Optimal Derivative Strategies with Discrete Rebalancing

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In this article, we determine the optimal investment strategy with derivatives for discrete rebalancing intervals. We find that the investor buys a more conservative portfolio and strongly reduces his exposure to volatility compared to the continuous-time case. This leads to much less extreme positions in the derivatives. Even with monthly rebalancing, the investor can nevertheless realize up to almost two-thirds of the utility gain from option trading in continuous time. Variance contracts are useful due to their stable exposures to volatility and jump risk.

There is a huge body of literature on dynamic asset allocation in continuous-time models, starting with Merton [1971]. Basically all the models in this area of the literature assume that the portfolio can be rebalanced continuously. In reality, however, the investor can only trade at discrete points in time, and the more often the portfolio is rebalanced, the larger the transaction costs, especially for retail investors. An ad hoc solution to this problem is to implement the continuous-time strategy in discrete time. The performance of these naive discretization strategies is analyzed by Rogers [2001] in a Black–Scholes setup and by Branger, Breuer, and Schlag [2007] in models with stochastic volatility, jumps, and derivatives. They show that even yearly rebalancing is nearly perfect if the investor only trades the stock and the money market account. With options, however, the picture changes completely. Discrete trading can cause huge utility losses, and the investor has to trade at least daily to benefit from derivatives at all.

In this article, we solve for the optimal investment strategy in discrete time when the investor uses derivatives. With discrete trading, the market is, technically speaking, incomplete, and the choice of derivatives matters. We consider the cases where the investor trades (in addition to the stock and the money market account) one or two options, or alternatively one option and a variance contract. There is no analytical solution for the optimal investment strategy in discrete time, and we use backward induction combined with numerical optimization and a Monte Carlo simulation to solve the portfolio planning problem.

We assume a standard parametrization for the stock price and volatility dynamics, which is taken from Liu, Longstaff, and Pan [2003]. Note that we consider a partial equilibrium and an asset allocation problem and do not intend to explore the asset pricing and equilibrium implications of our setup. Of course, a representative investor in our economy would just hold the stock (which represents the market portfolio) and would have no intention to hold or trade derivative contracts. So one could interpret our setup as one where heterogeneous investors are present in the economy, and for one of these different types of investors the trading strategies analyzed in our article could indeed be optimal.

It is furthermore important to note that we analyze *optimal* strategies which generate the highest possible expected utility of terminal wealth under the restriction of discrete trading. In contrast to *naive* strategies where a trader simply applies a strategy that is optimal in continuous time, in a discrete time setting we determine the portfolio holdings in the beginning of each sub-period in a dynamically consistent optimal way. As a consequence, there is no need for a trader who is following our strategy to adjust his position during the next time interval, simply because the initial portfolio was optimal. To highlight the importance of this dynamic optimality property of our strategies it may be helpful to consider the example of an option hedge in discrete time. In this context our strategies would correspond to the case where a trader, for example, minimizes the variance of the hedging error at the end of the hedging interval. For this trader there is no need to adjust his position during the hedging period. However, when a trader naively employs a hedging strategy derived in the context of a continuous-time model, such as Black–Scholes, in a discrete-time setting, a risk measure like the portfolio delta can indeed deviate significantly from the target value of zero over the hedge interval. In this case additional trades may be warranted to bring the position delta back into an acceptable range.

We find that optimal strategies in discrete time are significantly better than naive discretization approaches. Even with one option only, an investor who rebalances his portfolio once a month can realize about 56% of the maximal utility gain due to derivatives if he chooses the option carefully. This maximal gain is the difference in utility between the case of an incomplete market with only the stock and the money market account and a complete market with sufficiently many derivatives, where the investor implements his trading strategy in continuous time. With two derivative contracts, the utility gain increases slightly to 59% of the maximal value.

The best results are achieved with longer-term options, for which the sensitivities with respect to the risk factors are relatively stable over the investment horizon. The choice of the strike price is important when the investor can only use one call, but does not matter much in the case with two options. We also consider a variance contract, for which the payoff is equal to the realized variance over some time period. With the variance contract as the only derivative, the investor can realize 59% of the maximal utility gain, and with the variance contract and an additional call option, his utility gain increases to 63%.

Compared to the optimal strategy in continuous time, the investor buys a more conservative portfolio and does not take extreme positions in derivatives. As a consequence, his exposure to the risk factors is reduced. This is particularly true for volatility risk, where the exposure shrinks to about one-third compared to the continuous-time case. With a variance contract the investor can reduce the variability in his factor exposures across different levels of volatility, so that utility gains are higher with the variance contract than with options only. The degree to which the dependence of the exposure on volatility can be reduced, however, depends on the value of spot volatility.

Another difference between discrete and continuous trading is that the optimal exposures with respect to the risk factors are independent of volatility for continuous trading but depend on volatility for discrete trading. When volatility is high and rebalancing would become more important (but is impossible), the investor chooses a more conservative portfolio. Furthermore the optimal stock position changes from short in continuous time to long in discrete time.

We check the robustness of our results with respect to the investor's degree of risk aversion, the restriction of no borrowing, and the introduction of margin requirements. For a higher risk aversion, the investor gains less in absolute terms from having access to derivatives, but he can still realize more than half of the maximal utility gain from trading derivatives in continuous time. The no-borrowing restriction is binding and economically important, since with only one option available the investor can only realize about two-thirds of the utility gain in the unrestricted case. With two options, however, it hardly matters anymore.

We are not the first to consider discrete trading strategies involving options. Brandt and Santa-Clara [2006] show how to approximate dynamic investment strategies by static strategies in timing and conditioning portfolios. They use quadratic utility and extend the basic Markowitz approach to multiple periods and the use of conditioning information. Although they argue that their method works for derivatives as well, they use only stocks, bonds, and the money market account in their numerical examples. In our study, we focus on the use of derivatives, and we do not rely on approximating strategies but rather determine the truly optimal strategy in discrete time.

Another strand of the literature analyzes the empirical performance of option strategies to assess whether options offer "too good" a risk–return tradeoff, which would indicate mispricing. Doran and Hamernick [2006] explore the historical performance of several standard option investment strategies using S&P 500 Index options. They find that, based on the Sharpe ratio, many option strategies tend to outperform the index. Goetzmann, Ingersoll, Spiegel, and Welch [2007], however, argue that the Sharpe ratio may be misleading in case of option payoffs. Driessen and Maenhout [2007] investigate the optimal position in OTM puts or ATM straddles of standard expected utility maximizers and non-expected-utility investors using historical data on the S&P 500 Index and index options. They find that nearly all types of investors should optimally take short positions in puts and straddles. This result is robust with respect to transaction costs, margin requirements, and peso problems. While these articles analyze the historical performance of option strategies, we are interested in the effect of discrete trading. We therefore use a simulation setup where options are priced correctly by assumption, and we study to what degree the investor can benefit from derivatives, even if he cannot trade continuously.

A recent article that analyzes investment in the variance contract is by Egloff, Leippold, and Wu [2007]. They determine the optimal investment in variance contracts in a model with two diffusive variance factors (but no jumps) and estimate their model using data on the S&P 500 and S&P 500 variance swap rates. The utility improvements resulting within the model are substantial, and this result is also confirmed by an empirical analysis of the strategy for the period from January 2003 to January 2006. However, one has to keep in mind that the variance swap rate declined almost monotonically over this period, so that large gains from variance contracts are not surprising. Different from their approach, we consider a model with jumps in returns and volatility, and we take discrete trading into account when searching for the optimal strategy.

MODEL SETUP

We consider a model with stochastic volatility, jumps in the stock price, and jumps in volatility. In the context of option pricing, this model is also studied in, for example, Duffie, Pan, and Singleton [2000] or Broadie, Chernov, and Johannes [2007]. Liu, Longstaff, and Pan [2003] and Branger, Schlag, and Schneider [2008] analyze its implications for portfolio planning. The setup is very general and able to capture rather well the key properties of the time series of stock index returns as well as of the cross-section of index option prices.

