that these models are not more useful for cross-sectional pricing. In this paper, we therefore examine the cross-sectional performance of option pricing models based on deterministic volatility functions.

Our empirical analysis is based on a sample of FTSE 100 index option prices.[1] The FTSE 100 index options market is unusual because both European and American style options with the same maturities and written on the same underlying cash index are heavily exchange traded side by side.[2] Most other liquid option markets trade either only European or American style contracts on a given underlying security. The design of our experiment is straightforward. For each date in the sample, we fit an implied binomial tree to the panel of all European style options on the FTSE 100 index and then examine how well the model prices the corresponding American style options. This approach of fitting an implied binomial tree to one set of derivatives to then price another set (usually less liquid or exotic ones) mirrors the way deterministic volatility functions are commonly used in practice.

A key assumption of the Black and Scholes (1973) option pricing model is that the underlying stock price follows a geometric Brownian motion with constant volatility. This constant volatility assumption was initially believed to characterize the observed option prices reasonably well (see, for example, Rubinstein, 1985). However, since the 1987 market crash, the data appear to be at odds with the Black–Scholes model. In particular, using S&P 500 index option prices from 1986 to 1992, Rubinstein (1994) documents a persistent *implied volatility smile* and *implied volatility term structure*. The volatility smile refers to the observation that implied volatilities calculated by inverting the Black–Scholes formula for options with the same maturity date tend to decrease (at a decreasing rate) with the strike price of the option. The volatility term structure captures the variation of at-the-money implied volatilities from options with different maturity dates. Both phenomena are more pronounced for individual stock options than for index options and the phenomena are not confined to US markets. For example, Gemmill and Kamiyama (1997) document volatility smiles and term structures in the UK and Japan.

An ever growing number of alternative option pricing models relax the constant volatility assumption in an effort to explain the observed Black–Scholes implied volatility smile and term structure patterns. These extensions of the Black–Scholes model can be categorized into two groups: stochastic volatility models and deterministic volatility function models. In stochastic volatility models, the volatility of the underlying stock follows itself a random process. The problem with these models is that their parameters and especially the current state of the unobservable volatility process are difficult to estimate.[3]

[1] The FTSE 100 index is a capitalization-weighted index of the 100 largest companies traded on the London Stock Exchange. The index was introduced with a base level of 1000 on January 3, 1984.

[2] In our sample, the average daily trading volumes of at-the-money FTSE 100 index put and call options are 262 and 258 for European style contracts and 305 and 247 for American style contracts.

[3] Recent econometric methods for estimating stochastic volatility models from data on the underlying security include quasi-maximum likelihood (Harvey et al., 1994; Alizadeh et al., 2002) simulated maximum likelihood (Danielsson, 1994; Sandmann and Koopman, 1998), Markov-chain monte-carlo (Jacquier et al., 1994; Kim et al., 1998), and simulated method of moments (Duffie and Singleton, 1993; Andersen and Sorensen, 1994). Alternatively, the parameters of stochastic volatility models can be inferred from the cross-section of option prices (Bates, 1996; Bakshi et al., 1997) as well as from both option prices and returns on the underlying security (Chernov and Ghysels, 2000; Jones, 2000; Pan, 2002).

Deterministic volatility function models, in contrast, are more convenient because they specify the volatility as a deterministic function of the observable stock price and time. Typically, these models take the market prices of options as given and then generate implied binomial or trinomial trees (essentially discretizations of the state-price densities) that are consistent with these option prices.[4] For example, Rubinstein (1994) constructs an implied binomial tree that on a given day is consistent with the observed cross-section of option prices with different strike prices but the same maturity date. Derman and Kani (1994a,b) and Dupire (1993) generalize this approach by building implied trees that on a given day are consistent with the observed panel of all options with different strike prices and different maturity dates.[5]

Option pricing models serve two functions. First, they help traders and institutions hedge their option portfolios, which obviously is critical for risk management. DFW therefore examine the predictive and hedging performance of deterministic volatility functions models.[6] Using data on S&P 500 index options, they find that, for hedging purposes, deterministic volatility function models perform no better than an ad-hoc procedure of smoothing Black–Scholes implied volatilities across strike prices and maturities. The second function of option pricing models, however, is to contemporaneously price less liquid and exotic derivatives, such as American style options, Asian style options, barrier options, or look-back options. As a complement to DFW's time-series study, we examine whether deterministic volatility models are at least useful for cross-sectional pricing. Specifically, for each date in our sample of FTSE 100 index option prices, we fit a modified Derman–Kani implied binomial tree to the panel of all European style options with different strike prices and maturities and then test how well this model prices a corresponding panel of American style options.

Our empirical results are striking. As expected, we find that the Black–Scholes implied volatilities of FTSE 100 index options vary significantly along both the strike price and time-to-maturity dimensions. However, while the modified Derman–Kani implied tree model prices the American style options better than a standard Cox et al. (1979, CRR) tree model with constant volatility, it performs no better than an ad-hoc procedure of smoothing Black–Scholes implied volatilities across strike prices and maturities. In particular, the ad-hoc model delivers smaller root mean squared valuation errors than the implied tree model for both put and call options, although a formal statistical test indicates that this difference in root mean squared errors is not statistically significant. The ad-hoc model also outperforms the implied tree model in terms of generating theoretical prices that lie more frequently within the observed bid/ask spreads.

[4] There is an analogous class of so-called no-arbitrage term-structure models that are calibrated to fit exactly the observed cross-section of bond yields. See, for example, Black et al. (1990), Brandt and Yaron (2002), Heath et al. (1992), Ho and Lee (1986), and Hull and White (1993).

[5] Other references on deterministic volatility function models include Aït-Sahalia and Lo (1998), Barle and Cakici (1995), Breeden and Litzenberger (1978), Burashi and Jackwerth (2001), Chriss (1997), and Jackwerth and Rubinstein (1996a,b).

[6] DFW's approach is somewhat different from that of Derman and Kani (1994a,b), Dupire (1993), and Rubinstein (1994). Rather than calibrate an implied tree to the observed option prices, they approximate the local volatility function with a constant, a second-order polynomial in the strike price, and a second-order bivariate polynomial in the strike price and time-to-maturity.

The rest of this paper is organized as follows. In Section 2, we describe the FTSE 100 index options data and examine a representative implied volatility smile and term structure pattern. In Section 3, we describe the modified Derman–Kani implied binomial tree model and an ad-hoc benchmark model. We also outline a statistical test for whether one model significantly outperforms another. Section 4 presents the empirical results and Section 5 concludes with a summary of our results.

## 2. FTSE 100 index options

### 2.1. Data

The raw data are a panel of 249,670 daily closing prices of both European and American style put and call options on the FTSE 100 index traded on the London International Financial Futures and Options Exchange (LIFFE) from October 1995 to September 1997. Both European and American style options expire on the third Friday of the expiry month and are cash-settled. On each trading day, there are five or six maturities outstanding, ranging from less than 1 month to a year. We measure the time-to-maturity of an option by the number of calendar days between the valuation and expiration dates. The strike prices for a given style of option are spaced at intervals of 50 index points from each other and the strike prices for adjacent European and American style options are spaced at intervals of 25 index points. For example, there might be European style options with strike prices of 4225 and 4275 and American style options with strike prices of 4250 and 4300. We define the moneyness of an option as $|(\text{strike price})/(\text{forward price}) - 1|$.

The reported option price is the average traded price during the last 30 seconds of each trading session. If no trade occurs during this time period, the reported price is the average of the market makers' bid and ask quotes at the market close. Since these are bona fide quotes, meaning that the market makers are committed to transact at these prices, they are updated frequently and are likely to straddle the unobserved market price. In addition, the dataset provides a set of implied forward prices for the index (one for each option maturity) calculated by put–call-parity from at-the-money put and call options.

To discount future cash flows, we collect UK interest rates. We use 1-, 3-, 6-, and 12-month Sterling London Inter-Bank Offer Rates (LIBOR) from Datastream to construct a term structure of interest rates. LIBOR rates are close to the rates at which major market players in the FTSE index option market are funded, and traders therefore conventionally use LIBOR rates in the valuation of FTSE index options. We compute discount rates for periods other than the four maturities (1, 3, 6, and 12 months) through linear interpolation.

We also collect data on the daily cash dividends paid by all stocks in the FTSE 100 index from Bloomberg. We use the aggregate dividends of the index to compute the present value of all dividends paid over the remaining life of each option as PVD $= \sum_{k=1}^{n} D_k \exp\{-r_k \tau_k\}$, where $D_k$ is the $k$th cash dividend, $\tau_k$ is the time between the ex-dividend date of the $k$th dividend and the valuation date, $r_k$ is the discount rate for the period $\tau_k$, and $n$ is the total number of dividends paid between the valuation date and expiration. Since our data includes the forward values of the index inferred from the options through put–call-parity, we then use the present value of the dividends to compute the implied spot value of the index:

$$S = F\mathrm{e}^{-rT} + \mathrm{PVD}, \tag{1}$$

where $S$ denotes the implied spot value, $F$ is the forward value, and $T$ is the term of the forward. Following Aït-Sahalia and Lo (1998), we use this implied spot value as the price of the underlying security, rather than the closing level of the index, to avoid any non-synchronicities between the option prices and the underlying security price.

