Credit Risk Model with Lagged Information

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In this article, I propose a structural credit risk model with lagged information on a firm. Under the simple assumption that information on a firm's asset value is observed with a time lag, I show the model also has the properties of a reduced-form model with a default intensity process. In contrast to some previous studies, where both periodic and noisy accounting reports assumptions are necessary to derive the same result, I argue that lagged information is sufficient to generate a default intensity process. This difference in the assumptions has fundamentally different implications for the time series of credit spreads; under both the noisy accounting and lagged information assumptions, the model-implied credit spreads exhibit periodic patterns, whereas under my model assumptions they do not.

Credit risk modeling can be largely categorized into two groups; structural models, based on economic assumptions of a firm's dynamics, and reduced-form models, in which defaults are governed by an exogenous intensity process. The two approaches both have distinct advantages and disadvantages. For example, structural models in general cannot capture the short-term credit spread, while reduced-form models lack economic motivation.[1]

Several studies have tried to combine the benefits of the two models. In their seminal article, Duffie and Lando [2001], DL henceforth, showed that a structural model can generate a default intensity process when some restrictions are imposed on the information set of bond investors. Their assumption is that 1) periodic (and therefore lagged) and 2) noisy accounting reports are available to defaultable bond investors rather than complete information as in the usual structural model setting. But the paper is not clear on which information restriction is the main contributor for generating an intensity process. In other words, is it the lagged information release or the noisy accounting report that matters?

In this article, I propose a model showing that it is not necessary to make the two restrictive assumptions. Specifically, I show that one can obtain an intensity process from a structural model with only one assumption—the lagged information release of the firm. The only function of the noisy accounting information assumption in DL is to prevent the credit spread on a bond that is maturing immediately after an announcement from being zero when the firm's asset value is sufficiently above the default boundary. Considering the advent of the stricter corporate disclosure rules in the U.S. implemented following the accounting scandals early on in this decade, there seems to be less prevalence in the inaccurate accounting report than in the delay of the dissemination of the insider information.

Although the difference in the assumptions looks minor, it gives rise to the fundamentally different econometric implication in the time-series of credit spreads. In DL's model, the credit spread has a periodic pattern. It means that the credit spreads are narrower on average after the earnings announcement and get wider as time goes by. On the contrary, my model does not show any such patterns. Which model better describes the actual credit spreads is an open empirical question.

CREDIT MODEL WITH LAGGED INFORMATION

The setup is based on a simple first passage model, e.g., Back and Cox [1976]. Let us assume that the asset value of a firm A_t follows a lognormal process on a fixed probability space $(\Omega, \mathcal{F}, P)$.

$$dA_t = \sigma_a A_t dW_t \tag{1}$$

Here the drift of A_t is zero. One can interpret this as A_t being the ratio between the true asset value and the level of debt (distance to default) where the amount of debt increases at the same rate as the asset does. This assumption is economically reasonable as well, since the firm might use debt to finance their projects in some cases and therefore the overall debt amount will increase over time. Alternatively, one can view this as the firm adjusting its level of debt towards the optimal capital structure.

We define the default event at time τ as the first crossing of the constant barrier D by the process A_t. Investors are risk-neutral without the loss of generality. One can alternatively assume risk-averse investors and a risk-neutral process in Equation (1).

Imperfect Bond Market Information

We assume that bond investors only observe the information generated by a default event and the lagged firm's asset process. In other words, bond investors do not know the asset value at current time t, but do know the firm's asset value at time $t - l$, where l is some positive number which represents the information delay to the bond market and whether the firm has already defaulted ($\tau \leq t$). One can view this as the firm disseminating insider information in an untimely manner or making corporate announcements after some time delays. Then the information set available to bond investors can be written as:

$$\mathcal{G}_t = \{A_{u-l}, 1_{\{\tau \leq u\}}\} \tag{2}$$

Let s be the future time for which we want to calculate the survival probability. From the above definition of a default event, default is to occur when

$$A_s = A_{t-l}\exp\left(\sigma_a (W_s - W_{t-l}) - \frac{\sigma_a^2}{2}(s-(t-l))\right) < D$$