Eraker, Johannes, and Polson [2003] find strong evidence for jumps in volatility in the time series and the results in Broadie, Chernov, and Johannes [2007] provide support for jumps in volatility based on option prices. Eraker [2004] shows that jumps in volatility help to fit option and return data simultaneously. However, we restrict the model to constant jump sizes, both in returns and volatility, to retain comparability to the optimal portfolio strategies under continuous trading derived in the literature.

The dynamics of the stock price S_t and the local variance V_t are given by the following stochastic differential equations:

$$\begin{aligned} dS_t &= (r + \eta_1 V_t + \mu_X(\lambda^P - \lambda^Q)V_t)S_t dt \\ &\quad + \sqrt{V_t} S_t dB_t^{(1)} + S_{t-}\,\mu_X(dN_t - \lambda^P V_t dt) \end{aligned} \tag{1}$$

$$\begin{aligned} dV_t &= \kappa^P(\bar{v}^P - V_t)dt + \sigma_V\sqrt{V_t} \\ &\quad \times (\rho dB_t^{(1)} + \sqrt{1-\rho^2}\,dB_t^{(2)}) + \mu_Y(dN_t - \lambda^P V_t dt) \end{aligned} \tag{2}$$

The Poisson process N_t has a jump intensity of $\lambda^P V_t$. μ_X and μ_Y are the deterministic price and variance jump sizes. $\eta_1\sqrt{V_t}$ and $\eta_2\sqrt{V_t}$ are the risk premia for the diffusions $B_t^{(1)}$ and $B_t^{(2)}$, so that the investor receives a compensation of $\eta_1 V_t$ and $\eta_2 V_t$ for bearing one unit of $\sqrt{V_t}dB_t^{(1)}$ and $\sqrt{V_t}dB_t^{(2)}$, respectively. The difference $\mu_X(\lambda^P - \lambda^Q)V_t$ represents the instantaneous premium for jump risk.

Given these factor risk premia, the dynamics of the stock price and the volatility process under the risk-neutral measure Q are

$$dS_t = rS_t dt + \sqrt{V_t} S_t d\tilde{B}_t^{(1)} + S_{t-}\,\mu_X(dN_t - \lambda^Q V_t dt) \tag{3}$$

$$\begin{aligned} dV_t &= \kappa^Q(\bar{v}^Q - V_t)dt + \sigma_V\sqrt{V_t} \\ &\quad \times (\rho d\tilde{B}_t^{(1)} + \sqrt{1-\rho^2}\,d\tilde{B}_t^{(2)}) + \mu_Y(dN_t - \lambda^Q V_t dt) \end{aligned} \tag{4}$$

where

$$\kappa^Q = \kappa^P + \sigma_V(\rho\eta_1 + \sqrt{1-\rho^2}\eta_2) + (\lambda^P - \lambda^Q)\mu_Y$$
$$\kappa^Q\overline{v}^Q = \kappa^P\overline{v}^P.$$

Under Q, $\tilde{B}_t^{(1)}$ and $\tilde{B}_t^{(2)}$ are standard Brownian motions, and N_t is a Poisson process with intensity $\lambda^Q V_t$.

The investor can trade in the stock and the money market account. Furthermore, he has access to derivatives. The dynamics of the price of a derivative $O_t^{(i)} = g^{(i)}(t, S_t, V_t)$ are given by

$$\begin{aligned} dO_t^{(i)} &= rO_t^{(i)}dt + (g_s^{(i)}S_t + g_v^{(i)}\sigma_V\rho)(\eta_1 V_t dt + \sqrt{V_t}dB_t^{(1)} \\ &\quad + g_v^{(i)}\sigma_V\sqrt{1-\rho^2}(\eta_2 V_t dt + \sqrt{V_t}dB_t^{(2)}) \\ &\quad + \Delta g^{(i)} \times ((\lambda^P - \lambda^Q)V_t dt + dNt - \lambda^P V_t dt) \end{aligned}$$

where the exposures to the risk factors $dB_t^{(1)}$, $dB_t^{(2)}$, and dN_t follow from the sensitivities

$$g_s^{(i)} = \left.\frac{\partial g^{(i)}(t,s,v)}{\partial s}\right|_{(t,S_t,V_t)}$$
$$g_v^{(i)} = \left.\frac{\partial g^{(i)}(t,s,v)}{\partial v}\right|_{(t,S_t,V_t)}$$
$$\Delta g^{(i)} = g^{(i)}(t,(1+\mu_X)S_{t-}, V_{t-} + \mu_Y) - g^{(i)}(t, S_{t-}, V_{t-})$$

To calculate option prices and sensitivities in our model, we use Fourier inversion techniques as explained, for example, in Duffie, Pan, and Singleton [2000]. The resulting differential equations are solved numerically.

Besides options, we also consider a variance contract, which is analyzed in Bondarenko [2007]. The variance contract is equivalent to a certain portfolio of standard European options with maturity date T, which replicates the realized variance over the interval from $t = 0$ to $t = T$. The payoff of this contract is linear in variance, which makes it similar to a forward contract. When discrete returns are used to calculate the realized variance, its payoff at time T is given by

$$RV(0,T) = \int_0^T \left(\frac{dS_t}{S_{t-}}\right)^2 = \int_0^T V_u du + \int_0^T \mu_X^2 dN_u$$

These integrals would of course be replaced by discrete sums in practice, which is also what we do in our study, since the stock price and variance process have to be discretized anyway.

The variance contract is special in that its exposures to volatility risk and jump risk do not depend on the current levels of the stock price or volatility, but only on its time to maturity. We thus expect the exposure of a portfolio involving the variance contract instead of an option to be more stable over time. On the other hand, the relation between the exposure to volatility risk and jump risk is fixed for the variance contract, whereas it depends on the moneyness for options. With call options, the investor is thus able to choose the optimal mixture of these two exposures even if only one derivative is available.

METHODOLOGY

Theory

The investor derives utility from terminal wealth only, and we assume a CRRA utility function. This is the standard choice in the literature and allows us to perform comparisons with the continuous-time case, for which closed-form solutions have been derived. The level of risk aversion is γ, that is, $U(W) = W^{1-\gamma}/(1-\gamma)$. The investor maximizes his expected utility, and his indirect utility function is defined as

$$J(w,v,t) = \max_{\{\Phi_{[t,T]}\in A_t\}} E_t[U(W_T) \mid W_t = w, V_t = v]$$

$\Phi_{[t,T]}$ denotes his investment strategy over the time period between t and T. A_t is the set of admissible strategies, which depends on whether the investor can trade continuously or only at discrete points in time and on the assets available.

In the case of continuous trading and in a complete market, the solutions to the portfolio planning problem have been studied extensively in the literature. Liu and Pan [2003] derive optimal strategies for a model without jumps in volatility, and Branger, Schlag, and Schneider [2008] extend their analysis to the case of volatility jumps. When the market is complete, the investor can achieve any desired exposure to the risk factors, and it turns out to be useful to work with these exposures instead of asset positions. The dynamics of the investor's wealth can be written as

$$dW_t = rW_t dt + \theta_t^{B1} W_t(\eta_1 V_t dt + \sqrt{V_t} dB_t^{(1)}) + \theta_t^{B2} W_t(\eta_2 V_t dt + \sqrt{V_t} dB_t^{(2)}) + \theta_t^{N} W_{t-}(dN_t - \lambda^Q V_t dt) \tag{5}$$

where θ^{B1} describes the share of the investor's wealth invested in the $B1$-diffusion, θ^{B2} the position in the $B2$-diffusion, and θ^{N} the relative investment in jump risk. The corresponding positions in the assets follow from the sensitivities of the assets with respect to the risk factors.