We apply five filters to the options data. First, we exclude options with less than 6 days to maturity. These options have very small time-premiums and their implied volatilities are inaccurate since they are very sensitive to market micro-structure problems and measurement errors (see, for example Hentschel, 2001). Second, we eliminate options with moneyness greater than 10%. These deep in- and out-of-the-money options trade at close to their intrinsic values and contain little information about volatility. Furthermore, they are traded infrequently and their quotes are not updated as often. Third, we exclude options priced at less than 50 pence because for these options, the discreteness of the price quotes leads to disproportionally large bid/ask spreads and rounding errors.[7] Fourth, we only use out-of-the-money and at-the-money options in constructing the implied binomial trees because in-the-money options are less liquid and are, in the absence of measurement error, redundant by put–call-parity. Finally, we eliminate options that violate the no-arbitrage bounds.[8]

Tables 1 and 2 describe the cleaned data. Table 1 shows the average price, the average bid/ask spread, the average relative bid/ask spread, and the number of observations for short-, medium-, and long-dated out-of-, at-, and in-the-money European style options.[9] Table 2 shows the same statistics for American style options. There are approximately the same numbers of observations for both sets of options (59,603 and 58,005). Comparing the bid/ask spreads, either in absolute terms or as a fraction of the mid-point price, the American style options appear only slightly more liquid than the European style options.

### 2.2. Representative implied volatility pattern

To further illustrate the extent to which the observed FTSE 100 index option prices are at odds with the Black–Scholes model, we examine the Black–Scholes implied volatilities on October 5, 1995. The implied volatility pattern on this particular day is

[7] Alternatively, we could exclude options based on a measure of moneyness that is standardized by the volatility of the underlying over the life of the option. Indeed, in our data, the 50 pence filter is roughly equivalent to excluding all options that are more than 2.4 standard deviations from at-the-money.

[8] For European style calls $C \geq \max\ [0, S - \mathrm{PVD} - \mathrm{e}^{-rT}K]$ and puts $P \geq \max[0, \mathrm{e}^{-rT}K - S + \mathrm{PVD}]$. For American style calls $c \geq \max[C, S - K]$ and puts $p \geq \max[P, K - S]$.

[9] The strike prices of the European options and those of the American options are not identical but spaced at intervals of 25 index points. In addition, the longest maturity for the European options is 1 year while that for the American options is 9 months. These two factors, in addition to the exercise style, explain the difference of average option prices between the two samples.

Table 1
Summary statistics for the European style options

| Moneyness/Maturity | Short | Medium | Long | Subtotal |
|---|---|---|---|---|
| *Call options* | | | | |
| OTM | 8.31 (3.35) | 22.05 (4.52) | 76.85 (9.04) | |
| | 63.44% | 42.12% | 22.31% | |
| | 4301 | 3755 | 10,682 | 18,738 |
| ATM | 57.68 (4.99) | 93.54 (5.50) | 180.28 (8.93) | |
| | 10.99% | 6.70% | 5.51% | |
| | 1169 | 935 | 2256 | 4360 |
| ITM | 531.95 (5.82) | 516.66 (6.24) | 596.28 (13.03) | |
| | 2.54% | 2.22% | 3.37% | |
| | 10,062 | 7490 | 18,953 | 36,505 |
| Subtotal | 15,532 | 12,180 | 31,891 | 59,603 |
| *Put options* | | | | |
| OTM | 5.60 (3.06) | 14.40 (4.18) | 41.27 (8.04) | |
| | 76.58% | 64.05% | 46.71% | |
| | 10,062 | 7490 | 18,953 | 36,505 |
| ATM | 52.61 (4.99) | 80.75 (5.57) | 134.47 (8.75) | |
| | 12.14% | 7.99% | 7.21% | |
| | 1169 | 935 | 2256 | 4360 |
| ITM | 250.54 (5.65) | 270.42 (6.22) | 316.43 (13.58) | |
| | 3.96% | 3.61% | 5.64% | |
| | 4301 | 3755 | 10,682 | 18,738 |
| Subtotal | 15,532 | 12,180 | 31,891 | 59,603 |

This table shows the summary statistics for daily data on European style FTSE 100 index options from October 2, 1995 to September 30, 1997. For each moneyness and maturity category, we report (i) the average quoted bid/ask mid-point price in pounds, (ii) the average bid/ask spread in pounds (ask price minus bid price) in parentheses, (iii) the average percentage bid/ask spread (spread divided by the bid/ask mid-point) and (iv) the total number of observations. OTM, ATM, and ITM stand for out-of-the-money, at-the-money, and in-the-money, respectively. Short-, medium-, and long-terms refer to options with less than 40 days, with between 40 and 70 days, and with more than 70 days to expiration, respectively.

representative of that on any day in the sample. The closing index level on this day was 3544.

Fig. 1 plots the implied volatilities of all European style at- and out-of-the-money options as a function of the strike price and time-to-maturity. The implied volatility surface is smoothed using a third-order Legendre polynomial (for reasons discussed in Section 3).[10] Specifically, we regress the implied volatilities of options with different strike prices and times-to-maturity on first-, second-, and third-order Legendre polynomial terms in the strike price and time-to-maturity and then plot the fitted polynomial. The mean absolute error of this regression is only 0.2353%, suggesting that the fit is almost perfect.

The implied volatility smile and term structure patterns emerge clearly. Instead of a flat surface, which is consistent with the Black–Scholes formula, the implied volatilities

[10] Legendre polynomial belong to the class of orthogonal polynomials (other members of the class include Chebyshev, Hermite, and Laguerre polynomials), which have desirable asymptotic properties for approximating bounded and measurable functions (see Judd, 1998, or Powell, 1981).

Table 2
Summary statistics for the American style options

| Moneyness/Maturity | Short | Medium | Long | Subtotal |
|---|---|---|---|---|
| *Call options* | | | | |
| OTM | 8.47 (3.22) | 23.07 (4.40) | 61.84 (7.27) | |
| | 37.97% | 35.98% | 24.74% | |
| | 4082 | 3559 | 10,077 | 17,718 |
| ATM | 57.33 (4.59) | 93.93 (5.28) | 153.20 (7.61) | |
| | 10.19% | 6.32% | 5.92% | |
| | 1170 | 936 | 2334 | 4440 |
| ITM | 518.20 (5.73) | 501.27 (6.05) | 556.68 (10.96) | |
| | 2.67% | 2.39% | 3.32% | |
| | 9976 | 7493 | 18,378 | 35,847 |
| Subtotal | 15,228 | 11,988 | 30,789 | 58,005 |
| *Put options* | | | | |
| OTM | 5.75 (2.94) | 14.46 (4.10) | 35.07 (6.84) | |
| | 51.22% | 57.70% | 39.74% | |
| | 9976 | 7493 | 18,378 | 35,847 |
| ATM | 55.15 (4.67) | 84.26 (5.39) | 125.79 (7.55) | |
| | 9.87% | 6.98% | 6.73% | |
| | 1170 | 936 | 2334 | 4440 |
| ITM | 258.27 (5.85) | 281.28 (6.17) | 320.60 (10.61) | |
| | 4.04% | 3.66% | 4.58% | |
| | 4082 | 3559 | 10,077 | |
| Subtotal | 15,528 | 11,988 | 30,789 | 58,005 |

This table shows the summary statistics for daily data on American style FTSE 100 index options from October 2, 1995 to September 30, 1997. For each moneyness and maturity category, we report (i) the average quoted bid/ask mid-point price in pounds, (ii) the average bid/ask spread in pounds (ask price minus bid price) in parenthesis, (iii) the average percentage bid/ask spread (spread divided by the bid/ask mid-point) and (iv) the total number of observations. OTM, ATM, and ITM stand for out-of-the-money, at-the-money, and in-the-money, respectively. Short-, medium-, and long-terms refer to options with less than 40 days, with between 40 and 70 days, and with more than 70 days to expiration, respectively.

exhibit significant variations across both strike prices (the smile) and times-to-maturity (the term structure). For a given maturity, especially the shorter ones, the implied volatilities increase as the strike price decreases relative to the current level of the index (for out-of-the-money puts) and also increases slightly as the strike price increases (for out-of-the-money calls). This smile pattern becomes less severe as the time-to-maturity of the option increases. For most strike prices, the implied volatilities increase for longer-term options. For extremely out-of-the money puts, however, the volatility term structure is inverted, meaning that the implied volatilities are higher for short maturities than for long maturities.