Since bond investors' information set at time t is $\mathcal{G}_t$, the survival probability is given as $P[\tau > s \mid \mathcal{G}_t]$. Let this probability be $\psi(t, s)$. Then we can write

$$\psi(t,s) = P[\tau > s \mid A_{t-l}, \tau > t] = \frac{P[\tau > s \mid A_{t-l}]}{P[\tau > t \mid A_{t-l}]} \tag{3}$$

Let $p(s) = P[\tau > s \mid A_{t-l}]$. We only need to calculate $p(s)$ in order to calculate survival probability. The expression for $p(s)$ is a well-known result (Musiela and Rutkowski [1998]) and is given as

$$p(s) = N\left(-\frac{\sigma_a\sqrt{s-(t-l)}}{2} + \frac{\ln(A_{t-l}/D)}{\sqrt{s-(t-l)}}\right) - \frac{A_{t-l}}{D}N\left(-\frac{\sigma_a\sqrt{s-(t-l)}}{2} - \frac{\ln(A_{t-l}/D)}{\sqrt{s-(t-l)}}\right) \tag{4}$$

where $N(\cdot)$ is the cumulative density function for a standard normal random variable.

Let $a_s = \sigma_a\sqrt{s-(t-l)}$. Then we can rewrite the survival probability $\psi(t, s)$ as

$$\psi(t, s) = \frac{p(s)}{p(t)} \tag{5}$$

where

$$p(s) = N\left(-\frac{a_s}{2} + \frac{\ln(A_{t-l}/D)}{a_s}\right) - dN\left(-\frac{a_s}{2} - \frac{\ln(A_{t-l}/D)}{a_s}\right) \tag{6}$$

$$p(t) = N\left(-\frac{a_t}{2} + \frac{\ln(A_{t-l}/D)}{a_t}\right) - dN\left(-\frac{a_t}{2} - \frac{\ln(A_{t-l}/D)}{a_t}\right) \tag{7}$$

Numerical Illustration

The common drawback of structural models with complete information is that they predict much lower short-term credit spreads than the empirically observed ones. In those models, bond investors know how far the firm's asset is from the default boundary and therefore default is never a surprise. This implies that default probability is very small for short horizons and so is the credit spread. However, in the model with lagged information, default can be a surprise event because bond investors are uncertain about the firm's current asset value.

Default probability. In this section, I show numerically how incorporating information lag in the model improves the behavior of the short-term default probability.

Let's consider the following parameter set for the firm's asset process as a basis case:

$$A_{t-l} = 100; \quad D = 75; \quad \sigma_a = 15\%$$

Exhibit 1 shows short-term default probabilities for no lag, three-month lag and six-month lag. As we can see, adding information lag drastically increase the short-term default probability. It does not affect the long-term default probability very much, which can be seen from Exhibit 2. For the complete information case where no lag is assumed, it is impossible to achieve this high short-term default probability without having an absurd long-term default probability.

Exhibit 3 shows default probability for various information lags. Even in a short-term horizon, such as three months, the likelihood of default is not a trivial magnitude. We can see that the default probability is more then 1% for a lag of four months, whereas it is almost zero in the perfect information case.

Credit Spreads. We now turn to the implication of our model for the credit spread of a defaultable bond over the risk-free rate. For simplicity, I assume the risk-free rate and the recovery rate are constant. Let the maturity of the bond be T and survival probability up to T be p. Recovery at maturity is also assumed, which means that

EXHIBIT 1
Short-Term Default Probabilities for Cases of No Lag, Three Months Lag, and Six Months Lag

