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Modeling the volatility of the Heath–Jarrow–Morton model: a multifactor GARCH analysis

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Abstract

Based on the nonparametric study of Pearson and Zhou (2000), A non-parametric analysis of the forward rate volatilities, L. Hughston (ed), The New Interest Rate Models, Rsk Books, 2000], a Heath–Jarrow–Morton (HJM) model is developed for the forward-rate volatility. It allows the volatility to react in a flexible way with the forward rate, time to maturity and the forward spread. Specifically, the volatility is a power function of the forward rate for short-maturity forward rates, but becomes a U-shape curve for long maturities. The volatility tends to decrease with the time to maturity. After adjusting for the level and maturity effects, the spread effect becomes significant only for short maturities. The proposed volatility function outperforms all four alternative specifications. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Pioneered by Ho and Lee (1986), the arbitrage models have developed into one of the two major frameworks in the term structure literature. They take as given the initial term structure of the interest rate so that the stochastic process of the interest rate is arbitrage-free. This method takes advantage of the information of the entire term structure to price contingent claims.

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Heath et al. (1992) remarkably generalize the previous studies and develop a multifactor, continuous-time model. They build the model on a given initial forward-rate process and use the equivalent martingale method to facilitate contingent claims pricing. Given some regularity conditions and under the risk-neutral probability, the prices of contingent claims are exclusively determined by the initial term structure and the volatility function of the forward rate. Under this general framework, most of the previous arbitrage models are nested as special cases.

Given the generality of the Heath–Jarrow–Morton (HJM) model and the critical role the forward-rate volatility plays in the model, there have been quite a few studies that attempt to extend and test the HJM model by assuming certain functional forms for the forward-rate volatility. The most commonly used specifications are the Constant and Square-Root models. They are borrowed from the Vasicek (1977) and Cox et al. (1985) models in the interest rate literature and are simple enough to generate closed-form solutions of the options. However, studies such as Flesaker (1993) and Amin and Morton (1994) have rejected these models, as they do not adequately reflect the features of the real market data.

More recent studies realize the inadequacy of having a few simple volatility specifications, they propose more sophisticated and more realistic functional forms. Some of these models have been compared and tested by various methods and datasets. Cohen and Heath (1992) compare the performance of several forward-rate models by testing their ability to predicting future Treasury security prices. They find that the Proportional model performs significantly better than the Constant model. Amin and Morton (1994) study the implied interest rate volatilities of six-term structure models in the HJM class using Eurodollar futures and options data. It is shown that while the one-factor models tend to earn larger profits, the two-factor models give closer estimation to the options prices, especially for the long-term options. Abken and Cohen (1994) conduct a Generalized Method of Moments test on Treasury bond data, which strongly supports the exponentially damped proportional specification which is the combination of the simple exponential model and the linear proportional model. Amin and Ng (1997) study the information content of implied volatility from several volatility specifications of the HJM models. The exponential and linear proportional models outperform the Constant, the Square-Root and Proportional models. Bühler et al. (1999) compare some forward-rate models using German market data. They conclude that the one-factor linear proportional model outperforms the other three models examined including the two two-factor specifications.

It seems that the conclusion on the performance of a certain volatility specification depends on the method and data used. It is not clear which study generates more convincible results. Pearson and Zhou (2000) make the very first effort to estimate the forward-rate volatility function by conducting a nonparametric analysis so that the relationship between the forward-rate volatility and the forward-rate level as well as the forward spread is developed without being imposed any specification assumption. It provides important guidance for further parametric studies and also makes it possible to compare and test the parametric models in a general framework.

Using specifications suggested by the results of Pearson and Zhou (2000), this paper develops a parametric model for the forward-rate volatility of HJM class by conducting a multifactor GARCH analysis. In this paper, we construct time series of the instantaneous forward rates for a range of maturities, investigate the factors that drive the movement of the forward-rate volatility and compare the performance of several HJM volatility models in the GARCH family framework. Specifically, we select the GARCH model that best fits the data and imbed in it the five HJM volatility models. The performances of the five HJM models are then examined and compared. It is shown that the proposed volatility specification performs the best for all the time series. It is also confirmed that the volatility of forward rate with different maturities changes with the forward rate and the forward spread in a different way. Specifically, the volatility tends to be a simple power function of the forward rate for short-maturity forward rates, but becomes a U-shape curve for long-maturity forward rates. For all data series, the volatility tends to decrease with the time-to-maturity. After adjusting for the level and maturity effects, the forward-rate spread effect becomes significant only for short-maturity forward rates.

The balance of the paper is organized as follows. Section 2 briefly reviews the HJM framework. Section 3 combines the "level" and GARCH modeling methods to propose a generalized HJM volatility model in the GARCH framework. Section 4 introduces the econometric approach, while Section 5 explains data. Section 6 reports the results of GARCH-family HJM volatility estimation. Section 7 concludes.

2. The HJM framework

Following Heath et al. (1992), we consider a trading interval $[0, \tau]$ for a fixed $\tau > 0$. Let $f(t, T)$ be the instantaneous forward rate at time t for date $T(T > t)$. The period $(T - t)$ is referred to as the time to maturity. The probability space is represented by (Ω, F, P), where Ω is the state space, F is the σ-algebra and P is the probability measure. The instantaneous forward rate, $f(t, T)$, is defined by:

$$f(t,T) = -\partial \log P(t,T)/\partial T \qquad \forall T \in [0,\tau], t \in [0,T].$$

where $P(t, T)$ is the time t price of a zero-coupon bond that matures at T. $f(t, T)$ is assumed to follow the Ito process:

$$\mathrm{d} f(t,T) = \mu(\omega,t,T)\mathrm{d}t + \sigma(\omega,t,T)\mathrm{d}B(t), \tag{1}$$

where $B(t)$ is a Brownian motion, μ and σ are the "drift" and "volatility" (or diffusion) functions, respectively, and $\omega \in \Omega$ indicates the possible dependence of the drift and volatility functions on the entire history of the process.