Another way to visually judge the validity of the Black–Scholes model is to compare the shape of the risk-neutral distribution implied by the FTSE 100 index options to that of a log-normal distribution (the risk-neutral distribution corresponding to the Black–Scholes model). Fig. 2 plots as histograms the implied risk-neutral distribution of the index 14, 43, 71, 162, 260, and 351 days from October 5, 1995 implied by the observed option prices

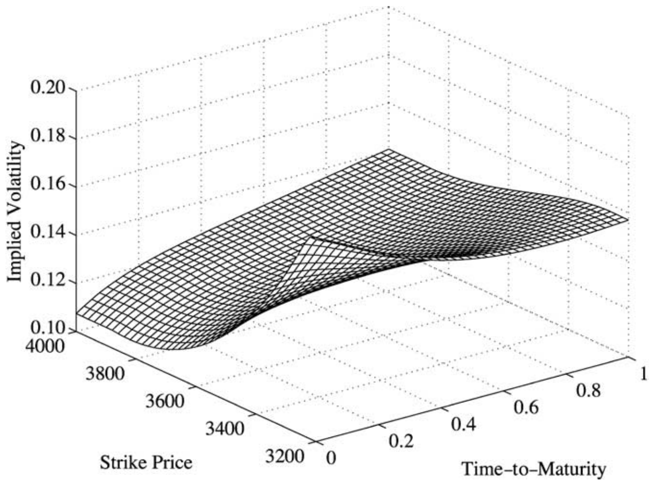


Fig. 1. Implied Black–Scholes volatility surface. This figure shows the Black–Scholes implied volatility surface for European style FTSE 100 index options on October 5, 1995. The implied volatilities are calculated by inverting the Black–Scholes formula and are smoothed using a third-degree polynomial in strike price and time-to-maturity.

and a 50-node Derman–Kani tree (we provide a detailed description of this model below). For comparison, the figure also plots for each maturity a log-normal distribution with the same mean and variance as the implied risk-neutral distribution. For all six maturities, the implied risk-neutral density exhibits a fat left tail relative to the log-normal density, which indicates that the market believes a large decrease in the index level is more likely than under a geometric Brownian motion model. This pattern in the implied risk-neutral densities is consistent with the findings of Jackwerth and Rubinstein (1996a) for the S&P 500 Index.[11]

## 3. Research methodology

### 3.1. Derman–Kani implied binomial tree

Motivated by the recent empirical evidence against the Black–Scholes model, Derman and Kani (1994a,b; DK) extend the standard Cox et al. (1979, CRR) binomial tree framework for constant volatility, which can be viewed as a discretization of the Black–Scholes model, to be consistent with almost any implied volatility pattern. In particular, DK show how to construct a binomial tree for the underlying stock price under the assumption that the volatility is a deterministic function of the stock price and

[11] Under risk-neutrality, the fat left tail of the risk-neutral distribution translates into returns with negative swekness and excess kurtosis, which is quantitatively consistent with empirical observations for the FTSE 100 index returns. See, for example, Gemmill and kamiyama (1997).

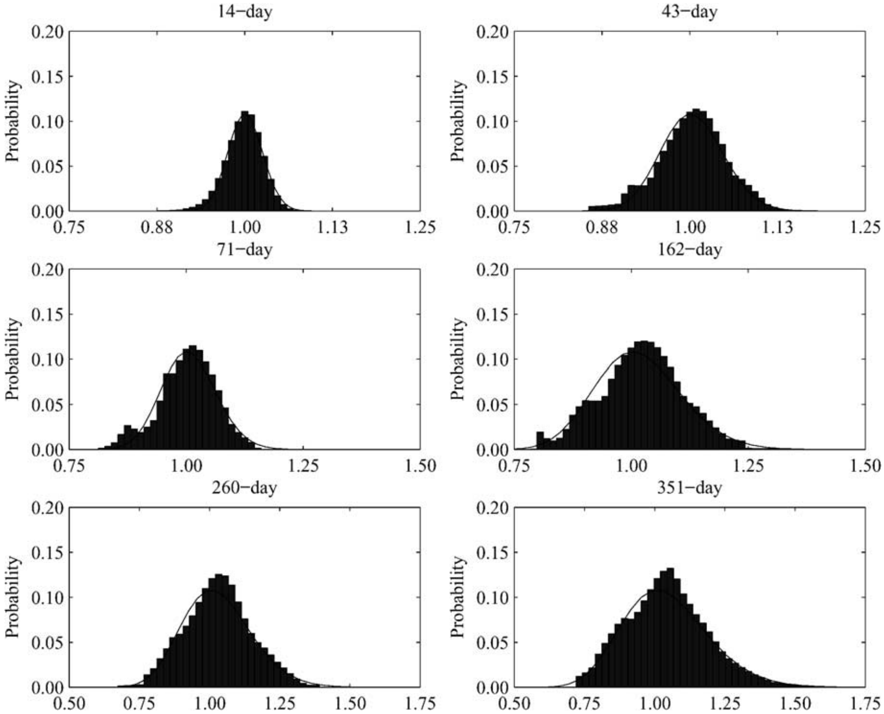


Fig. 2. Implied risk-neutral probability distributions. This figure shows the implied risk-neutral probability distributions of the FTSE 100 Index at various horizons on October 5, 1995. The probabilities are inferred from European style options with 14, 43, 71, 162, 260, and 351 days to maturity.

time such that the tree prices correctly all securities used to construct the tree. Since the DK tree eliminates any apparent arbitrage opportunities in liquid securities, it can then be used to price other less liquid and exotic derivatives, such as American style options, Asian style options, barrier options, or look-back options. The basic structure of the DK tree is the same as that of the CRR tree, except that the node positions and the up- and down-move probabilities are determined such that the generated risk-neutral distribution of the stock price is identical to that implied by the option prices. Instead of the log-normal risk-neutral distribution generated by the standard CRR tree (in the limit), the DK tree is flexible enough to accommodate almost any implied risk-neutral distribution or state-price density.

The DK model is related to the implied binomial tree approach used by Rubinstein (1994) to infer the risk-neutral distribution of the stock price at the maturity of a set of options with different strike prices (but with the same maturity date). The major advantage of DK's model is that it uses options with both different strike prices and different maturities as inputs. Thus, the DK tree not only captures the risk-neutral probabilities of the stock price at a single future date, but it also describes consistently the dynamics of the stock price in the interior of the tree. This feature of the model is particularly important for pricing American style options whose payoffs depend not only on the terminal stock price but also on the stock prices at the interior nodes of the tree because of the possibility of early exercise.

The first step in building a DK tree is to compute the Black–Scholes implied volatilities of all input options by inverting the Black–Scholes formula. Recall that we use only out-of-the-money and at-the-money options, since in-the-money options are notoriously illiquid and their quotes are therefore likely to be inaccurate. Specifically, we use out-of-the-money puts to compute the implied volatilities for strike prices below the current stock price and out-of-the-money calls for strike prices above the current stock price. The at-the-money implied volatility is computed as the average of the implied volatilities of the at-the-money puts and calls. Since the options only have a discrete set of strike prices and times-to-maturities, we then construct a continuous implied volatility surface by smoothing the implied volatilities across both dimensions using orthogonal Legendre polynomials (see also footnote 10).[12]

We extend the original DK algorithm to allow for a term structure of interest rates and for discrete dividends.[13] The second extension is particularly important because Harvey and Whaley (1992) find that the time-variations and seasonalities in aggregate daily cash dividends paid by the stocks in the S&P 100 index lead to serious mispricing of S&P 100 index options (which are American style) by a model that assumes a constant dividend yield. Since the FTSE 100 index has the same number of stocks as the S&P 100 index and the daily cash dividends paid by the stocks in the FTSE 100 index also vary significantly through time, we suspect that similar mispricing would arise in our case.

Starting from the current stock price $S_0$, we build the DK tree forward recursively at equal time steps $\Delta t = T/N$, where $T$ denotes the maturity of the option and $N$ is the total number of time steps in the tree.[14] Assuming we have already built the tree to the $(n-1)$th step, we show how to compute the stock prices at the nodes of the $n$th step. Let $S_{n-1,k}$ denote the stock price at the $k$th node from the bottom at the $(n-1)$th step. In the risk-neutral world, the stock price either moves down from $S_{n-1,k}$ to $S_d$ with probabilities $d$ or moves up to $S_u$ with probability $(1-d)$.[15]

Suppose that:

- the stock prices $\{S_{n-1,k}\}_{k=1}^{n-1}$ at the $(n-1)$th time step are known,
- the Arrow–Debreu prices $\{\lambda_{n-1,k}\}_{k=1}^{n-1}$ at the $(n-1)$th time step are known, and
- $S_u$ is known.

[12] Chriss (1997) suggests using bilinear interpolation instead of polynomial smoothing. However, in our empirical implementation, we find that bilinear interpolation often generates kinks in the implied volatility surface that lead to a higher number of nodes with bad transition probabilities (see our discussion below).