If the investor can only trade at discrete points in time, the set of admissible strategies is reduced to those where the positions are constant between the rebalancing dates, and the market becomes incomplete. With a naive discretization strategy, the investor just applies the continuous-time strategy in discrete time, while the alternative is to determine the truly optimal strategy in discrete time.

Branger, Breuer, and Schlag [2007] find that even with naive discretization, the loss due to discrete trading can almost be neglected in the case when only the stock and the money market account are traded. However, as soon as the investor has access to derivatives, utility losses for naive strategies can become very large. Even with weekly rebalancing, there is a significant bankruptcy risk for the investor, and he is better off not using derivatives at all. Of course, bankruptcy can only occur when the number of short calls exceeds the number of units held in the stock. To actually benefit from derivatives, the investor has to trade at least daily when he relies on a naive discretization strategy.

We denote the points in time where the investor adjusts his portfolio (plus the end of the investment horizon) by $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. Δ_i denotes the time between successive reblancing points t_{i-1} and t_i. In case the rebalancing dates are equally spaced, $\Delta_i \equiv \Delta$ for all i. The dynamics of wealth are

$$W_{t_i} = W_{t_{i-1}} \left[\phi_{t_{i-1}} \frac{S_{t_i}}{S_{t_{i-1}}} + \sum_{k=1}^{K} \psi_{t_{i-1}}^{k} \frac{O_{t_i}^{k}}{O_{t_{i-1}}^{k}} + \left(1 - \phi_{t_{i-1}} - \sum_{k=1}^{K} \psi_{t_{i-1}}^{k}\right) e^{r(t_i - t_{i-1})} \right] \tag{6}$$

where $\phi_{t_{i-1}}$ and $\psi_{t_{i-1}}^{k}$ denote the weight of the stock and k-th derivative ($k = 1, \ldots, K$) at time t_{i-1}, respectively.

The optimal portfolio of the investor in discrete time solves

$$J_{\text{disc}}(w, v, t_i) = \max_{\{\phi_{t_j}, \psi_{t_j}^{k}, k=1,\ldots,K, j=i,\ldots,n-1\}} E_{t_i}[U(W_T) \mid W_{t_i} = w, V_{t_i} = v]$$

and from the Bellman principle, we know that the optimal portfolio can be found by backward induction

$$J_{\text{disc}}(w, v, t_i) = \max_{\{\phi_{t_i}, \psi_{t_i}^{k}, k=1,\ldots,K\}} E_{t_i}[J_{\text{disc}}(W_{t_{i+1}}, V_{t_{i+1}}, t_{i+1}) \mid W_{t_i} = w, V_{t_i} = v] \tag{7}$$

where the dynamics of wealth are given by Equation (6).

Implementation

For discrete trading, there is no closed-form solution for the portfolio planning problem, so that one has to resort to numerical techniques on a discretized state space. Time discretization is naturally determined by the pre-specified trading dates. For the local variance V, we use a grid with 10 equally wide intervals between 0.003 and 0.12 and another 10 between 0.12 and 0.76, respectively. For variance levels in between grid points, conditional utility levels and optimal portfolio positions are interpolated linearly. For the other dimension of the state space, expressed in option moneyness (defined as strike price divided by current stock price), we also use a grid with subsequent linear interpolation. The moneyness levels we consider range from 40% to 200% of the current stock price, and this interval is divided into 5,000 subintervals. When the variance increases above the largest or falls below the smallest value on the grid, the utility levels' and optimal portfolio weights' respective points on the boundary are used. Finally, note that in the CRRA case $J_{\text{disc}}(w, v, t) = w^{1-\gamma} J_{\text{disc}}(1, v, t)$, so that it is sufficient to determine the optimal indirect utility function and the optimal portfolio weights for $w = 1$.

The portfolio planning problem is solved by backward induction. Starting at time $t_n \equiv T$ with a known indirect utility J, we have to find the optimal portfolio consisting of the stock, the money market account, and various derivatives, as well as the associated indirect utility, for every grid point characterized by a point in time t_i ($i = n-1, n-2, \ldots, 1, 0$) and a variance level v. The numerical optimization uses a sequential quadratic programming method. The optimizations are each carried out with two different sets of starting values, once with the optimal allocation from the previous variance grid point and once with the stock position initialized to one and the derivative positions initialized to zero. This was done to ensure the numerical accuracy of the output of the optimization routine. The results of our optimization were only considered reliable when they were virtually identical for the two sets of starting vaues.

Expected utility for a given candidate portfolio composition is calculated by Monte Carlo simulation, where we simulate the variance and the wealth level at time t_{i+1} and then rely on the already known indirect utility at time t_{i+1} (which may have to be interpolated), using Equation (7). The Monte Carlo simulation is based on 500,000 paths. To gain efficiency, we tabulate the prices of the options at the end of each period when they are sold using the above-mentioned variance and moneyness grids. Prices between the grid points are interpolated linearly. For the numerical examples, we use the parameter values from Liu, Longstaff, and Pan [2003] shown in Exhibit 1.

Performance Measurement

Our objective is to assess the performance of optimal trading strategies in discrete time. Furthermore, we want to identify which derivatives the investor should trade and how much he benefits from an increase in the frequency of trading. To do so, we measure the improvement of the investor's expected utility compared to the annualized percentage difference in certainty equivalent wealth between the strategy we want to evaluate and a reference strategy. This measure has been used, for example, in Liu and Pan [2003].

The first benchmark we consider is the optimal strategy with continuous trading in a complete market. This strategy gives the maximal expected utility the investor can achieve. Its *certainty equivalent wealth* $\mathcal{W}_{cont,w}$ is defined implicitly via

$$J_{cont,w}(W_0, V_0, 0) = \frac{\mathcal{W}_{cont,w}^{1-\gamma}}{1-\gamma}$$

Analogously, the certainty equivalent wealth $\mathcal{W}_{disc,w}$ of the discrete strategy with derivatives is given implicitly via

$$J_{disc,w}(W_0, V_0, 0) = \frac{\mathcal{W}_{disc,w}^{1-\gamma}}{1-\gamma}$$

The improvement the investor could achieve by continuous trading instead of discrete trading is then defined as the annualized percentage difference in certainty equivalent wealth,

$$\mathcal{R}_{disc \to cont,w} = \frac{\ln(\mathcal{W}_{cont,w} / \mathcal{W}_{disc,w})}{T} \cdot 100 \qquad (8)$$

This number will be positive, since the discrete-time strategies cannot be better than the optimal continuous-time strategies.

The second benchmark we consider is a discrete strategy where the investor trades the stock and the money market account only. The certainty equivalent wealth for this case without derivatives is denoted by $\mathcal{W}_{disc,w/o}$. The rebalancing frequency is now the same for the strategies with and without derivatives, and the utility improvement due to the availability of derivatives is measured by

$$\mathcal{R}_{disc,w/o \to w} = \frac{\ln(\mathcal{W}_{disc,w} / \mathcal{W}_{disc,w/o})}{T} \cdot 100 \qquad (9)$$

EXHIBIT 1
Calibrated Parameters for the SVCJ Model

κ^P	5.3
$\bar{v}^P$	0.02172
λ^P	1.84156
σ_V	0.22478
ρ	−0.57
μ_X	−0.25
μ_Y	0.22578
η_1	1.38117
η_2	−2.0
λ^Q	7.36623
κ^Q	3.50630
$\bar{v}^Q$	0.03283

This number will also be positive since the investor cannot be worse off with additional investment possibilities. Note that this is not true for naive discretization strategies like those analyzed in Branger, Breuer, and Schlag [2007]. There, the improvement due to additional assets could be more than offset by the utility loss from using a trading strategy which is optimal in continuous time but not in discrete time.