The evolution of the forward rate of all maturities is simultaneously and exogenously determined. Once the forward-rate term structure is derived, the dynamics of the bond price can be obtained as the following (see Heath et al., 1992):

$$\mathrm{d}P(t,T) = [f(t,t) + b(t,T)]P(t,T)\mathrm{d}t + a(t,T)P(t,T)\mathrm{d}B(t), \qquad (2)$$

where:

$$a(\omega,t,T) = -\int_t^T \sigma(\omega,t,\nu)\mathrm{d}\nu,$$

$$b(\omega,t,T) = -\int_t^T \mu(\omega,t,\nu)\mathrm{d}\nu + (1/2)a(\omega,t,T).$$

Given the initial term structure of the forward rate and some regularity conditions, there exists a unique equivalent martingale probability measure or risk-neutral probability, Q, which implies that the drift term can be expressed by the volatility functions:

$$\mu^Q(\omega,t,T) = \sigma(\omega,t,T)\int_t^T \sigma(\omega,t,\nu)\mathrm{d}\nu.$$

Thus, under Q, the forward rate, $f(t, T)$, evolves according to the process:

$$\mathrm{d}f(t,T) = \mu^Q(\omega,t,T)\mathrm{d}t + \sigma(\omega,t,T)\mathrm{d}\tilde{B}(t), \qquad (3)$$

where $\tilde{B}(t)$ is the Brownian motion under the risk-neutral probability. It is the difference between the original Brownian motion and the sum of the market price of risk over time, i.e.:

$$\tilde{B}(t) = B(t) - \int_0^t \gamma(\nu)\mathrm{d}\nu, \qquad (4)$$

where $\gamma(t)$ is the market price for risk at time t. Now, what determine the entire term structure of the forward rates are the initial term structure of the forward rate and the volatility function of the forward rate.

In modeling forward-rate volatility, there are, in general, two approaches. One is to assume that the volatility changes with factors such the forward-rate level, which we call the “level” method, the other is to assume that the volatility evolves from its own history, which we call the GARCH method.

3. Volatility modeling

3.1. The GARCH models

GARCH models are used as a successful treatment to the financial data which often demonstrate time persistence, volatility clustering and deviation from the normal distribution. Among the earliest models is Engle (1982) linear ARCH model, which captures the time varying feature of the conditional variance. Bollerslev (1986) develops Generalized ARCH (GARCH) model, allowing for persistency of the conditional variance and more efficient testing. Engle and Bollerslev (1986) invent the Integrated GARCH (IGARCH) model that provides consistent estimation under the unit root condition. Engle et al. (1987) design the ARCH-in-Mean (ARCH-M) model to allow for time varying conditional mean. Nelson's (1990) Exponential GARCH (EGARCH) model allows asymmetric effects and negative coefficients in the conditional variance function. The leveraged GARCH (LGARCH) model documented in Glosten et al. (1993) takes into account the asymmetric effects of shocks from different directions.

As far as the HJM volatility modeling is concerned, we are interested in finding some continuous-time models that best describe the dynamics of the time series. Being discrete-time by nature, how well can the GARCH class models approximate the continuous-time process suggested in the previous section? Nelson (1990) shows that the GARCH (1, 1) model and EGARCH model converge to continuous-time Ito diffusion processes, as the sample interval goes to zero. Nelson (1992) and Nelson and Foster (1994) also show that ARCH models fitted to high frequency data provide optimal and consistent estimates of true volatility underlying a given observation system. Fornari and Mele (1995) provide a summary of ARCH models and their diffusion limits. Duan (1995) shows that under some preference and distribution assumptions, the GARCH model converges to its diffusion limit and can explain some pricing error in the Black–Scholes option model. It seems that as long as the model and the data frequency are properly selected, the ARCH models approximate the true processes reasonably well.

3.2. The level models

The level models have been widely proposed and used in the interest-rate term structure literature, such as the Constant model (Ho and Lee, 1986), the Square-Root model (Cox et al., 1985) and the Exponential model (Vasicek, 1977).

In the HJM framework, the arbitrage-free assumption requires that the volatility function of the forward rate be bounded (Heath et al. (1992), p. 80, C1 (iii)). The Constant model and the exponentially damped model are both bounded. Heath et al. (1992) propose a constraint proportional function of the forward rate, which allows the volatility to change linearly with the forward rate when the forward rate is low, but is capped by a constant when the forward rate is high. The Square-Root model can be constrained in a similar manner. Thus, the four models that have been widely used in the existing literature become the benchmarks of our study and are, thus, included. Their detailed functional forms are provided in Table 1.

The nonparametric analyses of Pearson and Zhou (2000) document that the volatility functions of the forward rate adopt different forms for different maturi-

Table 1
HJM volatility models of the forward rate and the corresponding GARCH conditional variance functions

Model	Functional form
Panel A: Alternative HJM forward-rate volatility models	
Constant	σ_0
Square root	$\min[\sigma_0 f(t,T)^{1/2}, M]$
Proportional	$\min[\sigma_0 f(t,T), M]$
Exponential	$\min[\sigma_0 e^{\lambda(T-t)}, M]$
Combination	$\min[\sigma_0 \vert f(t,T)-c\vert^{\sigma_1} + \sigma_2 e^{\lambda(T-t)}, M]$
Panel B: The GARCH conditional variance functions that imbed the HJM models	
Constant	$h_t = c + ph_{t-1} + q\varepsilon_{t-1}^2$
Square root	$h_t = c + ph_{t-1} + q\varepsilon_{t-1}^2 + [\sigma_0 f(t,T)^{1/2}]^2$
Proportional	$h_t = c + ph_{t-1} + q\varepsilon_{t-1}^2 + [\sigma_0 f(t,T)]^2$
Exponential	$h_t = c + ph_{t-1} + q\varepsilon_{t-1}^2 + [\sigma_0 e^{\lambda(T-t)}]^2$
Combination	$h_t = c + ph_{t-1} + q\varepsilon_{t-1}^2 + [\sigma_0 \vert f(t,T)-c\vert^{\sigma_1} + \sigma_2 e^{\lambda(T-t)}]^2$

Panel A summarizes the five HJM forward-rate volatility models. Some of them are constrained by a large constant so as to fulfill the HJM regularity condition. Panel B reports the GARCH conditional variance functions that imbed the HJM volatility models listed in Panel A.

ties. They change with the forward-rate level and the forward-rate spread. Specifically, the volatility increases monotonically with the forward rate at a faster speed for short maturities, but becomes a combination of the convex and concave function for medium and long maturities, i.e., it increases with the forward rate and then decreases with it at moderate forward-rate levels. With respect to the forward-rate spread, the volatility increases with it, in general; however, for long maturities, it decreases at the moderate spread levels.