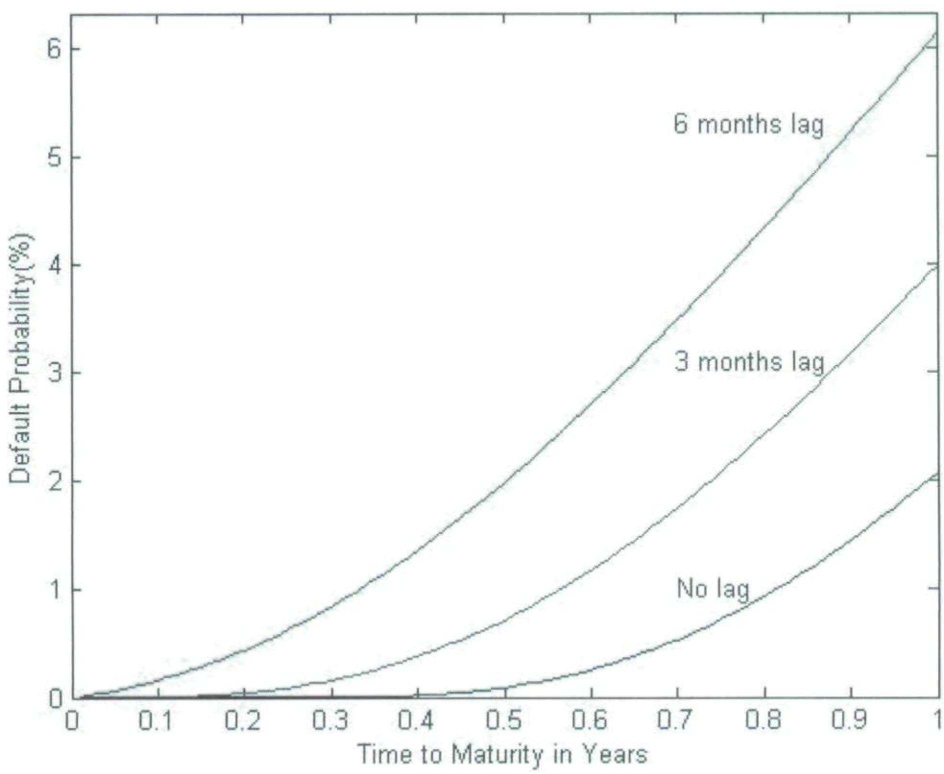


EXHIBIT 2
Default Probabilities for No Lag and Six Months Lag

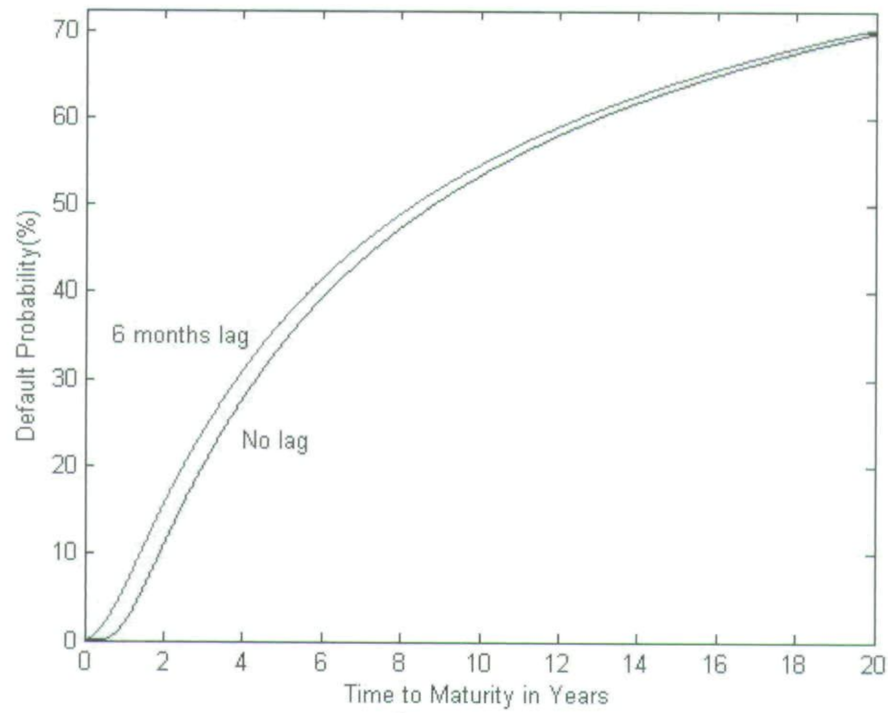
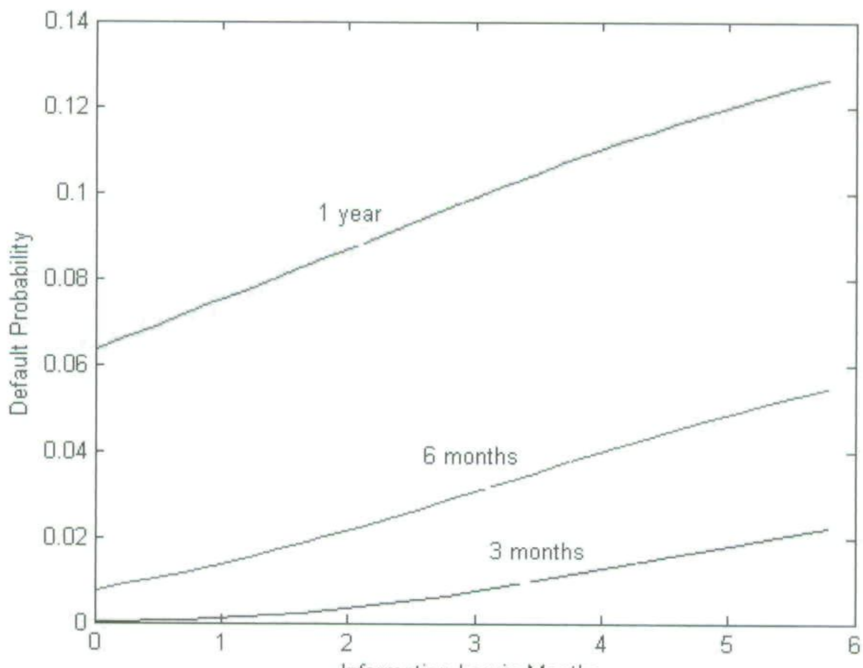


EXHIBIT 3
Probability of Default in Three Months, Six Months, and One Year



when the bond defaults at $\tau < T$, it is redeemed at R of the face value on the maturity date.

Then it can be easily shown that the credit spread S is given as follows:

$$S = -\frac{\log((1-p)R+p)}{T} \tag{8}$$

For numerical calculations, $R = 70\%$ is used for the recovery rate.

Exhibit 4 illustrates the credit spreads when information lags are zero, three and six months, respectively. The effect of the information lag on the firm's asset is clear. The short-term credit spreads are nonzero and increasing in the information lag.

Deriving an Intensity-Based Reduced-Form Model from the Lagged Information Model

We have seen numerically that our model can generate nonzero short-term spreads as the intensity-based reduced form models do. I show in this section that this model not only mimics the credit spread patterns of the reduced-form models but it actually is a reduced form model. Specifically, I prove that the model implies the existence of an intensity process, and therefore the information lag alone is sufficient to produce a reduced-form model from a structural model.

Before I start with a formal setting, I provide an intuitive meaning to intensity at default time τ. It is approximately the rate of default probability for a very short time period. For example, if the intensity at τ is h, then the probability of default during the interval $(t, t + \Delta t)$ is approximately $h\Delta t$. Mathematically,