[13] When the number of steps in the tree is non-divisible by the number of days to maturity, there may be dividends paid in-between nodes. In that case, the future values of the dividends are paid at the nodes immediately following the ex-dividend dates.

[14] In our implementation, we use $N = 50$. Our results are not much different with higher values of $N$.

[15] Note that $S_d \equiv S_{n,k}$ and $S_u \equiv S_{n,k+1}$.

We can then solve for the stock price $S_d$ as follows. Let $F$ denote the price at node $\{n-1,k\}$ of a forward contract on the stock for delivery at time step $n$. Risk-neutral pricing implies:[16]

$$F = (1-d)S_u + dS_d. \tag{2}$$

In addition, the standard cash-and-carry relationship between the spot and forward is:

$$F = S_{n-1,k}\exp(r_n \Delta t) - D_n, \tag{3}$$

where $r_n$ denotes the interest rate between times $t_{n-1}$ and $t_n$ and $D_n$ are the dividends with ex-dividend date $t_n$. Let $p_k(K,t_n)$ denote the price at node $\{n-1,k\}$ of a European style put option with strike price $K$ and expiration at time step $n$. If we set the strike price equal to the one-step ahead forward price $F$, the price of the one-step ahead put option is:

$$p_k(F,t_n) = \exp\{-r_n \Delta t\} d(F - S_d). \tag{4}$$

Solving Eq. (2) for $d$ and substituting the solution into Eq. (4) yields:

$$S_d = \frac{p_k(F,t_n)S_u + \exp\{-r_n \Delta t\}(F - S_u)F}{p_k(F,t_n) + \exp\{-r_n \Delta t\}(F - S_u)}. \tag{5}$$

Analogously, if $S_d$ is known, $S_u$ can be expressed as:

$$S_u = \frac{c_k(F,t_n)S_d + \exp\{-r_n \Delta t\}(S_d - F)F}{c_k(F,t_n) + \exp\{-r_n \Delta t\}(S_d - F)}, \tag{6}$$

where

$$c_k(F,t_n) = \exp\{-r_n \Delta t\}(1-d)(S_u - F), \tag{7}$$

denotes the value of a European style call option with strike price $K$ equal to the one-step ahead forward price $F$ and expiration date $t_n$ at node $\{n-1,k\}$.

If neither $S_u$ nor $S_d$ is known, in which case, we are determining the central node(s) for a new level of the tree, we set the central node to be the one-step ahead forward price $F$ if $n$ is odd. If $n$ is even, the two central nodes $S_u$ and $S_d$ must satisfy:

$$S_u S_d = F^2. \tag{8}$$

Solving Eq. (8) together with Eq. (6), the two central nodes are then given by:

$$S_u = Fu \text{ and } S_d = \frac{F}{u}, \tag{9}$$

[16] The fact that $0<d<1$ implies that $S_d<F<S_u$.

where

$$u = \frac{F + c_k(F, t_n)\exp\{r_n \Delta t\}}{F - c_k(F, t_n)\exp\{r_n \Delta t\}}. \tag{10}$$

To summarize, if we know the prices of the one-step ahead options $p_k(F,t_n)$ and $c_k(F,t_n)$ with strike prices equal to the forward prices at each node at the $(n-1)$th step, we can compute the stock prices of the $n$th step. In practice, however, we do not observe these one-step ahead options and instead need to infer their prices from the observed option prices. Let $P(K,t_n)$ be the *observed* time-0 price of a European put option with strike price $K$ and maturity $t_n$. The no-arbitrage price of this option can be expressed as the sum of the values of the option at all nodes of the $(n-1)$th step of the tree multiplied by the Arrow–Debreu prices corresponding to those nodes. Formally:

$$P(K, t_n) = \sum_{j=1}^{k} p_j(K, t_n)\lambda_{n-1,j}, \tag{11}$$

where for all $j>k$, the option $p_j(K,t_n)$ will expire out-of-the-money and for all $j<k$, we have from the payoff of a put option:[17]

$$p_j(K, t_n) = \exp\{-r_n \Delta t\}[d(K - S_{n,j}) + (1-d)(K - S_{n,j+1})]. \tag{12}$$

From Eqs. (2) and (3), we have:

$$dS_{n,j} + (1-d)S_{n,j+1} = S_{n-1,j}\exp\{r_n \Delta t\} - D_n. \tag{13}$$

Combining Eqs. (12) and (13), we get:

$$p_j(K, t_n) = \exp\{-r_n \Delta t\}K - S_{n-1,j} + \exp\{-r_n \Delta t\}D_n, \tag{14}$$

where every term on the right hand side is already computed. Therefore, we can rewrite the observed put option price in Eq. (11) as:

$$P(K, t_n) = \Sigma_p + p_k(K, t_n)\lambda_{n-1,k}, \tag{15}$$

with

$$\Sigma_p = \sum_{j=1}^{k-1}[\exp\{-r_n \Delta t\}K - S_{n-1,j} + \exp\{-r_n \Delta t\}D_n]\lambda_{n-1,j}, \tag{16}$$

[17] Note that $S_{n,k} < K = F < S_{n,k+1}$.

and solve for the one-step ahead put option price:

$$p_k(K,t_n) = \frac{P(K,t_n) - \Sigma_p}{\lambda_{n-1,k}}. \tag{17}$$

Similarly, for the one-step ahead call option, we have:

$$c_k(K,t_n) = \frac{C(K,t_n) - \Sigma_c}{\lambda_{n-1,k}}, \tag{18}$$

with

$$\Sigma_c = \sum_{j=k+1}^{n-1} [S_{n-1,j} - \exp\{-r_n \Delta t\}K - \exp\{-r_n \Delta t\}D_n]\lambda_{n-1,j}. \tag{19}$$

To finish the algorithm, we need the evolution of Arrow–Debreu prices:

$$\lambda_{n,j} = \begin{cases} \exp\{-r_n \Delta t\} d\lambda_{n-1,j} & \text{for } j = 1 \\ \exp\{-r_n \Delta t\}[(1-d)\lambda_{n-1,j-1} + d\lambda_{n-1,j}] & \text{for } 1 < j < n\,. \\ \exp\{-r_n \Delta t\}(1-d)\lambda_{n-1,j-1} & \text{for } j = n \end{cases} \tag{20}$$

Since we do not observe a continuum of option prices, $P(F,t_n)$ and $C(F,t_n)$ are really *interpolated* option prices. Recall that in the very first step of the algorithm, we constructed a two-dimension volatility surface $\sigma(K,t)$. To compute $P(F,t_n)$ and $C(F,t_n)$, we convert the smoothed implied volatility $\sigma(F,t_n)$ into option prices using the Black–Scholes formula.

The algorithm we just described is actually a modified version of the original DK approach, which centers the tree and evaluates the one-step ahead options $p_k(K,t_n)$ and $c_k(K,t_n)$ at the spot price instead of the forward price. The original algorithm suffers from a "bad probabilities" problem, where for high interest rates and severe volatility smiles the next nodes generated by Eqs. (5), (6) and (9) may violate the no-arbitrage conditions, resulting in up- and down-probabilities that lie outside the [0,1] interval.[18] Whenever this happens, the new nodes are overwritten in such a way that the logarithmic spacing of the tree is preserved. Centering the tree and evaluating the one-step ahead options at the forward price, which was originally suggested by Barle and Cakici (1995), reduces significantly the number of bad nodes. The few bad nodes that do occur are located along the bottom and top edges of the tree, where the index is either at extremely high or extremely low levels, leaving the economically interesting area of the tree unaffected.

Once the implied binomial tree is constructed, we first check that it prices correctly, or within marginal errors (and well within the bid/ask spread), the set of input options.[19] Then,

[18] In most cases, the bad probabilities probably reflect arbitrage opportunities embedded in the data. Whether these arbitrage opportunities can be exploited depends on transaction costs and other factors.

[19] Through this validation step, we determine that for smoothing the implied volatilities, in the very first step of the algorithm, the third-order Legendre polynomials perform better in terms of minimizing the "in-sample" pricing or calibration errors than other smoothing or interpolation methods. In particular, for all days in the sample, the mean absolute error of the third-order polynomial regression is less than 0.4%.

we can use the tree to price other derivatives just as with a standard CRR binomial tree. For example, to price American style options, we move recursively backward through the tree and set the value of the option equal to the greater of either its intrinsic value or the risk-neutral probability weighted values of the option at the two successive nodes discounted at the one-step ahead interest rate.[20]

### 3.2. Ad-hoc benchmark model

Since the 1987 stock market crash and the subsequent appearance of the implied volatility smile and term structure, many traders have adopted an ad-hoc procedure for pricing options (see Hull, 1999, p. 441). In particular, to price an American style or other exotic option on an index or stock with liquidly traded European style options, a trader first obtains a volatility rate for the option by interpolating the implied volatility surface constructed from the Black–Scholes implied volatilities of the European style options and then builds a standard CRR binomial tree with the interpolated volatility to price the American or exotic options. This procedure is ad-hoc because it is internally inconsistent. It uses the Black–Scholes model and a standard CRR binomial tree, which assume constant volatility, but takes different volatilities as inputs, depending on the strike price and maturity of the options being priced. Nonetheless, this ad-hoc procedure is simple and incorporates the observed implied volatility smile and term structure.