Given the features of the forward-rate volatility discovered in Pearson and Zhou (2000), we propose the following parametric model:

$$\sigma(\cdot) = \sigma_0 |f(t,T) - c|^{\sigma_1} + \sigma_2 e^{\lambda(T-t)}. \tag{5}$$

For all coefficients, the sign and size depend on the time series. For simplicity, we omit this dependency in Eq. (5). This specific functional form is adopted due to the following considerations: first, the term $|f(t,T) - c|^{\sigma_1}$ is included to allow for a combination of the concavity and convexity with respect to the forward rate. This is possible when c is positive and within the range of the forward-rate levels. When c is negative, this term will only show convexity or concavity, depending on the sign of σ_1. When c is positive, but bigger than the maximum forward-rate level, we need to guarantee the term $f(t,T) - c$ be raised by a non-integer power, such as 0.5, so we take its absolute value and let it be raised to power σ_1. Second, we allow the time to maturity $(T-t)$ plays a role in the movement of the forward rates, depending on the sign of λ, which we expect to be negative. Third, this model nests the four widely used HJM volatility specifications. Specifically, when

σ_1 and σ_2 are zero, the function collapses to the Constant model; when σ_0 is zero, it becomes the Exponential model; when σ_2 is zero, it can be either the Proportional model or the Square-Root model, depending on the parameter σ_1. Besides, it nests most of the specifications that previous studies have proposed and examined. A list of the existing specifications are provided in Table 2.

To qualify for the HJM volatility function, the proposed Combination model specified in Eq. (5) is constrained so that the volatility does not explode in finite time. Its constrained functional form and those of the four alternative HJM volatility models are provided in Table 1.

As suggested in Pearson and Zhou (2000), the forward-rate spread is another factor that affects the forward-rate volatility. Defined as the difference between a long-term forward rate and a short-term forward rate, the forward-rate spread is regarded as a measure of the slope of the forward yield curve. It represents a term premium, a compensation for the risk of holding a longer-term contract. As an extension to the model described in Eq. (5), we include the forward-rate spread as the third factor and examine its role in the volatility function:

$$\sigma(\cdot) = g\left(f(t,T), T-t, f(t,T)^{\mathrm{l}} - f(t,T)^{\mathrm{s}}\right), \tag{6}$$

where $f(t, T)^{\mathrm{l}}$ is a forward rate with long maturity and $f(t, T)^{\mathrm{s}}$ is a forward rate with short maturity.

3.3. GARCH imbedded level models

As both the level models and the GARCH models have their own merits in treating the volatilities, it is difficult to pick one model without comparing their contributions. To do this, we combine the two models into a more generalized

Table 2
Forward-rate volatility models in the existing literature

Specification	References
σ_0	Cohen and Heath (1992), Flesaker (1993), Amin and Morton (1994), Amin and Ng (1997), Bühler et al. (1999)
$\sigma_0 f^{1/2}$	Amin and Morton (1994), Amin and Ng (1997)
$\sigma_0 f$	Amin and Morton (1994), Amin and Ng (1997)
$\sigma_0 + \sigma_1(T-t)$	Amin and Morton (1994)
$[\sigma_0 + \sigma_1(T-t)]f$	Amin and Morton (1994), Amin and Ng (1997), Bühler et al. (1999)
$\sigma_0 \exp[-\lambda(T-t)]$	Amin and Morton (1994), Amin and Ng (1997)
$\sigma_0 \min(f, M)$	Cohen and Heath (1992), Abken and Cohen (1994)
$\sigma_0\, g(T-t)\min(f, M)$	Cohen and Heath (1992), Abken and Cohen (1994)
$\sigma_{0i}\, g_i(T-t)\min(f, M)$	Cohen and Heath (1992)

framework and examine its power in capturing the forward-rate volatility movement. Thus, the combined level and GARCH (p, q) model will look like the following:

$$f_1 = \alpha_0 + \alpha_i f_{t-1} + \varepsilon_t,$$
$$\varepsilon_t = \eta_t \sqrt{h_t},$$
$$h_t = c_0 + \sum_{i=1}^{p} p_i h_{t-i} + \sum_{i=1}^{q} q_i \varepsilon_{t-i}^2 + \text{Level Model}, \tag{7}$$

where f_t is the instantaneous forward rate at time t with the time to maturity of $(T-t)$, h_t is the conditional variance of the forward rate at time t, and η_t is assumed to follow the standard normal distribution. Besides the GARCH model, we also experiment with various extensions of the GARCH family models, such as EGARCH and LGARCH.[1] As mentioned before, the EGARCH model allows for negative coefficients and asymmetric effects in the conditional variance function; the LGARCH model allows for shocks from different directions to have different effects. The GARCH-family model that best fits the data will be chosen for further analyses.

3.3.1. The level and time-to-maturity effect

We first examine the effects of the forward-rate level and time to maturity (maturity effect, hereafter) on the conditional volatility of the forward rate. Specifically, we imbed the HJM volatility functions described in Section 2 into the GARCH family conditional variance functions. For example, the GARCH (1, 1) model that imbeds the Square-Root model has the following conditional variance function:

$$h_t = c_0 + p h_{t-1} + q \varepsilon_{t-1}^2 + \left(\sigma_0 f_{t-1}^{1/2}\right)^2. \tag{8}$$

The square term is due to the fact that the square-root specification is for the "standard deviation" of the forward rate and h_t is the variance of the forward rate. Similarly, the GARCH (1, 1) model that imbeds the Exponential model has the following conditional variance function:

$$h_t = c_0 + p h_{t-1} + q \varepsilon_{t-1}^2 + \left[\sigma_0 \mathrm{e}^{\lambda(T-t)}\right]^2, \tag{9}$$

and the GARCH (1, 1) model that imbeds the Combination model is:

$$h_t = c_0 + p h_{t-1} + q \varepsilon_{t-1}^2 + \left[\sigma_0 |f(t,T) - c|^{\sigma_1} + \sigma_2 \mathrm{e}^{\lambda(T-t)}\right]^2. \tag{10}$$

The conditional variance function of the Constant model coincides with its GARCH-family counterpart.

[1] IGARCH is not considered as it is used in the unit-root case, in which case the volatility grows without bound. This violates the regularity condition of the HJM framework.

3.3.2. The forward-rate spread effect

We next introduce the forward-rate spread into the conditional volatility functions as the third factor and examine its contribution. Specifically, the GARCH (1, 1) model that corresponds to the Combination model is:[2]

$$h_t = c_0 + p h_{t-1} + q\varepsilon_{t-1}^2 + \left[\sigma_0 |f(t,T) - c|^{\sigma_1} + \sigma_2 e^{\lambda(T-t)} + \sigma_3\left(f_{t-1}^{l} - f_{t-1}^{s}\right)\right]^2, \tag{11}$$

where f_{t-1}^{l} is the forward rate with a long maturity at time $t-1$, and f_{t-1}^{s} is the forward rate with a short maturity at time $t-1$.