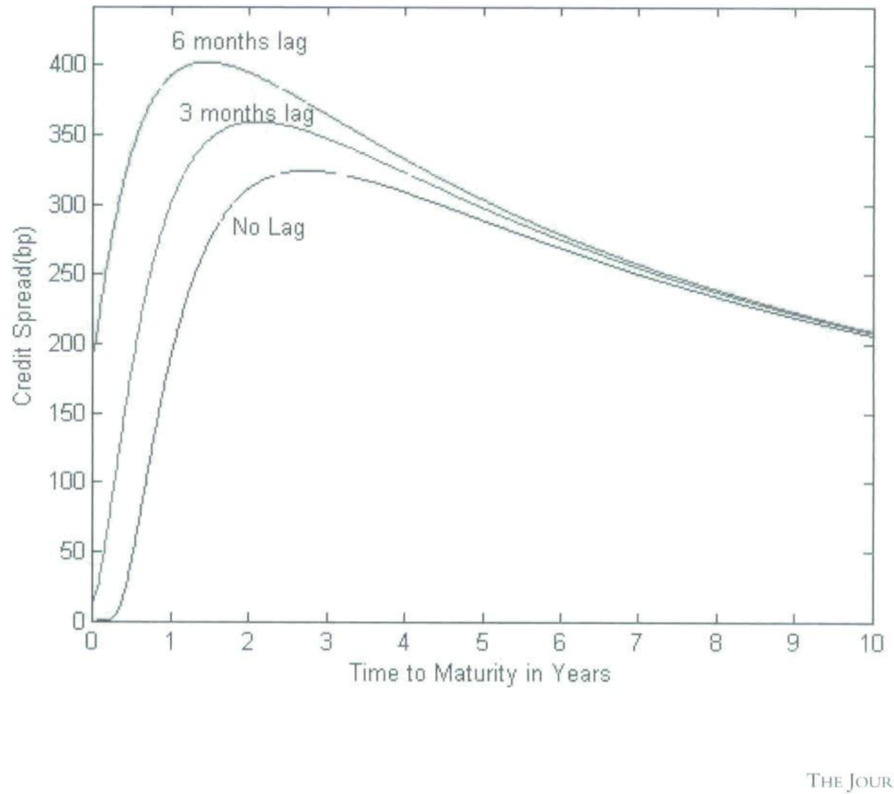
$$\lim_{\Delta t \downarrow 0} \frac{P[t < \tau \le t + \Delta t \mid \mathcal{G}_t]}{\Delta t} = h \tag{9}$$

Therefore we can just calculate the derivative of the default probability in our model for a quick check. From Equation (5),

$$\lim_{\Delta t \downarrow 0} \frac{P[t < \tau \le t + \Delta t \mid \mathcal{G}_t]}{\Delta t} = \frac{d(1-\psi(t,s))}{ds}\Big|_{s=t} = -\frac{\frac{dp(s)}{ds}\Big|_{s=t}}{p(t)} \tag{10}$$

EXHIBIT 4

Credit Spreads with No Time Lag, Three Months Lag, and Six Months Lag



From Equations (6) and (7) we can compute

$$\lim_{\Delta t \downarrow 0} \frac{P[t<\tau \leq t+\Delta t \mid \mathcal{G}_t]}{\Delta t} = \frac{\exp(-\frac{1}{2}(-\frac{\sigma_a\sqrt{l}}{2}+\frac{\ln(A_{t-l}/D)}{\sigma_a\sqrt{l}})^2)}{2\sqrt{2\pi l}\cdot p(t)} \tag{11}$$

Equation (11) tells us when the model has an intensity process and its magnitude. For any positive lag l, Equation (11) is a positive number and therefore we expect an intensity process to exist. But as $l \downarrow 0$, (11) becomes zero, which can be easily checked by l'Hôpital's Rule. Thus as long as there is a positive time delay l in the information of the firm, the model has an implicit intensity function and is a reduced-form model.

In the following, the above intuition is summarized formally.

Definition *The default time τ has an intensity process h_t with respect to the filtration $\mathcal{G}_t$ if ht is a non-negative process satisfying $\int_0^t h_s ds<\infty$ a.s. for all t, such that $\{1_{\{\tau\leq t\}}-\int_0^t h_s ds : t\geq 0\}$ is an $\mathcal{G}t$ -martingale (for details, see Bremaud [1981]).*

The following theorem says that the default stopping time in our model τ indeed has an intensity process.