### 3.3. Pricing error measures

We adopt the following eight pricing error measures:

(i) The mean valuation error (MVE) is the average difference between the market and model prices and reveals systematic biases of the model. The MVE is positive if the model overprices and negative if it underprices a set of options on average.

(ii) The root mean squared valuation error (RMSVE) is the square-root of the average squared difference between the market and model prices. The RMSVE measures how well the model fits in a statistical sense (with the usual bias vs. variance trade-off).

(iii) The frequency in bid/ask (FIBA) is the frequency the model price falls within the observed bid/ask spread. The FIBA measures how often the model suggests arbitrage opportunities in the observed option prices. Since the observed prices supposedly correspond to the true prices, these apparent arbitrage opportunities represent certain losses if the model is taken seriously.

(iv) The mean outside error (MOE) is the average error outside of the bid/ask spread, which is defined as follows. If the model price is below the bid or above the ask quote, the error is defined as the model price minus the bid or ask quote. If the model price is within the bid/ask spread, the error is set to zero. The MOE reveals whether the bid/ask violations measured by FIBA are symmetric, meaning the model overprices as much as

[20] The intrinsic value is $\max[S-K,0]$ for a call option and $\max[K-S,0]$ for a put option.

it underprices the options, or whether there are systematic biases in the mispricing relative to the bid/ask spread.

(v) The root mean squared outside error (RMSOE) to measure the variability of the errors outside the bid/ask spread.

(vi) The mean relative outside error (MROE) is the average outside error divided by the market price and measures the same form of mispricing as MOE, but in relative terms.

Since a number of bid and/or ask quotes are missing in our data, we also estimate a set of "effective" bid and ask quotes as a function of moneyness and time-to-maturity, using again third-order polynomial regressions. We then define an effective bid/ask spread as the quoted price $\pm 1/2$ of the estimated bid/ask spread. We use this effective bid/ask spread to construct a set of pricing error measures (iii′), (iv′), (v′), and (vi′), which are identical to (iii), (iv), (v), and (vi) above, except that we use the effective instead of the observed bid/ask spread. The three measures are denoted EFIBA, EMOE, ERMSOE and EMROE, respectively. Notice that the effective bid/ask spread is not only useful for filling in missing bid/ask quotes but also for cases where the traded price is far from the mid-point of the quoted bid/ask spread. In those cases, the model price can be very close to the traded price but is still outside the observed bid/ask band.

Table 3
Average valuation errors

| Model | RMSVE | ΔRMSVE | *t*-Ratio | MVE | RMSOE | MOE | FIBA (%) |
|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | |
| CRR | 14.10 | −8.08 | −31.81 | −4.04 | 6.79 | −1.33 | 30.7 |
| DK1 | 6.02 | −1.22 | −1.10 | 0.20 | 5.89 | 0.47 | 57.2 |
| DK2 | 4.80 | 0.13 | −0.16 | 0.03 | 5.47 | 0.19 | 66.8 |
| DK3 | 4.93 | 0.09 | 1.10 | −0.05 | 5.23 | 0.09 | 70.4 |
| DK4 | 5.02 | −0.51 | 0.47 | −0.15 | 5.63 | 0.05 | 70.1 |
| AH | 4.51 | −0.42 | 0.51 | 0.18 | 4.64 | 0.17 | 70.7 |
| *Put options* | | | | | | | |
| CRR | 13.60 | −7.66 | −20.23 | −5.10 | 6.49 | −4.74 | 19.1 |
| DK1 | 5.94 | −0.75 | −0.72 | −0.91 | 5.73 | −0.61 | 53.3 |
| DK2 | 5.19 | 0.02 | 0.30 | −1.13 | 5.66 | −0.58 | 64.6 |
| DK3 | 5.21 | 0.09 | 0.56 | −1.17 | 5.57 | −0.61 | 66.4 |
| DK4 | 5.30 | −0.47 | −3.61 | −1.29 | 5.78 | −0.67 | 65.5 |
| AH | 4.83 | −0.36 | −1.01 | −0.39 | 5.16 | −0.41 | 68.0 |

This table describes the pricing errors for American style FTSE 100 index options of the Cox–Ross–Rubinstein model constructed with at-the-money implied volatilities (CRR), the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a first- through fourth-order polynomial in the strike price and time-to-maturity (DK1–DK4), and the ad-hoc benchmark model (AH). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the root mean squared outside error (RMSOE), (iv) the mean outside error (MOE), and (v) the frequency in bid/ask spread (FIBA). ΔRMSVE is the change in RMSVE from the current model to the next model. The *t*-ratio measures the statistical significance of this change. The last ΔRMSVE and *t*-ratio are for the change in RMSVE from the best model among the first five to the ad-hoc model.

### *3.4. Statistical test*

Like DFW, we apply West's (1996) $t$-test of out-of-sample predictive equivalence to the set of competing option pricing models. Let $\theta_t$ be an $(N \times 1)$ vector of root mean squared pricing errors at time $t$ corresponding to $N$ alternative models. Let $E[\theta]$ denote the population value of $\theta_t$ and let $\bar{\theta}$ be the sample average. If the population values of all parameters are known, then $(\bar{\theta} - E[\theta])$ is asymptotically normal with covariance matrix:

$$\Sigma = \sum_{k=-\infty}^{\infty} E[(\theta_t - E[\theta])(\theta_{t-k} - E[\theta])']. \tag{21}$$

Using GMM reasoning, we thus find a vector $\hat{\theta}$ that minimizes:

$$\frac{1}{T}\sum_t [(\theta_t - \hat{\theta})' S^{-1}(\theta_t - \hat{\theta})], \tag{22}$$

where $S$ is the Newey–West estimator of the covariance matrix $\Sigma$. We use the resulting estimate of $E[\theta]$ and its asymptotic covariance matrix to draw statistical

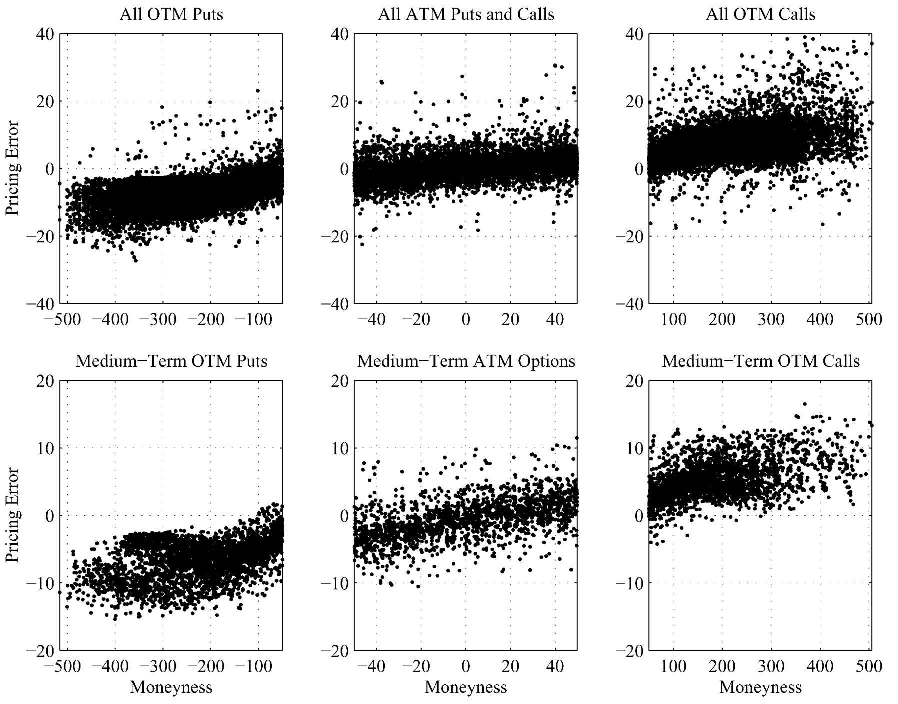


Fig. 3. Pricing errors of CRR model. This figure shows the pricing errors for American style FTSE 100 index options of the Cox–Ross–Rubinstein binomial tree model constructed with at-the-money implied volatilities. The first row shows the pricing errors for all maturities and the second row shows the pricing errors for options with 40–70 days to maturity.