4. The econometric approach

4.1. Maximum likelihood method

As pointed out by Bera and Higgins (1993), the GARCH models are most often estimated by maximum likelihood method. We, thus, adopt it in this study as well. The log likelihood function of the GARCH model based on previous period's information $f_t|\psi_{t-1} \sim N(\alpha_0 + \alpha_1 f_{t-1},\ h_t)$ is given by:

$$l(\theta) = \frac{1}{T}\sum_{t=1}^{T} l_t(\theta),$$

where $\theta = (\xi',\ \gamma')'$ with ξ and γ the conditional mean and conditional variance parameters, respectively, and:

$$l_t(\theta) = \text{const.} - \frac{1}{2}\log(h_t) - \frac{\varepsilon_t^2}{2h_t^2}.$$

The likelihood function provided above is maximized using Berndt, Hall, Hall and Hausman (BHHH) numerical algorithm (Berndt et al., 1974).

4.2. Tests

Specification test is conducted by checking the normality of the standardized residual:

$$\varepsilon_t^* = \varepsilon_t / \sqrt{h_t}.$$

If the conditional variance function is well specified, then the standardized residual should be close to the white noise. Jarque and Bera (1980) test is used to

[2] We have tried several nonlinear specifications of the forward spread. However, they cannot generate convergence.

detect the deviation from normality. Ljung and Box (1978) Q-test is used to detect the serial correlation in the standardized residual and the squared standardized residual. F-test is used to detect remaining ARCH effect.

Engle and Ng (1993) develop the sign bias test to detect the asymmetric impact of the lagged negative and positive shocks on the conditional variance. They also design the negative size bias test and positive size bias test to determine whether shocks from different directions and with different magnitudes have different impacts on the conditional variance. These tests are also used in this paper.

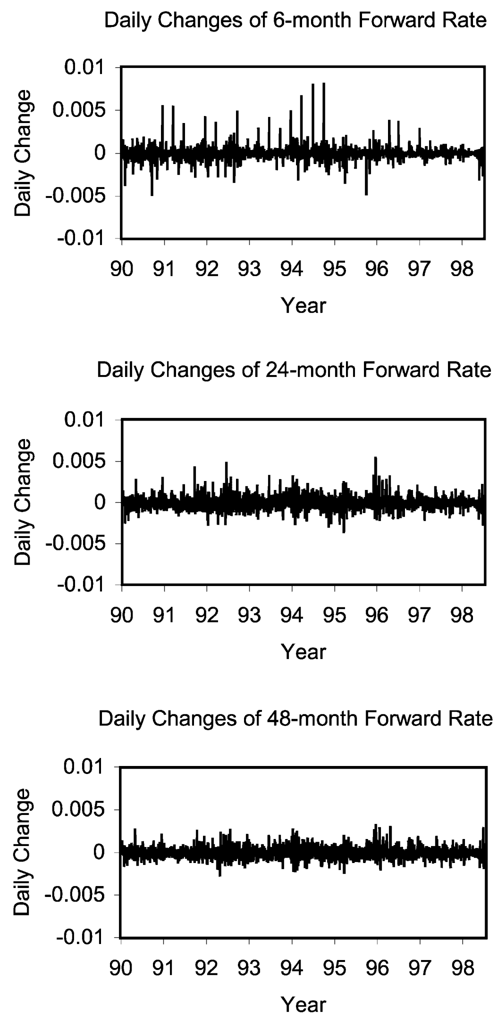

Fig. 1. The daily changes of the 6-, 24- and 48-month instantaneous forward rates. The instantaneous forward rates are derived from the daily Eurodollar futures prices from April 1990 to October 1998. There are 2142 observations in each series.

Likelihood ratio test is used to examine the relative performance of each of the four HJM models with respect to the Combination model.

Lastly, we test the equality of the coefficient estimates between each pair of series. Since the residuals of different series do not have equal or known variances, in general, we use the modified the Chow-type test proposed by Toyoda (1974), which only requires larger sample size for at least one series.

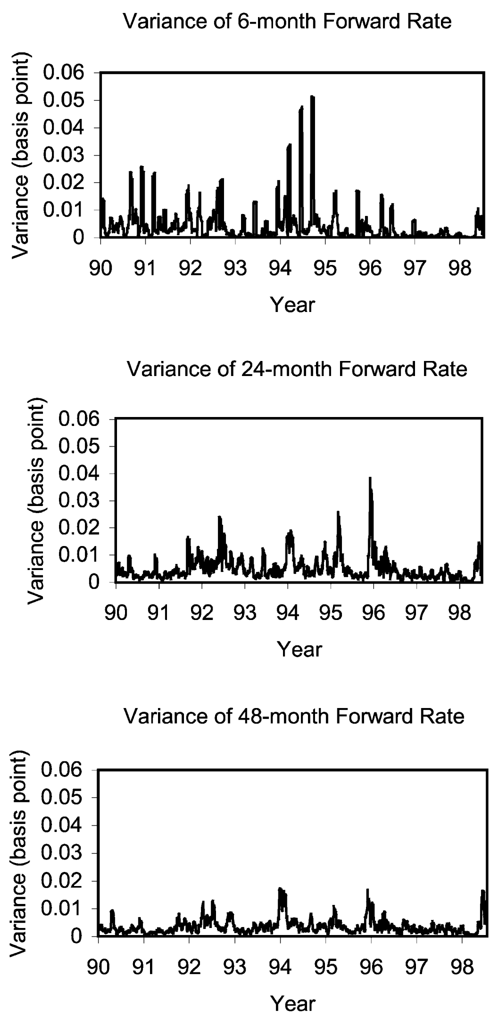

Fig. 2. The 14-day moving averages of squared daily changes of the 6-, 24- and 48-month instantaneous forward rates. The instantaneous forward rates are derived from the daily Eurodollar futures prices from April 1990 to October 1998. There are 2143 observations in each series.

5. The data

We estimate the volatility functions using the estimates of the "instantaneous" forward rates for a range of maturities. These estimates of the "instantaneous" forward rates are constructed from the daily settlement prices of the Eurodollar futures contracts (based on 3-month LIBOR) 14- and 1-month LIBOR futures contracts traded on the Chicago Mercantile Exchange. The Eurodollar futures contracts started trading in December 1981, when only 3- and 6-month contracts were available. Starting in 1990, the maturities of the Eurodollar contracts have extended out to at least 4 years, and currently extend out to 10 years. The 1-month Eurodollar futures contracts are also available from 1990.