The sum $\mathcal{R}_{total} \equiv \mathcal{R}_{disc,w/o \to w} + \mathcal{R}_{disc \to cont,w}$ gives the utility gain when going from the optimal discrete-time strategy, with only the stock and the money market account, to the optimal continuous-time strategy in a complete market. This sum only depends on the rebalancing frequency and decreases the more often the portfolio is rebalanced, but it does not depend on the choice of derivatives used to complete the market. The fraction $\mathcal{R}_{disc,w/o \to w}/\mathcal{R}_{total}$ then shows what percentage of the total utility gain due to derivatives can already be realized with discrete trading.

BASIC RESULTS

Strategies with Call Options

We first consider the case where the investor uses only one type of call option and rebalances his portfolio once a month. The results for a risk aversion of $\gamma = 3$ are given in Exhibit 2. We also performed the analysis for $\gamma = 8$, and the results were qualitatively similar.

Note first that the optimal position in the call is always negative, irrespective of moneyness and maturity. The highest utility gains are obtained for options with a time to maturity of six months and a moneyness of $M = 1.04$ and for a time to maturity of one year and $M = 1.16$. In these two cases, $\mathcal{R}_{disc,w/o \to w}$ is approximately equal to 1.85%, i.e., the investor's utility gain corresponds to a risk-free annualized excess return of 1.85% compared to the optimal strategy without options. As can be seen from the last column in the table, this represents about 56% of the maximal gain $\mathcal{R}_{total}$. For shorter term options, the corresponding numbers are generally smaller.

The poor performance of options with one month to maturity can be explained intuitively. Options allow the investor to achieve an exposure to volatility risk (which he cannot obtain with the stock) and to unbundle the fixed relation between diffusion risk and jump risk embedded in the stock. With a rebalancing frequency of one month and a time to maturity of the option of one month, the investor holds the option until expiry, where the exposure to volatility risk has vanished completely and where the exposure to the other two risk factors is either equal to that of the stock or equal to zero. The actual exposure of the portfolio will thus move away from the initial one rather fast, which reduces the performance of the strategy. For an option with longer time to maturity, the exposures will be much more stable over time.

Another insight from the numbers presented in Exhibit 2 is that the optimal moneyness of the call changes with increasing time to maturity. While an ATM call is optimal for 3 months to maturity, the investor should use an OTM call with $M = 1.04$ for 6, and one with $M = 1.16$ for 12 months to maturity. The intuition for this dependence on the moneyness is that the short position in the call serves two purposes. First the investor wants a negative exposure to volatility risk due to the negative volatility risk premium, and, second, he wants to deviate from the fixed relation between jump risk and stock diffusion risk offered by the stock. In our example, the investor needs additional negative exposure to jump risk. To achieve these two goals, the investor has to choose a call with the appropriate ratio of jump risk exposure to $B2$ exposure after hedging for $B1$ exposure, and this ratio depends both on the time to maturity and on the moneyness of the call.

The results for the case when the investor can include two options in his portfolio are shown in Exhibit 3, where we have assumed that one of the options is the ATM call. The maximum utility gain is hardly larger than with one option. The largest value for $\mathcal{R}_{disc,w/o \to w}$ of about 1.99% is again obtained using calls with a time to maturity of six months or one year. The strike price for the second option is less important, since utility differences between different strike prices are very small.

It is not surprising that the utility gain becomes larger when the investor rebalances his portfolio more often. We redid the analysis for rebalancing intervals from three months down to two weeks, and found that the maximal annualized additional return grows from 1.34% for quarterly to 2.19% for bi-weekly rebalancing. As one may expect, the optimal portfolio also becomes more extreme for more frequent rebalancing, since the investor has to worry less about the stability of the factor exposures.

EXHIBIT 2

Optimal Initial Asset Allocation with Stock, Money Market Account, and Call Option (Six-Month Investment Horizon, Monthly Rebalancing)

M	ϕ_0^*	ψ_0^*	θ_0^{B1}	θ_0^{B2}	θ_0^N	$\mathcal{R}_{disc\to cont,w}$	$\mathcal{R}_{disc,w/o\to w}$	$\frac{\mathcal{R}_{disc,w/o\to w}}{\mathcal{R}_{total}}$
Option Maturity: 1 Month								
1.00	1.282	−0.025	0.629	−0.081	−0.297	2.529	0.811	0.243
0.96	1.461	−0.055	0.528	−0.050	−0.312	2.295	1.045	0.313
0.92	1.615	−0.096	0.552	−0.025	−0.312	2.254	1.086	0.325
Option Maturity: 3 Months								
1.08	1.328	−0.031	0.599	−0.230	−0.313	2.130	1.210	0.362
1.04	1.502	−0.053	0.505	−0.220	−0.336	1.788	1.552	0.465
1.00	1.573	−0.077	0.454	−0.165	−0.332	1.759	1.581	0.473
0.96	1.658	−0.109	0.451	−0.114	−0.326	1.875	1.466	0.439
0.92	1.815	−0.156	0.478	−0.079	−0.327	1.982	1.358	0.407
Option Maturity: 6 Months								
1.20	1.237	−0.029	0.624	−0.252	−0.308	2.253	1.087	0.325
1.16	1.387	−0.043	0.574	−0.277	−0.331	1.932	1.409	0.422
1.12	1.455	−0.059	0.493	−0.266	−0.332	1.660	1.681	0.503
1.08	1.626	−0.084	0.448	−0.258	−0.352	1.514	1.827	0.547
1.04	1.711	−0.109	0.421	−0.219	−0.351	1.494	1.846	0.553
1.00	1.728	−0.133	0.416	−0.169	−0.335	1.585	1.755	0.525
0.96	1.889	−0.179	0.428	−0.140	−0.341	1.671	1.669	0.500
0.92	1.915	−0.215	0.453	−0.102	−0.322	1.831	1.509	0.452
Option Maturity: 1 Year								
1.20	1.624	−0.101	0.432	−0.253	−0.344	1.536	1.805	0.540
1.16	1.698	−0.118	0.443	−0.231	−0.348	1.486	1.854	0.555
1.12	1.609	−0.124	0.430	−0.186	−0.319	1.506	1.834	0.549
1.08	1.849	−0.169	0.415	−0.192	−0.347	1.523	1.817	0.544
1.04	1.849	−0.191	0.417	−0.160	−0.331	1.599	1.742	0.521
1.00	2.012	−0.239	0.426	−0.147	−0.338	1.827	1.513	0.453
0.96	2.033	−0.272	0.443	−0.120	−0.322	1.753	1.587	0.475
0.92	2.171	−0.334	0.448	−0.105	−0.317	1.827	1.513	0.453

Note: The optimal asset allocations and the associated indirect utility values are found by backward induction searching over 500,000 simulated stock price paths. In the exhibit, M represents the moneyness of the call, defined as strike price divided by current stock price. ϕ_0^* and ψ_0^* denote the optimal fractions of wealth invested in the stock and the call, respectively. θ_0^{B1}, θ_0^{B2}, and θ_0^N represent the optimal exposures to stock diffusion risk, volatility diffusion risk, and jump risk at time $t = 0$. $\mathcal{R}_{disc\to cont,w}$ (expressed in percent) is defined in Equation (8) and measures the utility gain from continuous instead of discrete rebalancing. $\mathcal{R}_{disc,w/o\to w}$ (expressed in percent) is defined in Equation (9) and measures the utility improvement due to the availability of one call option. $\mathcal{R}_{total}$ is the sum of $\mathcal{R}_{disc\to cont,w}$ and $\mathcal{R}_{disc,w/o\to w}$.