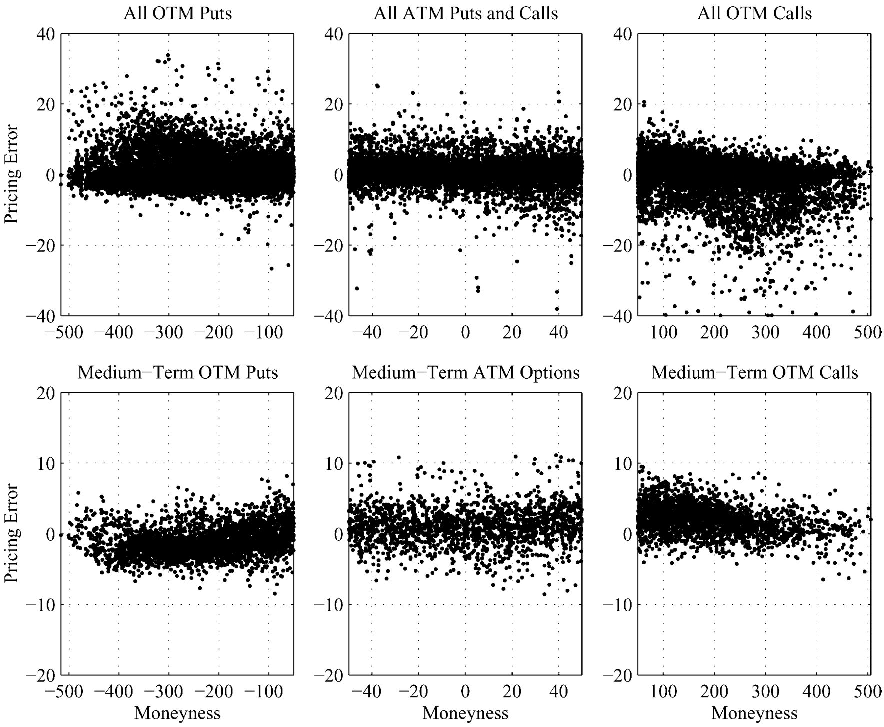


Fig. 4. Pricing errors of first-order DK model. This figure shows the pricing errors for American style FTSE 100 index options of the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a first-degree polynomial in the strike price and time-to-maturity. The first row shows the pricing errors for all maturities and the second row shows the pricing errors for options with 40–70 days to maturity.

inferences about the differences in root mean squared pricing errors across the competing models.

## 4. Empirical results

We compare six different models. First, we consider the standard CRR tree model, the discrete-time equivalent of the Black–Scholes model with constant volatility, to illustrate the importance of incorporating the implied volatility smile and term structure.[21] Then, we examine four versions of the DK tree model (DK1–DK4) that use first-, second-, third-, and fourth-order Legendre polynomials for smoothing the Black–Scholes implied volatilities, respectively. Finally, we consider the ad-hoc benchmark model AH. For each date in the sample, we calibrate the above six models to the set of European style options and then use the model to price all observed American style options.

[21] We use a 30-day at-the-money implied volatility as the constant volatility input for the CRR tree.

### 4.1. Aggregate results and test statistics

Table 3 reports the root mean squared valuation errors RMSVE and the incremental root mean squared valuation errors ΔRMSVE of each model compared to the next model on the list (CRR vs. DK1, DK1 vs. DK2, etc.) as well as the corresponding $t$-ratios. The last ΔRMSVE and $t$-ratio correspond to the increment from the best-fitting DK model to the ad-hoc benchmark model. The results in Table 3 show that the DK1 model yields a substantial improvement over the standard CRR model. The root mean squared error is decreased by £8.08 and £7.66 for the call and put options, respectively. Among the four DK models, DK2 performs best at the aggregate level, with the smallest root mean squared valuation errors (£4.80 for calls and £5.19 for puts). However, the ad-hoc benchmark model has even smaller root mean squared errors of 4.51 for calls and 4.83 for puts, representing an improvement of 42 and 36 pennies, or 6.4 and 6.9%, over DK2, respectively. The $t$-ratios reveal that only the improvements from CRR to DK1 and from DK4 to AH (for puts only) are statistically significant. The other improvements are not.

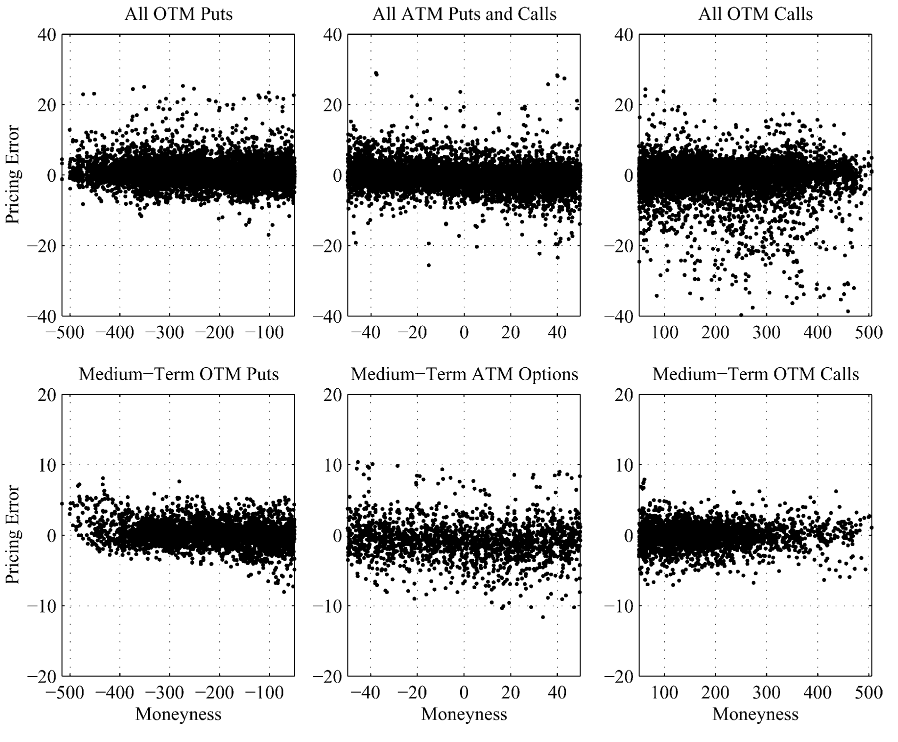


Fig. 5. Pricing errors of third-order DK model. This figure shows the pricing errors for American style FTSE 100 index options of the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a third-degree polynomial in the strike price and time-to-maturity. The first row shows the pricing errors for all maturities and the second row shows the pricing errors for options with 40–70 days to maturity.

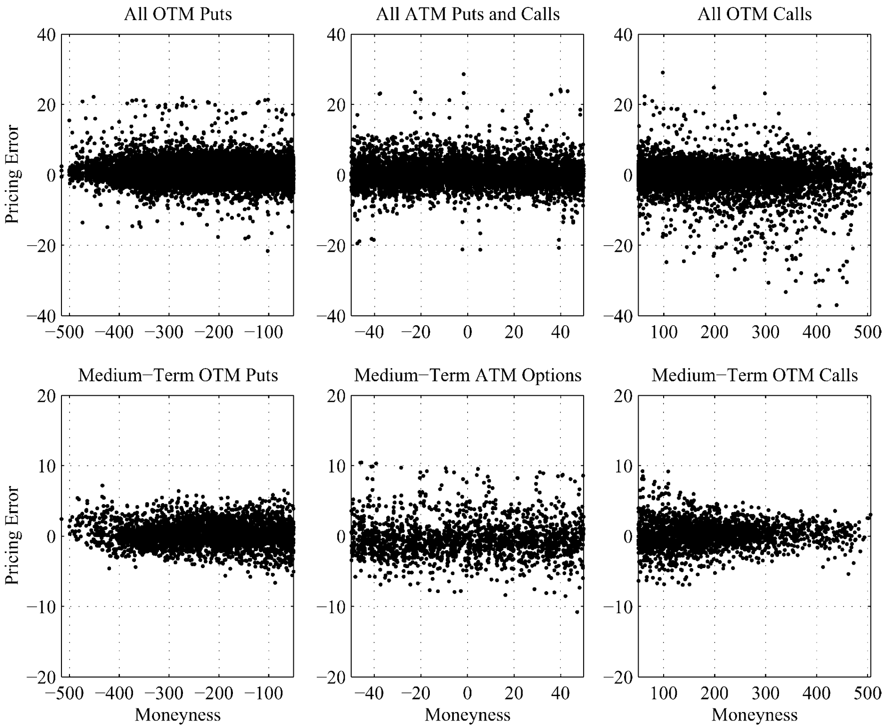


Fig. 6. Pricing errors of ad-hoc model. This figure shows the pricing errors for American style FTSE 100 index options of the ad-hoc benchmark model. The first row shows the pricing errors for all maturities and the second row shows the pricing errors for options with 40–70 days to maturity.

### *4.2. Pricing errors of individual models*

Figs. 3–6 plot the pricing errors for the CRR, DK1, DK3, and AH models.[22] The first row of each figure plots the pricing errors for all maturities while the second row plots the pricing errors only for options with maturities between 40 and 70 days. The three plots in each row correspond to out-of-the-money puts, at-the-money puts and calls, and out-of-the-money calls. Comparing the first and third columns of Fig. 3, we notice that the CRR model systematically over-prices out-of-the-money calls and under-prices out-of-the-money puts. These mispricing patterns are to be expected since the CRR model ignores the implied volatility smile. The other models that take into account the smile do not exhibit such biases.