Some advantages of using Eurodollar futures data are discussed in Jegadeesh and Pennacchi (1996). Eurodollar and 1-month LIBOR futures contracts are very actively traded with a very small bid-ask spread. Trading stops in all contracts at the same instant, at which time final settlement prices for all contracts are determined essentially simultaneously, eliminating concerns about the possible

Table 3
Summary statistics

Maturity	Mean	Standard deviation	Skewness	Kurtosis	AR1	AR3	AR6
(A) Some summary statistics for the forward rates							
3 months	0.053	0.0134	0.23	2.90	0.996	0.989	0.978
6 months	0.056	0.0130	0.17	2.75	0.996	0.987	0.974
12 months	0.064	0.0131	0.26	2.68	0.994	0.979	0.960
18 months	0.072	0.0136	0.40	2.56	0.994	0.982	0.965
24 months	0.076	0.0135	0.44	2.44	0.996	0.987	0.974
30 months	0.078	0.0139	0.49	2.35	0.995	0.984	0.967
36 months	0.079	0.0142	0.46	2.25	0.995	0.985	0.972
42 months	0.081	0.0146	0.44	2.15	0.994	0.982	0.964
48 months	0.082	0.0139	0.37	2.08	0.993	0.982	0.966
(B) Some summary statistics for the forward-rate changes							
3 months	−0.00002	0.00072	1.58	98.92	0.050	0.015	−0.021
6 months	−0.00002	0.00070	2.30	34.23	0.030	0.001	−0.022
12 months	−0.00002	0.00081	0.76	10.25	0.068	−0.012	−0.045
18 months	−0.00002	0.00080	0.80	9.26	0.109	−0.005	−0.046
24 months	−0.00002	0.00074	0.70	8.16	0.118	−0.019	−0.053
30 months	−0.00002	0.00067	0.50	6.57	0.123	−0.016	−0.057
36 months	−0.00002	0.00063	0.50	6.25	0.124	−0.026	−0.063
42 months	−0.00002	0.00062	0.45	5.90	0.124	−0.010	−0.058
48 months	−0.00002	0.00061	0.47	5.76	0.119	−0.015	−0.055

Means, standard deviations, skewness and kurtosis of the daily Eurodollar instantaneous forward rates and their daily changes with selected maturities are computed. Up to six-lagged autocorrelation coefficients of selected forward-rate series are also provided. The forward rates are from April 1990 to October 1998. There are 2143 observations in each series.

non-synchroneity of prices. In addition, because 3-month LIBOR is a common index for floating rate instruments such as interest rate swaps and floating rate notes, the Eurodollar contracts are widely used for hedging and arbitrage, linking the Eurodollar term structure to the term structure of swap rates. A further advantage pointed out by Amin and Morton (1994) is that Eurodollar and 1-month LIBOR futures contracts are cash-settled, which avoids some delivery and timing problems that are inherent in the Treasury bond and note futures contracts.

We construct the daily instantaneous forward rates with initial times to maturity from 3 up to 48 months. The details of the data construction are provided in Appendix A. Fig. 1 shows the time series of the 3-, 12-, 24- and 48-month instantaneous forward rates, while Fig. 2 shows plots of daily rate changes. Some summary statistics of the selected series of forward rates are provided in Table 3. We observe that the sample mean increases with maturity, all series are slightly skewed to the right and have slightly thinner tails. The up to six-lagged autocorrelation coefficients are all greater than 0.96, showing a strong time persistency. This verifies the necessity of adopting the GARCH-family model. The autocorrelation coefficients of the rate changes are small and change signs from time to time.

6. The empirical results

6.1. GARCH-family model selection

We start with the naive GARCH family models[3] to examine whether these models fit the forward-rate data well. Specifically, we estimate the AR (1)–GARCH (1, 1) family models for all 16 series of forward rates by using the maximum likelihood method and BHHH algorithm, without inserting the HJM volatility specifications.

The results of the 48-month series are provided in Table 4, which shows the pattern that most other data series possess. As we can see, in general, all three models perform well in capturing the ARCH effect. Though there is still some serial correlation left in the residual, the serial correlation in the squared residual and the size and sign bias are very well rectified. Though the EGARCH model and the LGARCH model have slightly higher log likelihood functional value, the leverage term in neither model is significant.[4] As a result, we select GARCH model for the following analyses.

[3] For the generalized AR (s)–GARCH (p, q) model, we have tried several combinations of s, p and q and found that the AR(1)–GARCH (1, 1) family models provide good approximation in most cases. This is consistent with some related studies such as Milhøj (1990).

[4] The computation is done with the software Rats. For the BHHH method, the p-values are computed using the product of the first derivatives. As shown in related studies, it is consistent for the matrix being estimated, and it converges to the “true” value as sample size grows. Given the relatively large number of observations, this should not be a concern.

Table 4
Preliminary model selection

	GARCH (1, 1)	EGARCH (1, 1)	LGARCH (1, 1)
c_0	6.58e − 9 (0.00)	− 0.25 (0.00)	5.65e − 9 (0.69)
q	0.04 (0.00)	0.10 (0.00)	0.03 (0.00)
p	0.95 (0.00)	0.98 (0.00)	0.95 (0.00)
l		0.09 (0.15)	0.01 (0.09)
skew	0.34	0.31	0.34
kurt	5.36	5.16	5.33
$\log L$	12,871.9	12,876.7	12,872.6
$Q(4)_{\varepsilon}$	45.4 (0.00)	44.2 (0.00)	44.9 (0.00)
$Q(4)_{\varepsilon^2}$	2.6 (0.45)	3.2 (0.36)	2.7 (0.44)
Bias test	0.4 (0.73)	0.3 (0.84)	0.6 (0.64)

Results are for the 48-month forward-rate series from April 1990 to October 1998, which represent those for the other series. For all models, the mean equation is:

$f_t = \alpha_0 + \alpha_1 f_{t-1} + \varepsilon_t$, where $\varepsilon_t = \eta_t \sqrt{h_t}$, $\eta_t \sim N(0,1)$.

The conditional variance equation in the GARCH model is:

$h_t = c_0 + p h_{t-1} + q \varepsilon_{t-1}^2$.

The conditional variance equation in the EGARCH model is:

$h_t = \exp\left(c_0 + p \log(h_{t-1}) + q g_{t-1}\right)$,

$g_{t-1} = |\varepsilon_{t-1}/\sqrt{h_{t-1}}| - \sqrt{2/\pi} - l \varepsilon_{t-1}/\sqrt{h_{t-1}}$.

The conditional variance equation in the LGARCH model is:

$h_t = c_0 + p h_{t-1} + q \varepsilon_{t-1}^2 + l I_{t-1} \varepsilon_{t-1}^2$.