Theorem *Define a process h_t* by

$$h_t=-\frac{d\phi(t,s)}{ds}\Big|_{s=t}=\frac{\exp(-\frac{1}{2}(-\frac{\sigma_a\sqrt{l}}{2}+\frac{\ln(A_{t-l}/D)}{\sigma_a\sqrt{l}})^2)}{2\sqrt{2\pi l}\,p(t)}$$

for $t\leq\tau$ and $h_t=0$ for $t>\tau$,
where

$$p(t)=N\left(-\frac{\sigma_a\sqrt{l}}{2}+\frac{\ln(A_{t-l}/D)}{\sigma_a\sqrt{l}}\right)$$
$$-dN\left(-\frac{\sigma_a\sqrt{l}}{2}-\frac{\ln(A_{t-l}/D)}{\sigma_a\sqrt{l}}\right)$$

Then h_t is an $\mathcal{G}_t$ -intensity process of τ.

In summary, if we assume that investors can only observe the information filtration $\mathcal{G}_t$ then the default time admits an intensity process. This implies that the lagged information model has a corresponding reduced-form model. Therefore, it is obvious that the lagged information model doesn't have drawbacks as other structural models do.

COMPARISON WITH DUFFIE AND LANDO [2001]

My model and DL's model share some common features; they both can have nonzero short-term credit spreads and generate the default intensity process endogenously. However, the econometric implications for the time series of credit spreads are fundamentally different. My model is time-homogeneous, i.e., the predictions for credit spreads are not time-dependent. This is because the time lag l is constant and information is assumed to be released continuously.[2]

On the contrary, the credit risk model of DL implies that the credit spreads are periodic and jagged. In DL, they require the two restrictions on the bond investor's information set to generate nonzero short-term credit spreads. They assume that both periodic and noisy accounting reports are available to the bond investors. The main driver of their results between the announcement dates is the periodic, and therefore lagged, accounting reports. In addition to the nonzero short-term credit spread, the periodicity also generates credit spreads that are narrower on average after the earnings announcement and become wider afterwards. The only function of the noisy accounting reports assumption is to prevent the short-term credit spread immediately after an announcement from being zero when the firm's asset value is sufficiently higher than the default barrier.

Of course, corporations do make quarterly announcements, and I do not rule out the the possibility of lumpy information shocks to the market. However, by the time the announcements are made, they will have already become lagged information themselves. It can take a few days or weeks for the actual corporate actions or events to be made public. Therefore my model does not necessarily exclude the periodic information announcement case.[3]

In the next subsections, I explain why the implied credit spreads are periodic in DL's model.

Survival Probability in DL

In DL's incomplete information setting, rather than observing the firm's exact asset value, bond investors are assumed to receive noisy and periodic information at times t_1, t_2, ..., and so forth. At each observation time t_i, there is a noisy accounting report of assets, $Y_t = \ln(A_t) + U_t$, where U_t is normally distributed and independent of A_t. And bond investors can tell whether the firm has defaulted. Formally, the information set (H_t) available to the bond investors is given as

$$\mathcal{H}_t = \{Y(t_1), \ldots, Y(t_n), 1_{\{\tau \le s\}}\} \tag{12}$$

where τ is the default event time, t is the current time, and t_n is the most recent observation time. Default boundary is D, and the firm defaults as soon as A_t crosses D.

To show that the model-implied credit spreads are periodic, we need to first verify that the default probability is periodic. Applying Bayes' rule, we can write the survival probability as

$$P[\tau \ge s | \mathcal{H}_t] = \int_D^\infty P[\tau \ge s | A_t = x, \tau > t] \times P[A_t = x | Y_t, \tau > t] dx \tag{13}$$

where $P[\tau = s | A_t = x, \tau > t]$ is the survival probability under complete information and the notation $P[A_t = x | Y_t, \tau > t]$ means the probability density function of A_t given Y_t and $\tau > t$.