EXHIBIT 3

Optimal Initial Asset Allocation with Stock, Money Market Account, and Two Call Options (Six-Months Investment Horizon, Monthly Rebalancing)

M_1	ϕ_0^*	$\psi_{1,0}^*$	$\psi_{2,0}^*$	θ_0^{B1}	θ_0^{B2}	θ_0^{N}	$\mathcal{R}_{disc \to cont,w}$	$\mathcal{R}_{disc,w/o \to w}$	$\frac{\mathcal{R}_{disc,w/o \to w}}{\mathcal{R}_{total}}$
Option Maturity: 1 Month									
1.01	1.310	0.032	–0.063	0.607	–0.063	–0.298	2.459	0.881	0.264
0.97	1.493	–0.061	0.007	0.541	–0.053	–0.322	2.312	1.028	0.308
0.93	1.537	–0.069	–0.007	0.517	–0.046	–0.312	2.249	1.092	0.327
0.89	1.624	–0.093	–0.011	0.557	–0.046	–0.308	2.275	1.065	0.319
0.85	1.628	–0.107	–0.015	0.553	–0.056	–0.295	2.381	0.959	0.287
0.81	1.747	–0.139	–0.017	0.582	–0.061	–0.298	2.443	0.897	0.269
Option Maturity: 3 Months									
1.09	1.574	0.001	–0.079	0.444	–0.163	–0.331	1.572	1.769	0.529
1.05	1.562	–0.011	–0.059	0.486	–0.179	–0.335	1.735	1.605	0.481
1.01	1.459	–0.180	0.124	0.484	–0.189	–0.322	1.703	1.637	0.490
0.97	1.591	0.041	–0.112	0.460	–0.187	–0.342	1.679	1.662	0.497
0.93	1.711	0.003	–0.089	0.457	–0.187	–0.360	1.662	1.678	0.502
0.89	1.402	0.075	–0.100	0.488	–0.191	–0.331	1.706	1.634	0.489
0.85	1.374	0.089	–0.099	0.463	–0.195	–0.335	1.720	1.621	0.485
0.81	1.453	0.074	–0.093	0.466	–0.190	–0.346	1.702	1.639	0.491
Option Maturity: 6 Months									
1.09	1.596	–0.063	–0.026	0.432	–0.244	–0.341	1.532	1.808	0.541
1.05	1.553	–0.179	0.113	0.462	–0.257	–0.347	1.465	1.876	0.562
1.01	1.390	–1.575	1.554	0.459	–0.274	–0.340	1.420	1.921	0.575
0.97	1.516	–0.576	–0.612	0.429	–0.266	–0.358	1.391	1.950	0.584
0.93	1.578	0.307	–0.339	0.426	–0.264	–0.374	1.354	1.987	0.595
0.89	1.142	0.416	–0.322	0.453	–0.271	–0.353	1.381	1.959	0.587
0.85	1.009	0.437	–0.280	0.454	–0.266	–0.358	1.377	1.963	0.588
0.81	0.735	0.552	–0.272	0.442	–0.273	–0.365	1.370	1.970	0.590
Option Maturity: 1 Year									
1.09	1.537	–0.414	0.379	0.430	–0.273	–0.363	1.403	1.937	0.580
1.05	1.316	–0.991	1.024	0.436	–0.271	–0.353	1.409	1.932	0.578
1.01	1.247	–6.401	6.479	0.418	–0.280	–0.363	1.371	1.970	0.590
0.97	1.092	2.696	–2.553	0.427	–0.273	–0.362	1.364	1.977	0.592
0.93	0.829	1.589	–1.337	0.422	–0.276	–0.362	1.358	1.983	0.594
0.89	0.749	1.217	–0.912	0.423	–0.264	–0.367	1.351	1.989	0.596
0.85	0.598	1.117	–0.730	0.417	–0.256	–0.370	1.354	1.987	0.595
0.81	–0.029	1.354	–0.692	0.414	–0.260	–0.366	1.369	1.972	0.590

Note: The optimal asset allocations and the associated indirect utility values are found by backward induction searching over 500,000 simulated stock price paths. In the exhibit, M_1 represents the moneyness of the first call, defined as strike price divided by current stock price, the second call is at the money ($M_2 = 1$). ϕ_0^*, $\psi_{1,0}^*$, and $\psi_{2,0}^*$ denote the optimal fractions of wealth invested in the stock and the two calls, respectively. θ_0^{B1}, θ_0^{B2}, and θ_0^{N}, represent the optimal exposures to stock diffusion risk, volatility diffusion risk, and jump risk at time $t = 0$. $\mathcal{R}_{disc \to cont,w}$ (expressed in percent) is defined in Equation (8) and measures the utility gain from continuous instead of discrete rebalancing. $\mathcal{R}_{disc,w/o \to w}$ (expressed in percent) is defined in Equation (9) and measures the utility improvement due to the availability of the two call options. $\mathcal{R}_{total}$ is the sum of $\mathcal{R}_{disc \to cont,w}$ and $\mathcal{R}_{disc,w/o \to w}$.

Interpretation: Factor Exposures

To interpret the findings, we take a look at the exposure of the strategies to the different risk factors. The initial exposures to the three risk factors $B1$, $B2$, and N shown in the tables are computed for the case when the local variance V is equal to its long-run mean. They are compared to two benchmarks. The first is given by a continuous trading strategy in an incomplete market where only the stock and the money market account are used. In this case, the optimal exposures at $t = 0$ are given by $\theta^{B1} = 0.81$, $\theta^{B2} = 0$, and $\theta^{N} = -0.20$. The second benchmark is represented by the optimal exposure in a complete market, where we have $\theta^{B1} = 0.51$, $\theta^{B2} = -0.73$, and $\theta^{N} = -0.42$ (see Liu, Longstaff, and Pan [2003] and Branger, Schlag, and Schneider [2008]).

In case of discrete trading the investor faces a tradeoff between two objectives. First, the factor risk exposures of his portfolio should be stable over time. Second, with CRRA utility and $\gamma > 1$, the utility of the investor is unbounded from below, and he is particularly afraid of ending up with very low levels of terminal wealth.

With two options, the investor could always set up a portfolio with an initial exposure equal to that in the continuous-time case. The results in Exhibit 3, however, show that the initial exposure with discrete rebalancing is smaller in absolute terms than in the continuous case. The differences are particulary large for the exposure to diffusive volatility risk, where we observe exposures of less than -0.3 in absolute value, compared to more than -0.7 in the case of continuous rebalancing. The exposures to both jump risk and diffusive stock price risk, on the other hand, are quite close to the continuous-time case.

Since the exposure to volatility risk can only be achieved via options, these positions are consequently also much smaller in absolute terms for discrete trading, which can be seen from Exhibit 4. The primary rationale for this is that the investor wants to ensure more stable exposures over time, the cost of which is to forgo a certain part of the volatility risk premium.

Interestingly, also the optimal position in the stock differs significantly between continuous and discrete trading. From Exhibit 4 it can be seen that, with continuous rebalancing, a short position in the stock is optimal, irrespective of the choice of options. In the discrete model, where the investor can only rebalance monthly, a long position is optimal. To get the intuition, assume that the optimal portfolio is put together step by step. First, the investor uses a long position in the stock to achieve the desired positive $B1$ exposure, and this also creates a certain jump exposure. Second, the $B2$ exposure and the remaining jump exposure (which are both negative in our example) are achieved by a portfolio of two options which is hedged by a position in the stock so that it has zero $B1$ exposure. This induces a long position in the call with the lower strike price and a short position in the call with the higher strike price. The hedge consists of a long position in the stock if the volatility risk exposure is moderate. This is the case with discrete trading, where the resulting overall stock position is thus positive. On the other hand, for a large volatility risk exposure, as in the case of continuous trading, a short position in the stock is needed to hedge the $B1$ exposure of the options, which results in an overall short position in the stock.