Tables 4–9 break out the pricing errors measures for the six models into three moneyness categories (in-, at-, and out-of-the-money). The pricing errors for the CRR

[22] The plots for DK2 and DK4 are not shown here to save space. They are similar to those for DK1 and DK3. For details, see subsequent discussions on tables.

Table 4
Average valuation errors for CRR model

| Moneyness | RMSVE | MVE | MOE | MROE (%) | FIBA (%) | EMOE | EMROE (%) | EFIBA (%) |
|---|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | | |
| All | 14.10 | − 4.03 | − 1.33 | 5.9 | 30.7 | − 2.64 | 3.2 | 23.5 |
| ITM | 14.86 | − 9.51 | − 4.85 | − 2.1 | 17.4 | − 6.37 | − 2.2 | 17.5 |
| OTM | 13.71 | 2.69 | 1.01 | 13.8 | 33.3 | 1.93 | 11.6 | 25.4 |
| ATM | 12.55 | − 3.90 | − 1.66 | − 0.1 | 42.4 | − 2.56 | − 1.3 | 37.5 |
| *Put options* | | | | | | | | |
| All | 13.60 | − 5.10 | − 4.74 | − 18.7 | 19.1 | − 3.81 | − 12.8 | 24.0 |
| ITM | 12.68 | 0.46 | − 2.33 | − 1.0 | 37.8 | 0.29 | − 0.0 | 38.9 |
| OTM | 14.70 | − 9.91 | − 6.30 | − 30.1 | 5.0 | − 7.47 | − 26.4 | 6.2 |
| ATM | 12.60 | − 4.16 | − 2.46 | − 2.5 | 42.6 | − 2.76 | − 2.5 | 42.3 |

This table describes the pricing errors for American style FTSE 100 index options of the Cox–Ross–Rubinstein model constructed with at-the-money implied volatilities (CRR). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the mean outside error (MOE), (iv) the mean relative outside error (MROE), and (v) the frequency in bid/ask spread (FIBA). EMOE, EMROE, and EFIBA are identical to (iii), (iv), and (v), except that we use the effective instead of the observed bid/ask spread.

model in Table 4 are significantly larger than those for the other models (in Tables 5–9). The mispricing is more severe for out-of-the money options than for at-the-money options. In particular, the CRR model overprices out-of-the-money calls by an average of £2.69 and underprices out-of-the-money puts by an average of £9.91. As a result, only 33.3% of the out-of-the-money calls and 5.0% of the out-of-the-money puts are priced inside the bid/ask spread. This disappointing performance of the CRR model

Table 5
Average valuation errors for first-order DK model

| Moneyness | RMSVE | MVE | MOE | MROE (%) | FIBA (%) | EMOE | EMROE (%) | EFIBA (%) |
|---|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | | |
| All | 6.02 | 0.20 | 0.47 | 3.5 | 57.2 | 0.01 | 0.9 | 55.4 |
| ITM | 5.96 | 0.56 | − 0.05 | 0.0 | 71.7 | 0.22 | 0.1 | 55.9 |
| OTM | 6.60 | − 0.68 | 0.64 | 6.5 | 47.9 | − 0.47 | 1.9 | 51.6 |
| ATM | 4.38 | 1.31 | 0.77 | 1.9 | 57.8 | 0.56 | 1.0 | 63.8 |
| *Put options* | | | | | | | | |
| All | 5.94 | − 0.91 | − 0.61 | − 3.1 | 53.3 | − 0.45 | − 0.8 | 57.5 |
| ITM | 7.91 | − 3.10 | − 2.09 | − 0.8 | 42.8 | − 1.61 | − 0.5 | 52.4 |
| OPM | 4.44 | 0.49 | − 0.37 | − 5.0 | 52.1 | 0.32 | − 1.4 | 57.5 |
| ATM | 3.82 | 0.33 | 0.11 | 0.4 | 66.9 | 0.10 | − 0.3 | 70.9 |

This table describes the pricing errors for American style FTSE 100 index options of the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a first-order polynomial in the strike price and time-to-maturity (DK1). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the mean outside error (MOE), (iv) the mean relative outside error (MROE), and (v) the frequency in bid/ask spread (FIBA). EMOE, EMROE, and EFIBA are identical to (iii), (iv), and (v), except that we use the effective instead of the observed bid/ask spread.

Table 6
Average valuation errors for second-order DK model

| Moneyness | RMSVE | MVE | MOE | MROE (%) | FIBA (%) | EMOE | EMROE (%) | EFIBA (%) |
|---|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | | |
| All | 4.80 | 0.03 | 0.19 | 1.3 | 66.8 | 0.10 | 0.2 | 67.1 |
| ITM | 5.29 | 0.21 | 0.06 | 0.1 | 75.1 | 0.33 | 0.2 | 63.4 |
| OTM | 4.49 | − 0.29 | 0.20 | 2.2 | 62.9 | − 0.23 | 0.1 | 70.4 |
| ATM | 3.84 | 0.27 | 0.36 | 1.0 | 64.2 | 0.22 | 0.5 | 70.5 |
| *Put options* | | | | | | | | |
| All | 5.19 | − 1.13 | − 0.58 | − 1.1 | 64.6 | − 0.57 | − 0.0 | 68.4 |
| ITM | 7.44 | − 2.99 | − 2.31 | − 1.1 | 44.0 | − 1.58 | − 0.6 | 54.5 |
| OTM | 2.91 | 0.23 | − 0.13 | − 1.5 | 69.9 | 0.14 | − 0.1 | 77.4 |
| ATM | 3.56 | − 0.66 | − 0.26 | − 0.2 | 68.9 | − 0.25 | − 0.2 | 75.6 |

This table describes the pricing errors for American style FTSE 100 index options of the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a second-order polynomial in the strike price and time-to-maturity (DK3). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the mean outside error (MOE), (iv) the mean relative outside error (MROE), and (v) the frequency in bid/ask spread (FIBA). EMOE, EMROE, and EFIBA are identical to (iii), (iv), and (v), except that we use the effective instead of the observed bid/ask spread.

illustrates the importance of incorporating the volatility smile, especially for out-of-the-money options.

Moving on to the model DK1 in Table 5, we notice a significant pricing improvement for out-of-the-money options. The root mean squared errors are reduced from £13.71 to £6.60 for out-of-the-money calls and from £14.73 to £4.44 for out-of-the-money puts. The frequency of the model price falling between the bid and ask quotes increases to

Table 7
Average valuation errors for third-order DK model

| Moneyness | RMSVE | MVE | MOE | MROE (%) | FIBA (%) | EMOE | EMROE (%) | EFIBA (%) |
|---|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | | |
| All | 4.93 | − 0.05 | 0.09 | 0.8 | 70.4 | 0.04 | 0.1 | 68.9 |
| ITM | 5.35 | 0.32 | 0.11 | 0.1 | 74.2 | 0.34 | 0.2 | 63.2 |
| OTM | 4.82 | − 0.57 | 0.04 | 1.3 | 69.5 | − 0.38 | − 0.2 | 74.3 |
| ATM | 3.78 | 0.11 | 0.19 | 0.6 | 67.2 | 0.16 | 0.3 | 73.6 |
| *Put options* | | | | | | | | |
| All | 5.21 | − 1.17 | − 0.61 | − 0.7 | 66.4 | − 0.62 | − 0.2 | 69.7 |
| ITM | 7.54 | − 3.19 | − 2.49 | − 1.2 | 41.8 | 0.34 | − 0.7 | 53.6 |
| OTM | 2.89 | 0.34 | − 0.07 | − 0.6 | 74.1 | 0.12 | − 0.2 | 80.6 |
| ATM | 3.53 | − 0.83 | − 0.36 | − 0.4 | 67.6 | − 0.30 | − 0.3 | 76.2 |

This table describes the pricing errors for American style FTSE 100 index options of the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a third-order polynomial in the strike price and time-to-maturity (DK3). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the mean outside error (MOE), (iv) the mean relative outside error (MROE), and (v) the frequency in bid/ask spread (FIBA). EMOE, EMROE, and EFIBA are identical to (iii), (iv), and (v), except that we use the effective instead of the observed bid/ask spread.