Periodicity in the Predicted Credit Spreads

I will show the periodicity comes from the term $P[A_t = x | Y_t, \tau > t]$ in Equation (13). The other term in the right-hand side of Equation (13), $P[\tau = s | A_t = x, \tau > t]$, is the survival probability under complete information, therefore it does not take any periodic pattern. The main intuition is that as time goes by after the announcement, investors become more uncertain about the value of the firm and the conditional densities of A_t are more dispersed. Then on the announcement day, they obtain up-to-date, but noisy, information on the firm and the conditional density becomes narrower.

Let us first consider the case when $t = t_i$, i.e., on the announcement date. Then the term $P[A_t = x | Y_t, \tau > t]$ is

$$P[A_{t_i} = x | Y_{t_i} = y, \tau > t_i] = P[A_{t_i} = x | A_{t_i} + U_{t_i} = y, \tau > t_i] \tag{14}$$

when $t > t_i$ (after the announcement date), the investors have the noisy accounting report Y_{ti} from time t_i and know whether the firm has defaulted or not ($1_{\{\tau > t\}}$) at t.

Then,

$$P[A_t = x | Y_{t_i} = y, \tau > t] \tag{15}$$

$$= P[A_t = x | A_{t_i} + U_{t_i} = y, \tau > t] \tag{16}$$

$$= P[A_t = x | A_t + (A_{t_i} - A_t) + U_{t_i} = y, \tau > t] \tag{17}$$

$$= P[A_t = x | A_t + U_t' = y, \tau > t] \tag{18}$$

where $U_t' \equiv (A_{t_i} - A_t) + U_{t_i}$.

Comparing the right-hand side of Equations (14) and (18), we can interpret them as the following: when $t > t_i$, it is as if investors receive noisy information on the firm, $A_t + U_t'$. Since the variance of U_t' $\text{var}(U_{t_i}) + (t - t_i) \cdot \sigma_a^2$ and the variance of U_{ti} is $Var(U_{ti})$, we can view this as the noise in information at t being greater than that at t_i. Furthermore, the variance of U_t', $\text{var}(U_{t_i}) + (t - t_i) \cdot \sigma_a^2$ is an increasing and continuous function in $t \in (t_i, t_{i+1})$; the noise gradually increases as time goes by and drops suddenly after the next information release at t_{i+1}. We therefore expect the noise in information on the firm's value to be jagged and periodic.

The increase in the variance of the noise term makes the conditional density $P[A_t = x | Y_t, \tau > t]$ in Equation (13) more dispersed. It results in the decrease in the survival probability in Equation (13) because it is an integration above some constant level D. Summarizing, the jagged pattern of the noise in the information set propagates to the survival probability and therefore these credit spreads also take a jagged and periodic form, on average.

I emphasize here that they are periodic *on average*. The default probability after the earnings announcement is smaller than before the announcement, other things being equal. Of course the default probability will be higher and the credit spread will be wider on the earnings announcement, if the firm announces a sufficiently negative earnings report, e.g., Y_{t_i} is revealed to be much smaller than $Y_{t_{i-1}}$. But if the firm announces that $Y_{t_i} = Y_{t_{i-1}}$ then the credit spread will become narrower on the announcement date. Whether this phenomenon can be found from the data is an interesting future research area.

CONCLUSION

We have shown that the lagged information model possesses the property of both the structural model and the reduced-form model. The advantage of this model is that it can be derived utilizing a simple restriction on the investor's information set. In contrast to the model of DL [2001], my model does not imply any periodic pattern in credit spreads. Whether credit spreads have any periodic pattern is an open, interesting question.

APPENDIX

Proof of Theorem

The proof uses the following theorem of Aven [1985]:

Let $|Y_n(s) = \frac{1}{k_n} P[s < \tau \leq s + k_n \mid \mathcal{G}_t]$ and $k_n \downarrow 0$ as $n \to \infty$. If there exists nonnegative and measurable processes h_t and y_t such that

i. for each t

$$\lim_{n\to\infty} Y_n(t) = h_t, \ a.s.$$

ii. for each t there exists for almost all ω *and* $n_0 = n_0(t, \omega)$ *such that*

$$|Y_n(s, \omega) - h(s, \omega)| \leq y(s, \omega)$$

where $s \leq t$, $n \geq n_0$.

iii. $\int_0^t y(s)ds < \infty$, *a.s.*, $0 \leq t < \infty$.