Until now we have only looked at the initial factor exposures when volatility is equal to the long-run mean. Exhibit 5 now shows the exposures conditional on the remaining number of time periods (months) and on the spot level of volatility. We consider the case with two options, both with a time to maturity of one year, and initial moneyness values $M_1 = 0.89$ and $M_2 = 1$. This represents the optimal combination from Exhibit 3. There is a slight tendency for the optimal $B1$ and N exposures to decrease in absolute terms with a decreasing remaining investment horizon, although this effect is rather weak. The optimal exposure to $B2$ risk remains more or less constant, so that, overall, time is not an important explanatory factor for exposures.

On the other hand, the dependence of the optimal factor exposures on the current level of volatility is much more pronounced. This is somewhat surprising given that the optimal exposure with continuous rebalancing does not depend on volatility at all. The demand for $B1$ exposure increases in volatility, and for very high levels of volatility, it even exceeds the optimal exposure of 0.51 from the continuous case. The demands for $B2$ and N decrease in volatility in absolute terms and approach the optimal exposure from a continuous time model for low volatility levels (-0.73 and -0.42, respectively). To get the intuition here, consider the jump component first. The intensity of jumps is proportional to the level of variance, so that the risk of multiple jumps until the next rebalancing date increases in variance. Since jumps might drive the investor into bankruptcy, he will reduce his jump exposure when volatility is high. Second, the exposure of

EXHIBIT 4

Optimal Initial Asset Positions for Continuous and Monthly Rebalancing

M_1	ϕ_0^*	$\psi_{1,0}^*$	$\psi_{2,0}^*$	$\theta^{B1}(\phi)$	$\theta^{N}(\phi)$	$\theta^{B1}(\psi_1)$	$\theta^{B2}(\psi_1)$	$\theta^{N}(\psi_1)$	$\theta^{B1}(\psi_2)$	$\theta^{B2}(\psi_2)$	$\theta^{N}(\psi_2)$
					Continuous Rebalancing						
1.09	–0.680	–2.042	2.867	–0.680	0.170	–17.827	–2.490	1.394	19.013	1.758	–1.982
1.05	–1.212	–4.646	5.684	–1.212	0.303	–35.972	–4.218	3.208	37.690	3.486	–3.929
1.01	–1.903	–30.019	31.336	–1.903	0.476	–205.391	–19.950	20.767	207.800	19.218	–21.661
0.97	–2.809	13.227	–11.536	–2.809	0.702	79.814	6.343	–9.095	–76.498	–7.075	7.974
0.93	–4.006	7.665	–5.074	–4.006	1.002	40.788	2.623	–5.202	–36.276	–3.355	3.782
0.89	–5.599	6.743	–3.860	–5.599	1.400	31.699	1.635	–4.486	–25.594	–2.367	2.668
0.85	–7.732	6.975	–3.144	–7.732	1.933	29.084	1.196	–4.525	–20.847	–1.928	2.173
0.81	–10.593	7.908	–2.760	–10.593	2.648	29.399	0.960	–4.974	–18.301	–1.693	1.908
					Monthly Rebalancing						
1.09	1.369	–0.468	0.474	1.369	–0.342	–4.082	–0.571	0.319	3.143	0.291	–0.328
1.05	1.417	–0.950	0.960	1.417	–0.354	–7.352	–0.862	0.655	6.365	0.589	–0.663
1.01	1.138	–6.840	6.954	1.138	–0.285	–46.790	–4.547	4.728	46.103	4.266	–4.803
0.97	0.968	2.972	–2.787	0.968	–0.242	17.933	1.426	–2.043	–18.476	–1.709	1.925
0.93	0.719	1.753	–1.449	0.719	–0.180	9.327	0.600	–1.189	–9.609	–0.889	1.001
0.89	0.812	1.210	–0.918	0.812	–0.203	5.690	0.294	–0.805	–6.086	–0.563	0.634
0.85	0.723	1.054	–0.711	0.723	–0.181	4.395	0.181	–0.684	–4.711	–0.436	0.491
0.81	0.099	1.267	–0.662	0.099	–0.025	4.711	0.154	–0.797	–4.391	–0.406	0.457

Note: The exhibit shows the positions in the stock and two call options as well as their contributions to the factor exposures at $t = 0$ for continuous and monthly rebalancing. The B2 exposure generated by the stock is always equal to zero and therefore not shown. The time to maturity of the options is one year, the investment horizon is six months. The moneyness of the first call option is given in Column 1 as M_1, the second option is an ATM call ($M_2 = 1$).

the portfolio to volatility risk is less stable over time when volatility is high, and the investor reduces his optimal investment into risky assets. The reduced exposure to volatility risk and jump risk reduces the expected excess return on the portfolio. This induces him to slightly increase the position in the stock (which offers a stable risk exposure) and so the $B1$ exposure increases in volatility.

The relative weights of the assets in a portfolio can change over time due to differences in their returns. If optimal allocations were constant, all one would have to do is to bring back the portfolio composition to its original value. With stochastic volatility, however, allocations also have to be adjusted for the difference between current and initial volatility. The question that arises here is which share of the utility gain is due to the first step and which to the second. So we carried out only the first step—i.e., we just rebalanced the portfolio at each trading date back to the allocations which would be optimal for a volatility level equal to the long-run mean. For an investment horizon of six months, monthly rebalancing, and a call option with $M = 1.08$, the value of adjusting the portfolio to the current volatility amounts to 12% of the utility gain $\mathcal{R}_{disc,w/o \to w}$. Rebalancing every two months increases this share to 19%, and for quarterly portfolio adjustments it reaches 22%. So the value of taking volatility as a state variable explicitly into account becomes larger when the portfolio is rebalanced less often.

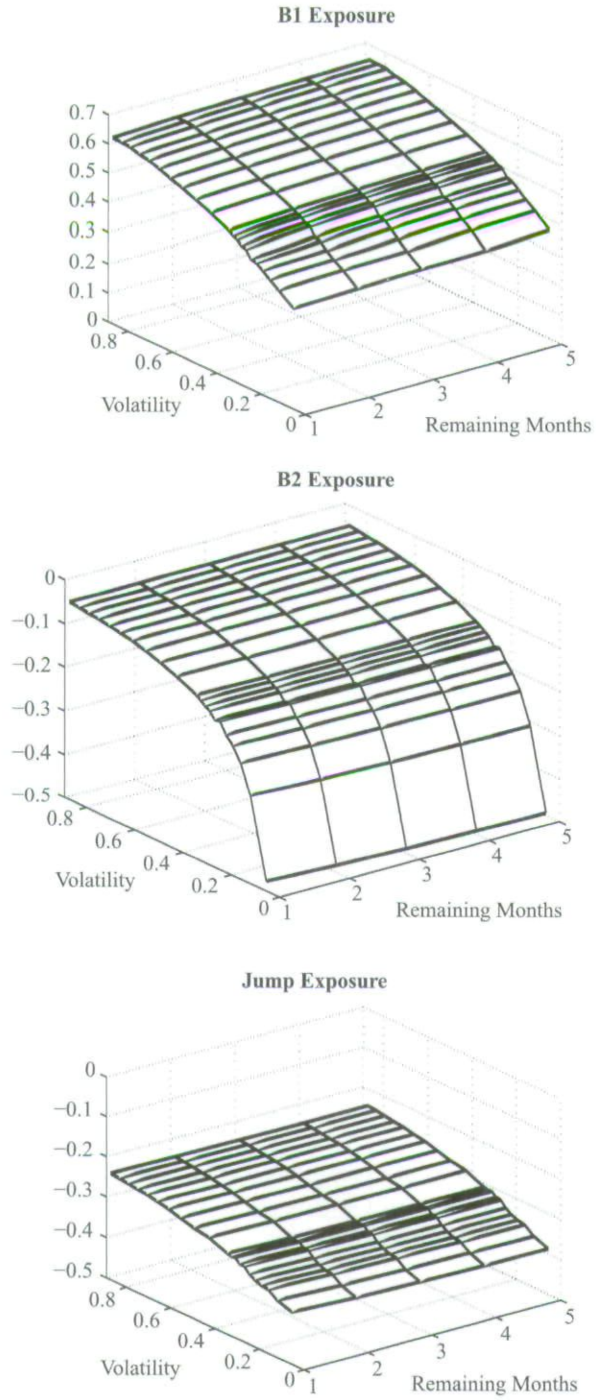
FURTHER ANALYSES

Variance Contract

Up to now, the investor had only access to call options. Now we assume that the variance contract described previously is available. The question is which of the two types of contracts is the better choice for the investor. The results of the performance analysis are given in Exhibit 6.