Table 8
Average valuation errors for fourth-order DK model

| Moneyness | RMSVE | MVE | MOE | MROE (%) | FIBA (%) | EMOE | EMROE (%) | EFIBA (%) |
|---|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | | |
| All | 5.02 | − 0.15 | 0.05 | 0.5 | 70.1 | − 0.00 | 0.0 | 69.1 |
| ITM | 5.52 | 0.20 | 0.11 | 0.1 | 72.9 | 0.27 | 0.2 | 63.1 |
| OTM | 4.76 | − 0.61 | − 0.03 | 0.7 | 69.4 | − 0.39 | − 0.4 | 74.2 |
| ATM | 3.87 | − 0.03 | 0.14 | 0.5 | 68.0 | 0.12 | 0.2 | 74.7 |
| *Put options* | | | | | | | | |
| All | 5.30 | − 1.29 | − 0.67 | − 0.8 | 65.5 | − 0.69 | − 0.4 | 69.6 |
| ITM | 7.42 | − 3.29 | − 2.55 | − 1.3 | 41.2 | − 1.70 | − 0.7 | 54.1 |
| OTM | 3.30 | 0.22 | − 0.14 | − 0.7 | 73.0 | 0.02 | − 0.1 | 80.1 |
| ATM | 3.64 | − 0.97 | − 0.42 | − 0.5 | 66.8 | − 0.33 | − 0.3 | 76.0 |

This table describes the pricing errors for American style FTSE 100 index options of the Derman–Kani implied binomial tree model constructed with implied volatilities that are smoothed using a fourth-order polynomial in the strike price and time-to-maturity (DK4). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the mean outside error (MOE), (iv) the mean relative outside error (MROE), and (v) the frequency in bid/ask spread (FIBA). EMOE, EMROE, and EFIBA are identical to (iii), (iv), and (v), except that we use the effective instead of the observed bid/ask spread.

47.9% for calls and to 52.1% for puts. Similar observations apply to at-the-money options.

Comparing Tables 5 and 6, there appear to be significant benefits from using second-order polynomials instead of first-order polynomials for smoothing the implied volatilities. Relative to the model DK1, the model DK2 reduces the root mean squared errors from £6.60 and £4.44 to £4.49 and £2.91 for out-of-the-money call and put options, respectively. The corresponding improvements for at-the-money options are a drop from

Table 9
Average valuation errors for ad-hoc model

| Moneyness | RMSVE | MVE | MOE | MROE (%) | FIBA (%) | EMOE | EMROE (%) | EFIBA (%) |
|---|---|---|---|---|---|---|---|---|
| *Call options* | | | | | | | | |
| All | 4.51 | 0.18 | 0.17 | 0.8 | 70.7 | 0.19 | 0.3 | 71.0 |
| ITM | 5.28 | 0.52 | 0.19 | 0.1 | 70.0 | 0.45 | 0.2 | 63.5 |
| OTM | 3.74 | − 0.22 | 0.13 | 1.4 | 71.7 | − 0.14 | 0.3 | 79.1 |
| ATM | 3.57 | 0.14 | 0.23 | 0.5 | 69.2 | 0.22 | 0.3 | 74.0 |
| *Put options* | | | | | | | | |
| All | 4.83 | − 0.39 | − 0.41 | − 0.5 | 68.0 | − 0.33 | − 0.1 | 71.4 |
| ITM | 6.82 | − 1.70 | − 1.82 | − 0.9 | 50.0 | − 1.00 | − 0.4 | 58.5 |
| OTM | 2.99 | 0.58 | − 0.04 | − 0.4 | 72.3 | 0.12 | 0.1 | 78.9 |
| ATM | 3.09 | − 0.13 | − 0.18 | − 0.3 | 72.6 | − 0.07 | − 0.1 | 80.9 |

This table describes the pricing errors for American style FTSE 100 index options of the ad-hoc benchmark model (AH). We measure pricing errors by (i) the root mean squared valuation error (RMSVE), (ii) the mean valuation error (MVE), (iii) the mean outside error (MOE), (iv) the mean relative outside error (MROE), and (v) the frequency in bid/ask spread (FIBA). EMOE, EMROE, and EFIBA are identical to (iii), (iv), and (v), except that we use the effective instead of the observed bid/ask spread.

£4.38 and £3.82 to £3.84 and £3.56. The frequency of the model price falling inside the bid/ask spread increases to 62.9% for out-of-the-money calls, 64.2% for at-the-money calls, 69.9% for out-of-the-money puts, and 68.9% for at-the-money puts.

The race between the models DK2 and DK3 is a close one, with DK3 (in Table 7) dominating slightly for at- and out-of-the-money options. Model DK3 reduces the root mean squared errors for out-of-the-money puts and for at-the-money calls and puts. It generates a slightly larger root mean squared error for out-of-the-money calls, but at the same time increases the frequency of the model price falling inside the bid/ask spread and the effective bid/ask spread by 6.6% and 3.9%, respectively. Model DK3 also dominates model DK2 slightly in the FIBA and EFIBA measures for the other moneyness categories.

Does this imply that the performance of the DK model increases monotonically in the order of the polynomial used to smooth the implied volatilities? Table 8 shows that this is not the case. The model DK4 performs worse than DK3 across all pricing error measures. Clearly, fourth-order polynomials over-fit the implied volatilities leading to binomial trees that incorporate nuances of the data that are not priced into the American style options.

We now compare the best DK model for pricing at- and out-of-the-money options, which is the model DK3, to the ad-hoc benchmark model. Comparing the results in Table 7 (for DK3) to those in Table 9 (for AH) indicates that the implied binomial tree model performs no better (but also not much worse) than the ad-hoc procedure in pricing the American style options. For out-of-the-money calls, the RMSVE is £3.74 for model AH and £4.82 for model DK3. For out-of-the-money puts, in contrast, the RMSVEs are £2.99 and £2.89, respectively. Therefore, model AH has a smaller RMSVE for calls but a slightly higher one for puts. The mean outside error (MOE) and relative mean outside error (RMOE) for both models suggest that out-of-the-money calls are slightly overpriced and out-of-the-money puts are slightly underpriced relative to the bid/ask quotes. The frequency of the AH model price falling inside the observed bid/ask spread is slightly higher for out-of-the-money calls (71.7% vs. 69.5%) and somewhat lower for out-of-the-money puts (72.3% vs. 74.1%). The same pattern holds for the effective bid/ask spread, except that the frequencies for both models are higher (79.1% vs. 74.3% for calls and 78.9% vs. 80.6% for puts).

For at-the-money options, model AH performs slightly better than model DK3 in terms of the RMSVEs (£3.57 vs. £3.78 for calls and £3.09 vs. £3.53 for puts). The MOE measure shows that both models price calls slightly above and puts slightly below the observed bid/ask spread. The overpricing of calls is more severe for model AH, while the underpricing of puts is more severe for model DK3. The frequency of the model prices falling inside the observed bid/ask spread is also higher for model AH than for model DK3 (69.2% vs. 67.2% for calls and 72.6% vs. 67.6% for puts). Looking at the effective bid/ask spread, we again see that the performance of both models improves in terms of generating model prices that fall between the estimated bid and ask quotes, but the relative ranking remains the same. Overall, the mispricing is less severe for at-the-money than for out-of-the-money options.

Finally, combining the results from Tables 4–9, we conclude that there is little evidence that the implied binomial tree model performs better than the ad-hoc procedure. However, the model DK3 also does not perform much worse the model AH.

Furthermore, both models perform surprisingly well, producing very small pricing errors and model prices that fall inside the bid/ask spread more than 75% of the time.

## 5. Conclusion

Since the 1987 stock market crash, the constant volatility assumption of the Black–Scholes model has come under increasing attack. The appearance and persistence of the implied volatility smile and term structure have prompted the development of both stochastic volatility and deterministic volatility function models. Among the deterministic volatility function models, Derman and Kani (1994a,b) introduce an implied binomial tree model that can be calibrated to be consistent with a set of observed option prices. This model, like all option pricing models, can be used for both hedging and cross-sectional pricing. Dumas et al. (1998) investigate the hedging performance of deterministic volatility function models and conclude that these models perform no better than an ad-hoc procedure of smoothing Black–Scholes implied volatilities across strike prices and maturities. However, as we argued in the introduction, this failure of the deterministic volatility function models along the time-series dimension does not automatically imply that the models are also useless for cross-sectional pricing of illiquid and exotic derivatives.

We study the ability of deterministic volatility function models, and in particular the Derman–Kani implied binomial tree model, to price derivatives cross-sectionally. For each date in our sample of FTSE 100 index option prices, we fit an implied binomial tree to the panel of all European style options and then examine how well the model prices the corresponding American style options. This approach of fitting an implied binomial tree to one set of derivatives to then price another set (usually less liquid or exotic ones) mirrors the way deterministic volatility functions are commonly used in practice.

We find that although the implied binomial tree model prices the American style options better than a standard Cox et al. (1979) tree model with constant volatility, it performs no better than an ad-hoc procedure of smoothing Black–Scholes implied volatilities across strike prices and maturities. The ad-hoc model delivers smaller root mean squared valuation errors than the implied tree model for both put and call options, although a formal statistical test indicates that this difference in root mean squared errors is not statistically significant. The ad-hoc model also outperforms the implied tree model in terms of generating theoretical prices that lie more frequently within the observed bid/ask spreads. Our results both complement and confirm the finding of Dumas et al.

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