The skew and kurt are the skewness and kurtosis of the residuals. The $Q(12)_{\varepsilon}$ is the Ljung and Box (1978) Q-test for the serial corraltion in the residuals, $Q(12)_{\varepsilon^2}$ is the Ljung and Box (1978) Q-test for the serial corraltion in the squared residuals and the bias test is the joint test for sign and size bias. The numbers in the parenthesis are the p-values.

6.2. The level and maturity effects

We imbed the five HJM models in the GARCH model as shown in Panel B of Table 1 and compare their performance. The results are summarized in Tables 5–7.

Table 5 reports the GARCH estimation results for the 6-month forward rate from April 1990 to October 1998. All the five HJM models have successfully captured the serial correlation as well as sign and size bias in the conditional variance function. The Constant model has the lowest log likelihood functional value. The Square-Root and Proportional models have slightly higher log likelihood functional values, but their coefficient σ_0 is insignificantly different from zero, indicating a trivial contribution from either specification. Among the four alternative HJM volatility models, the Exponential model has the highest log likelihood functional value, with a negative and significant λ, indicating a decreasing relationship between the volatility and the maturity of the forward rate.

Table 5
HJM model comparison: 6-month rate

	Constant	Square root	Proportional	Exponential	Combination
c_0	3.95e − 7(0.00)	4.06e − 7 (0.79)	3.89e − 7 (0.00)	2.66e − 7 (0.00)	1.50e − 7 (0.00)
q	0.02 (0.03)	0.0009 (0.87)	0.02 (0.04)	0.04 (0.00)	0.07 (0.00)
p	0.18 (0.00)	3.87e − 7 (1.00)	0.18 (0.00)	9.59e − 10 (1.00)	0.02 (0.49)
σ_0		0.0012 (0.59)	0.0012 (0.30)	1.84 (0.21)	− 7.65e − 4 (0.00)
c					0.03 (0.00)
σ_1					0.17 (0.00)
σ_2					5.41 (0.00)
λ				− 0.21 (0.00)	− 0.085 (0.00)
$\log L$	12,514.8	12,520.5	12,515.2	12,953.4	12,965.5
Likelihood ratio test	901.4 > 11.1	890 > 9.49	900.6 > 9.49	24.2 > 7.81	
$Q(4)_{\varepsilon}$	5.0 (0.18)	8.6 (0.65)	4.9 (0.18)	12.4 (0.01)	13.9 (0.00)
$Q(4)_{\varepsilon^2}$	0.5 (0.93)	1.0 (1.00)	0.5 (0.93)	3.6 (0.31)	4.2 (0.24)
Bias test	0.3 (0.83)	0.2 (0.89)	0.3 (0.81)	0.2 (0.88)	1.1 (0.33)

The maximum likelihood estimates for the imbedded GARCH models are for the 6-month forward-rate series from 1990 to 1998. Numbers in the parentheses are the p-values. For all models, the mean equation is:

$$f_t = \alpha_0 + \alpha_1 f_{t-1} + \varepsilon_t, \quad \text{where } \varepsilon_t = \eta_t \sqrt{h_t}, \eta_t \sim N(0,1).$$

The conditional variance equations that imbed the HJM volatility models are as summarized in Panel B of Table 1.

The Combination model has the highest log likelihood value, with all coefficients significant. The power coefficient estimate, σ_1, is positive, indicating a positive relationship between the volatility and the forward rate level. The coefficient estimate, c, is positive, but lower than most of the data observations. The maturity effect, λ, is negative, but is much smaller than both σ_1 and that in the Exponential model. It is shown that the volatility of 6-month forward rate is more sensitive to the level than to the time to maturity. The likelihood ratio test rejects all four alternative models, showing that none of the specifications represent the correct model to describe the volatility of the short-term forward rate.

Table 6 reports the GARCH estimation results for the 24-month forward rate from April 1990 to October 1998. The Square-Root and Proportional models have the lowest log likelihood functional values, with coefficient estimates σ_0 insignificant. The Constant and Exponential models have the same log likelihood functional value that is higher than the Square-Root and Proportional models. For the Exponential model, the maturity effect λ is negative, but is much smaller than that of the 6-month rate. The Combination model again has the highest log likelihood functional value. Though insignificant, the estimate of σ_1 becomes negative. The estimate of c is significant and close to the minimum value of the data series. The maturity effect, λ, is again negative, but becomes insignificant. The likelihood ratio test rejects all four alternative models.

Table 6
HJM model comparison: 24-month rate

	Constant	Square root	Proportional	Exponential	Combination
c_0	7.92e − 9 (0.00)	3.81e − 7 (0.06)	4.26e − 7 (0.00)	7.92e − 9 (0.00)	1.15e − 7 (0.72)
q	0.04 (0.00)	0.0018 (0.21)	0.02 (0.14)	0.04 (0.00)	0.05 (0.00)
p	0.95 (0.00)	0.27 (0.45)	0.16 (0.01)	0.95 (0.00)	0.94 (0.00)
σ_0		8.38e − 7 (1.00)	8.62e − 5 (1.00)	2.27 (0.00)	1.52e − 4 (0.75)
c					0.04 (0.00)
σ_1					− 9.52e − 3 (0.81)
σ_2					1.26e − 3 (0.73)
λ				− 0.09 (0.00)	− 2.74e − 3 (0.64)
$\log L$	12,509.8	12,435.4	12,434.8	12,509.8	12,529.6
Likelihood ratio test	39.6 > 11.1	188.4 > 9.49	189.6 > 9.49	39.6 > 7.81	
$Q(4)_{\varepsilon}$	44.7 (0.00)	39.7 (0.00)	39.5 (0.00)	44.7 (0.00)	50.1 (0.00)
$Q(4)_{\varepsilon^2}$	2.7 (0.44)	4.0 (0.26)	4.3 (0.23)	2.7 (0.44)	4.6 (0.20)
Bias test	0.9 (0.46)	0.03 (0.99)	0.1 (0.97)	0.9 (0.46)	1.6 (0.20)

The maximum likelihood estimates for the imbedded GARCH models are for the 24-month forward-rate series from 1990 to 1998. Numbers in the parentheses are the p-values. For all models, the mean equation is:

$$f_t = \alpha_0 + \alpha_1 f_{t-1} + \varepsilon_t, \quad \text{where } \varepsilon_t = \eta_t \sqrt{h_t}, \eta_t \sim N(0,1).$$

The conditional variance equations that imbed the HJM volatility models are as summarized in Panel B of Table 1.