In the first case, the investor is only allowed to trade the variance contract—no options. The optimal $\mathcal{R}_{disc,w/o \to w}$ is 1.98% for a maturity of one year. The utility gain is thus greater than the one obtained with just one call option, where Exhibit 2 shows a maximum utility gain of 1.85% across all option maturities considered. Like for call options, the investor is best off if he uses claims with longer maturities.

EXHIBIT 5
Factor Exposures with Monthly Rebalancing (Two Call Options)



The exhibits show the optimal exposures θ^{B1}, θ^{B2}, and θ^{N} conditional on the number of remaining months and on spot volatility. Both options have a time to maturity of one year and moneyness values of $M_1 = 0.89$ and $M_2 = 1$, respectively. The length of the investment horizon is six months.

The reason for the superiority of the variance contract is that its factor exposures do not change when volatility changes. Deviations of realized factor exposures from optimal factor exposures will thus be small. Therefore, the investor will take a larger position and earn a larger part of the risk premia even when volatility is high.

In the case when a call option is available, the best results are obtained for a maturity of six months and $M = 1.12$ for the call, as well as for a maturity of one year and $M = 1.16$. In both cases the investor can realize about 63% of the utility gain due to derivatives in a continuous-time model. This value is again higher than with the optimal choice of two call options. For the optimal portfolio, the investor takes a short position in the variance contract, while the position in the call changes from a short position for low volatility levels to a long position for high volatility levels.

Exhibit 7 shows the factor exposure of the optimal portfolios as a function of the remaining investment horizon and volatility. With the variance contract, the exposure to jump risk is much closer to the optimum from the continuous-time model. For volatility risk the effect is mixed. For low spot volatilities, the amount of volatility risk is larger with options only, while for higher spot volatilities it is larger with the variance contract. Furthermore, the dependence of the optimal exposure on volatility is much less pronounced than with two calls, so that the desire to rebalance the portfolio due to volatility changes is much weaker.

Transaction Costs

Transaction costs reduce the overall utility and tend to make less frequent rebalancing more attractive. In this section, we analyze the impact of transaction costs on options for values between 0.5% and 3% of the option price for a one-way transaction. Bakshi, Cao, and Chen [1997] consider round-trip transaction costs of 6% for OTM puts and 4% for ATM options. Transaction costs for the stock and the money market account are set to zero. Our optimization setup requires that the whole option position is turned over at every rebalancing date and replaced by a position in options with the original maturity and a defined strike price. This induces the somewhat extreme assumption that the investor actually liquidates all assets in the portfolio before setting up the new position, implying that we are likely to overestimate the impact of transactions costs for options.

EXHIBIT 6

Optimal Initial Asset Allocation with Stock, Money Market Account, and Variance Contract (Six-Months Investment Horizon, Monthly Rebalancing)

M	ϕ_0^*	$\psi_{v,0}^*$	$\psi_{2,0}^*$	θ_0^{B1}	θ_0^{B2}	θ_0^N	$\mathcal{R}_{disc \to cont,w}$	$\mathcal{R}_{disc,w/o \to w}$	$\frac{\mathcal{R}_{disc,w/o \to w}}{\mathcal{R}_{total}}$
variance contract only, time to maturity 3 months									
	0.351	–0.025		0.434	–0.120	–0.402	1.501	1.840	0.551
variance contract only, time to maturity 6 months									
	0.324	–0.047		0.427	–0.149	–0.409	1.377	1.963	0.588
variance contract only, time to maturity 1 year									
	0.304	–0.088		0.410	–0.152	–0.389	1.362	1.979	0.592
variance contract and call option, time to maturity 3 months									
1.08	0.583	–0.011	–0.022	0.406	–0.189	–0.423	1.355	1.986	0.594
1.04	0.792	–0.019	–0.024	0.409	–0.193	–0.426	1.308	2.032	0.608
1.00	0.848	–0.013	–0.031	0.442	–0.128	–0.350	1.431	1.909	0.572
0.96	0.426	–0.023	–0.005	0.449	–0.118	–0.396	1.475	1.865	0.558
0.92	0.142	–0.028	0.025	0.449	–0.122	–0.409	1.483	1.858	0.556
variance contract and call option, time to maturity 6 months									
1.16	0.780	–0.025	–0.022	0.429	–0.216	–0.359	1.299	2.042	0.611
1.12	0.748	–0.034	–0.026	0.394	–0.224	–0.406	1.227	2.113	0.633
1.08	0.823	–0.030	–0.036	0.390	–0.202	–0.389	1.242	2.099	0.628
1.04	0.832	–0.032	–0.045	0.375	–0.189	–0.397	1.250	2.090	0.626
1.00	0.676	–0.036	–0.036	0.398	–0.159	–0.392	1.321	2.020	0.605
0.96	0.299	–0.047	0.001	0.412	–0.147	–0.402	1.357	1.983	0.594
0.92	0.694	–0.030	–0.045	0.458	–0.117	–0.353	1.415	1.925	0.576
variance contract and call option, time to maturity 1 year									
1.20	0.823	–0.058	–0.043	0.381	–0.207	–0.383	1.248	2.092	0.626
1.16	0.846	–0.062	–0.050	0.387	–0.205	–0.398	1.230	2.110	0.632
1.12	0.863	–0.058	–0.055	0.410	–0.182	–0.383	1.258	2.082	0.623
1.08	1.153	–0.034	–0.093	0.407	–0.164	–0.346	1.319	2.021	0.605
1.04	0.513	–0.061	–0.019	0.440	–0.121	–0.331	1.355	1.986	0.594
1.00	0.450	–0.087	–0.022	0.407	–0.162	–0.404	1.298	2.043	0.611
0.96	0.212	–0.100	0.013	0.406	–0.167	–0.416	1.308	2.033	0.609
0.92	–0.093	–0.103	0.079	0.435	–0.153	–0.395	1.335	2.006	0.600