Then $1_{\{\tau > t\}} - \int_0^t h_s ds$ is an $\mathcal{G}_t$ martingale.

Let us show if (i) is satisfied. Following the similar algebra done in the previous section, we can get

$$\frac{d\phi(t,s)}{ds}\Big|_{s=t} = \frac{\frac{dp(s)}{ds}\big|_{s=t}}{p(t)}$$

After some algebra we can compute

$$\frac{dp(s)}{ds} = \frac{1}{\sqrt{2\pi}\sigma}\left(-\frac{da_s}{ds}\right)\exp\left(-\frac{1}{2}\left(-\frac{a_s}{2} + \frac{\ln(d)}{a_s}\right)^2\right)$$

$$\frac{dp(s)}{dt}\Big|_{s=t} = \frac{1}{\sqrt{2\pi}\sigma}\left(-\frac{\sigma_a}{2\sqrt{l}}\exp\left(-\frac{1}{2}(-\frac{\sigma_a\sqrt{l}}{2} + \frac{\ln(d)}{\sigma_a\sqrt{l}}\right)^2\right)$$

Therefore,

$$-\frac{d\phi(t,s)}{ds}\Big|_{s=t} = \frac{\exp\left(-\frac{1}{2}\left(-\frac{\sigma_a\sqrt{l}}{2} + \frac{\ln(d)}{\sigma_a\sqrt{l}}\right)^2\right)}{2\sqrt{2\pi l}\,p(t)} = h_t$$

And from the definition of $Y_n(t)$,

$$-\frac{d\phi(t,s)}{ds}\Big|_{s=t} = \lim_{n\to\infty} Y_n(t)$$

which means,

$$\lim_{n\to\infty} Y_n(t) = h_t$$

Thus (i) is satisfied.

Let $-d\phi(t,s)/ds = p_2(t,s)$ and $h_t = p_2(t, t)$. Since $p(t, s)$ is continuously differentiable, there exists $u(n) \in [s, s + k_n]$ such that

$$Y_n(t) = p_2(t, u(n))$$

from the mean-value theorem. Then,

$$|Y_n(s,\omega) - h(s,\omega)|$$

$$= |p_2(t, u(n)) - p_2(t, t)|$$

$$\leq p_2(t, u(n)) + p_2(t, t)$$

$$\leq \frac{1}{p(t)}\frac{\sigma_a^2}{a_{u(n)}^3} f\left(-\frac{a_{u(n)}}{2} + \frac{\ln(d)}{a_{u(n)}}\right) + \frac{1}{p(t)}\frac{\sigma^2}{2\lambda^3} f\left(-\frac{\lambda}{2} + \frac{\ln(d)}{\lambda}\right)$$

$$\leq \frac{1}{p(t)}\frac{\sigma_a^2}{a_{u(n)}^3} f\left(-\frac{a_{u(n)}}{2} + \frac{\ln(d)}{a_{u(n)}}\right) + \frac{1}{p(t)}\frac{\sigma^2}{2\lambda^3} f\left(-\frac{\lambda}{2} + \frac{\ln(d)}{\lambda}\right)$$

$$\leq \frac{2}{p(t)}\frac{1}{2\sigma_a\sqrt{l^3}}$$

Let $y(\omega) = \frac{2}{p(t)}\frac{1}{2\sigma_a\sqrt{l^3}}$. Then $y(\omega)$ is integrable since it doesn't depend on s. Therefore (ii) and (iii) are satisfied, and this completes the proof.

ENDNOTES

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[1]Giesecke [2004] provides a summary of the two classes of models in detail.

[2]The model can easily extend to the one with time-varying lag l_t.

[3]One can generalize the model to incorporate periodic information announcements. For example, the time lag l can be modeled as an increasing function of the time to the next announcement date.

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