Table 7 reports the GARCH estimation results for the 48-month forward rate from April 1990 to October 1998. The four alternative models are again rejected by the likelihood ratio test. The Constant and Exponential model have significantly higher functional values than the other two alternative models. The Combination model has a negative and significant estimate of σ_1. The relatively large and positive estimate for c indicates the existence of the “hump” in the middle range of the forward rate. The maturity effect is again insignificant.

Compare the results of Combination model in Tables 5–7, we observe obviously different patterns for different data series.[5] For shorter maturities, the forward-rate volatility tends to be less sensitive to the maturity effect, while increases monotonically with the forward rate. For medium maturities, the forward-rate volatility tends to be relatively independent of both the maturity and the

[5] To confirm this, we conduct a Chow-type test which allows unequal and/or unknown variances to compare the coefficient estimates for different series. The results show that they are significantly different. For each pair of the three series studied, the F-statistics are significantly larger than the critical value, strongly reject the hypothesis that the coefficients of different time series are the same. This indicates that the volatility of different forward rate series reacts in a different way to the changes of its driving forces.

Table 7
HJM model comparison: 48-month rate

	Constant	Square root	Proportional	Exponential	Combination
c_0	6.58e−9 (0.00)	2.87e−7 (0.00)	2.52e−8 (0.01)	6.58e−9 (0.00)	3.0e−9 (0.75)
q	0.04 (0.00)	0.02 (0.03)	0.03 (0.00)	0.04 (0.00)	0.04 (0.00)
p	0.95 (0.00)	0.18 (0.15)	0.92 (0.00)	0.94 (0.00)	0.92 (0.00)
σ_0		2.55e−4 (0.83)	7.74e−7 (1.00)	0.61 (0.00)	7.34e−5 (0.13)
c					0.075 (0.00)
σ_1					−0.11 (0.03)
σ_2					−1.29 (0.00)
λ				−0.07 (0.00)	−2.77e−3 (0.93)
$\log L$	12,871.9	12,817.0	12,841.3	12,871.9	12,880.4
Likelihood ratio test	17 > 11.1	126.8 > 9.49	78.2 > 9.49	17 > 7.81	
$Q(4)_{\varepsilon}$	45.4 (0.00)	40.0 (0.00)	42.4 (0.00)	45.4 (0.00)	45.5 (0.00)
$Q(4)_{\varepsilon^2}$	2.6 (0.45)	12.2 (0.01)	2.3 (0.51)	2.6 (0.45)	3.5 (0.32)
Bias test	0.4 (0.73)	2.2 (0.09)	1.3 (0.27)	0.4 (0.74)	0.4 (0.76)

The maximum likelihood estimates for the imbedded GARCH models are for the 48-month forward-rate series from 1990 to 1998. Numbers in the parentheses are the p-values. For all models, the mean equation is:

$$f_t = \alpha_0 + \alpha_1 f_{t-1} + \varepsilon_t, \quad \text{where } \varepsilon_t = \eta_t\sqrt{h_t}, \eta_t \sim N(0,1).$$

The conditional variance equations that imbed the HJM volatility models are as summarized in Panel B of Table 1.

forward rate. For long maturities, the volatility decreases and then increases with the forward-rate level, and independent of the maturity effect.

6.3. The spread effect

We define the forward-rate spread as the difference between the 48-month forward rate and the 3-month forward rate. We first isolate the forward-rate spread effect by assuming that in Eq. (11), all the other coefficients are zeros except s_0; thus, the HJM volatility function looks like the following:

$$\sigma(\cdot) = s_0\left(f(t,T)^{l} - f(t,T)^{s}\right). \tag{12}$$

The results are reported in Table 8. For all of the three series, the spread effect is significant. The coefficient estimate of s_0 is the highest for the 6-month rate and becomes lower for longer maturity data series.

We next combine the level effect, the maturity effect and the spread effect and estimate them in one model as specified in Eq. (11). The results are reported in Table 9. For all three series, coefficient c is still positive and significant. However, it becomes much smaller for the 48-month forward rate that it does not create a U-shape curve. The estimate of level power coefficient σ_1 is negative for all series, but only significant for the 6-month forward rate. The maturity effect is

Table 8
Forward-rate spread effect

Series	6-month	24-month	48-month
c_0	3.32e − 7 (0.18)	9.26e − 9 (0.58)	6.59e − 9 (0.65)
q	1.09e − 8 (1.00)	0.04 (0.00)	0.04 (0.00)
p	1.98e − 4 (1.00)	0.94 (0.00)	0.94 (0.00)
s_0	0.02 (0.01)	2.83e − 3 (0.00)	1.17e − 3 (0.00)
$\log L$	12,562.8	12,517.0	12,872.7
$Q(4)\varepsilon$	8.1 (0.04)	46.3 (0.00)	45.6 (0.00)
$Q(4)_{\varepsilon^2}$	0.3 (0.97)	2.5 (0.48)	2.6 (0.45)
Bias test	0.4 (0.79)	0.8 (0.48)	0.4 (0.76)

The maximum likelihood estimates for the imbedded forward-rate spread GARCH model are reported for the 6-, 24- and 48-month forward-rate series from 1990 to 1998. Numbers in the parentheses are the p-values. For all models, the mean equation is:

$$f_t = \alpha_0 + \alpha_1 f_{t-1} + \varepsilon_t, \quad \text{where } \varepsilon_t = \eta_t \sqrt{h_t}, \eta_t \sim N(0,1).$$

The conditional variance equation is:

$$h_t = c_0 + ph_{t-1} + q\varepsilon_{t-1}^2 + (s_0 \text{spread}_{t-1})^2,$$

where:

$$\text{spread}_t = f_{48\text{-month},t} - f_{6\text{-month},t}.$$

Table 9
Forward-rate level, maturity and spread effect

Series	6-month	24-month	48-month
c_0	1.92e − 7 (0.00)	− 2.83e − 7 (0.96)	4.24e − 9 (0.21)
q	0.016 (0.02)	0.04 (0.00)	0.03 (0.00)
p	4.70e − 3 (0.80)	0.94 (0.00)	0.95 (0.00)
σ_0	− 6.55e − 5 (0.00)	5.29e − 4 (0.92)	4.62e − 6 (0.89)
c	0.03 (0.00)	0.04 (0.00)	0.05 (0.00)
σ_1	− 0.27 (0.00)	− 4.97e − 3 (0.96)	− 0.44 (0.74)
σ_2	847.4 (0.21)	1.58e − 3 (0.96)	213.17 (0.00)
λ	− 0.14 (0.00)	− 0.01 (0.00)	− 0.19 (0.00)
s_0	0.02 (0.00)	3.12e − 3 (0.91)	9.31e − 4 (0.25)
$\log L$	12,996.2	12,520.6	12,882.8
$Q(12)_{\varepsilon}$	14.9 (0.00)	46.9 (0.00)	40.1 (0.00)
$Q(12)_{\varepsilon^2}$	7.7 (0.05)	2.4 (0.49)	2.7 (0.43)
Bias test	0.2 (0.92)	0.9 (0.41)	0.14 (0.93)

The maximum likelihood estimates for the imbedded forward-rate three-factor GARCH model are reported for the 6-, 24- and 48-month forward-rate series from 1990 to 1998. Numbers in the parentheses are the p-values. For all models, the mean equation is:

$$f_t = \alpha_0 + \alpha_1 f_{t-1} + \varepsilon_t, \quad \text{where } \varepsilon_t = \eta_t \sqrt{h_t}, \eta_t \sim N(0,1).$$

The conditional variance equation for the forward spread GARCH model is:

$$h_t = c_0 + ph_{t-1} + q\varepsilon_{t-1}^2 + \left(\sigma_0 |f(t,T) - c|^{\sigma_2} + \sigma_2 e^{\lambda(T-t)} + s_0 \text{spread}_{t-1}\right)^2,$$

where:

$$\text{spread}_t = f_{48\text{-month},t} - f_{6\text{-month},t}.$$

still negative and significant for all three series. The spread effect is significant only for short-maturity series.

7. Conclusion

This paper proposes a more general and realistic parametric model for the forward-rate volatility, based on the nonparametric results in Pearson and Zhou (2000). It is confirmed that the volatility changes with the forward-rate level in a different way for forward rates with different maturities. Specifically, before counting for the spread effect, the volatility increases mainly with the forward-rate level for short-maturity data series, but increases and then decreases with it for the long-maturity data series. For all data series, the volatility tends to decrease with the time to maturity. After adjusting for the level and maturity effects, the forward-rate spread effect becomes significant only for short-maturity data series.

The contribution of this study is to propose a more general and realistic volatility model for the forward rate in the HJM framework. It improves the understanding on the driving forces of the forward-rate volatility and their influences on the forward-rate volatility across a relatively long-maturity span. On the basis of this new HJM forward-rate volatility model, the options can be priced in a more efficient and accurate way.

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Appendix A. Instantaneous forward-rate construction

We first need to understand the relationship between the futures rate and the corresponding forward rate. In the context of specific models, some previous studies (e.g., Grinblatt and Jegadeesh, 1996) determine the difference between the interest rate implied from the futures contracts and the actual implied forward rate, commonly known as the “convexity bias.” This bias is due primarily to the fact that the futures contracts settle gains and losses daily, while forward contracts are settled only at maturity. This, together with the asymmetric effect of the interest rate changes on bond prices, results in a gap between the interest rate implied from the futures contracts and the “true” implied forward rates. The former is usually a few basis points higher than the latter, and the difference increases with maturity.

Burghardt and Hoskins (1995) document this relationship, and suggest an approximate procedure to adjust the implied futures interest rates that does not depend on any specific model. In constructing our estimates of instantaneous forward rates, we obtain the forward rates using the procedure of Burghardt and Hoskins (1995).

To construct the instantaneous forward rates, we start with the daily prices of the Eurodollar and 1-month LIBOR contracts from April 1990 to October 1998, convert the futures prices into (continuously compounded) yields based on the following:

$$y(T) = 1 - F(T,T)/100,$$

where $F(T, T)$ is the price of futures that matures at time T, $y(T)$ is the 3-month yield on Eurodollar time deposits. We then convert the yields into the corresponding forward rates based on the well known "convexity bias" (see, e.g., Burghardt and Hoskins, 1995). We then "chain" together the forward rates in order to build the term structure. Specifically, for day t, let $t+s_1,\ t+s_2,\ \dots,\ t+s_l$ denote the last trading dates of the 1-month LIBOR contracts, and let $t+\tau_1,\ t+\tau_2,\ \dots,\ t+\tau_m$ denote the last trading dates of the Eurodollar contracts.[6] Starting from the last trading date $t+s_1$ of the first 1-month LIBOR contract, we construct the forward rates $f(t,\ t+s_1,\ t+s_2),\ f(t,\ t+s_1,\ t+s_3),\ \dots,\ f(t,\ t+s_1,\ t+\tau_1)$, where we stop using the 1-month LIBOR contracts at $t+\tau_1$, the last trading date of the first Eurodollar contract.[7,8] From that date, we use the Eurodollar contracts to construct the forward rates $f(t,\ t+s_1,\ t+\tau_2),\ f(t,\ t+s_1,\ t+\tau_3),\ \dots,\ f(t,\ t+s_1,\ t+\tau_m)$. The result of this process is a set of forward rates from $t+s_1$ to $t+s_2,\ \dots,\ t+\tau_1,\ t+\tau_2,\ \dots,\ t+\tau_m$. To these forward rates, we fit a cubic spline[9] to obtain the entire term structure from $t+s_1$ to $t+\tau_m$. Finally, we differentiated the spline function at the points that correspond to actual time to maturity for each Eurodollar contract. At the same time, we keep track of the maturity of the Eurodollar contract each forward rate is converted from and sort

[6] The final settlement value of the Eurodollar and 1-month LIBOR contracts is based on either 3- or 1-month LIBOR quoted on the last trading date, for a deposit period beginning two business days after the last trading date. Thus, the forward rates we construct are actually for the times $t+t_1+2$ business days, $t+t_2+2$ business days, etc. The discussion in the text ignores this settlement convention in order to prevent the description from becoming needlessly complicated. However, the algorithms used to construct the forward rates incorporated the settlement conventions of the interbank market.

[7] This involves using at most three of the 1-month LIBOR futures contracts.

[8] Because both contracts stop trading two business days before the third Wednesday of the month, the last trading date of the first Eurodollar contract always coincides with the last trading date of one of the 1-month LIBOR contracts. One issue is that the maturity date of the deposit underlying a contract often does not coincide exactly with the last trading date of the next contract. In constructing the forward rates, we assumed that it does. This has virtually no impact on the term structures we construct.

[9] We use the "natural" boundary condition that the second derivative of the spline function be zero at the endpoints.

the forward rates based on the maturity of the Eurodollar contracts. This gives us 16 series of daily instantaneous forward rates with maturity of 3 through 48 months, with actual time to maturity, covering the time period from April 1990 to October 1998.[10]

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[10] The shortest time to maturity is at least 30 days from the first 1-month LIBOR contract so that the forward rates we constructed would not be affected by the spline boundary condition.

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