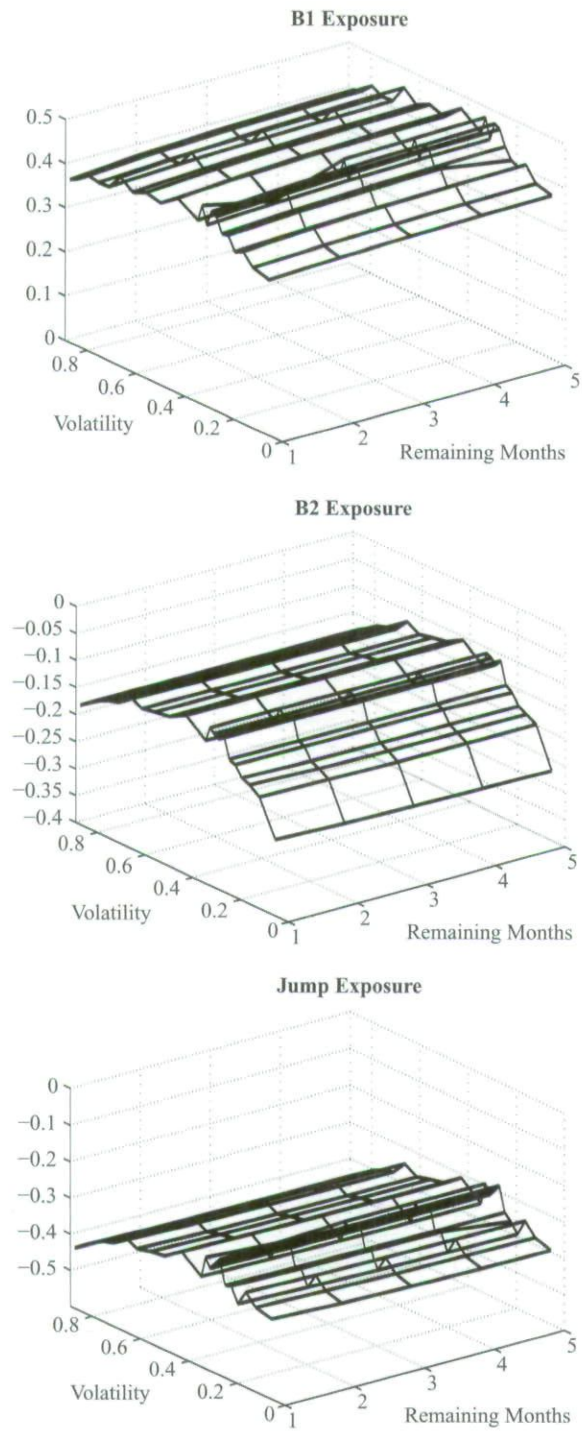
Note: The optimal asset allocations and the associated indirect utility values are found by backward induction searching over 500,000 simulated stock price paths. In the exhibit, M represents the moneyness of the call option (if available), defined as strike price divided by current stock price. ϕ_0^*, $\Psi_{v,0}^*$, *and* $\Psi_{2,0}^*$ *denote the optimal fractions of wealth invested in the stock, the variance contract, and the call option, respectively.* θ_0^{B1}, θ_0^{B2}, *and* θ_0^N *represent the optimal exposures to stock diffusion risk, volatility diffusion risk, and jump risk at time t = 0.* $\mathcal{R}_{disc \to cont,w}$ *(expressed in percent) is defined in Equation (8) and measures the utility gain from continuous instead of discrete rebalancing.* $\mathcal{R}_{disc,w/o \to w}$ *(expressed in percent) is defined in Equation (9) and measures the utility improvement due to the availability of the two call options.* $\mathcal{R}_{total}$ *is the sum of* $\mathcal{R}_{disc \to cont,w}$ *and* $\mathcal{R}_{disc,w/o \to w}$.

The following discussion is thus mainly meant to give a first impression of the impact of transaction costs on our results. A fully detailed analysis of this issue would go way beyond the scope of this article.

The results for the case of one option are given in Exhibit 8. Already moderate one-way costs of 0.5% for the option reduce the achievable $\mathcal{R}_{disc,w/o \to w}$ from 1.827% to 1.072%. However, monthly rebalancing is still better than less frequent portfolio changes. With transaction costs of 1% and 2%, respectively, rebalancing every two

EXHIBIT 7

Factor Exposures with Monthly Rebalancing (ATM Call and Variance Contract)



Note: The exhibit shows the optimal exposures θ^{B1}, θ^{B2}, and θ^{N} conditional on the number of remaining months and on spot volatility. The option and the variance contract have a time to maturity of one year, the option has a moneyness value of $M = 1.16$. The length of the investment horizon is six months.

months is optimal with $\mathcal{R}_{disc,w/o\to w}$ equal to 0.804% and 0.520%, respectively. With transaction costs of 3% it is only worthwhile to rebalance every three months with $\mathcal{R}_{disc,w/o\to w}$ equal to 0.373%. The optimal strike price of the option increases with the amount of transaction costs, since calls with higher strikes are cheaper and provide a higher exposure to volatility risk per unit of cost. Nevertheless, since our results slightly overstate the influence of transaction costs, further research is necessary to determine optimal derivative positions under more realistic assumptions on portfolio turnover.

No-Borrowing Constraint

As shown previously, the optimal portfolios can be quite extreme. To see whether our results significantly depend on unrestricted borrowing, we conduct a robustness check by not permitting the investor to take short positions in the money market account.

In the case with one option only, this no-borrowing constraint reduces the optimal $\mathcal{R}_{disc,w/o\to w}$ from the previously optimal value of about 1.85% in Exhibit 2 to about 1.17%. The investor can still realize approximately the same stock price diffusion exposure as in the unconstrained case but only much smaller volatility and jump exposures. He is not able to increase his volatility exposure further in absolute terms, since a larger short position in the call would also lower his $B1$ exposure which now cannot be compensated by leveraging the stock position.

If the investor is allowed to invest in two options, he will go long one option and short the other. Then, the required amount of borrowing is small, and the restriction causes basically no utility losses.

Margin Requirements

An investor may face margin requirements which limit the positions he can take in risky assets and, in particular, in derivatives. The structure of margin requirements and some computational details are given in the appendix. In a certain sense, margin requirements are equivalent to restrictions on how much an investor can borrow, so that the analysis here is closely related to the one in the previous subsection.

With one option, the optimal $\mathcal{R}_{disc,w/o\to w}$ is reduced to 1.83%, compared to 1.85% in the base case without margins. In terms of the optimal initial asset allocation, both the long stock position and the short call position

EXHIBIT 8

Optimal Initial Asset Allocation with Stock, Money Market Account and Call Option Under Transaction Costs (Six-Months Investment Horizon)

Δ	Optimal M	ϕ_0^*	ψ_0^*	θ_0^{B1}	θ_0^{B2}	θ_0^N	$\mathcal{R}_{disc \to cont,w}$	$\mathcal{R}_{disc,w/o \to w}$	$\frac{\mathcal{R}_{disc,w/o \to w}}{\mathcal{R}_{total}}$
				No Transaction Costs					
1	1.08	1.626	–0.084	0.448	–0.258	–0.352	1.514	1.827	0.547
2	1.12	1.544	–0.115	0.454	–0.172	–0.309	1.828	1.513	0.453
3	1.12	1.434	–0.101	0.474	–0.152	–0.291	1.999	1.341	0.401
				One-Way Transaction Costs: 0.5%					
1	1.12	1.385	–0.049	0.587	–0.220	–0.320	2.268	1.072	0.321
2	1.08	1.317	–0.053	0.569	–0.164	–0.295	2.294	1.047	0.313
3	1.04	1.278	–0.061	0.557	–0.123	–0.277	2.438	0.902	0.270
6	1.00	1.155	–0.056	0.608	–0.070	–0.249	2.692	0.649	0.194
				One-Way Transaction Costs: 1.0%					
1	1.16	1.200	–0.028	0.679	–0.177	–0.290	2.687	0.653	0.196
2	1.12	1.199	–0.034	0.639	–0.154	–0.281	2.536	0.804	0.241
3	1.08	1.186	–0.041	0.616	–0.125	–0.270	2.614	0.726	0.217
6	1.04	1.081	–0.037	0.649	–0.073	–0.245	2.788	0.552	0.165
				One-Way Transaction Costs: 2.0%					
2	1.16	1.062	–0.019	0.714	–0.119	–0.259	2.821	0.520	0.156
3	1.12	1.068	–0.023	0.685	–0.106	–0.254	2.841	0.500	0.150
6	1.08	0.997	–0.021	0.699	–0.065	–0.236	2.914	0.427	0.128
				One-Way Transaction Costs: 3.0%					
3	1.16	0.981	–0.013	0.730	–0.085	–0.240	2.967	0.373	0.112
6	1.08	0.968	–0.018	0.721	–0.054	–0.231	2.992	0.349	0.104

Note: The optimal asset allocations and the associated indirect utility values are found by backward induction searching over 500,000 simulated stock price paths. In the exhibit, Δ is the rebalancing interval in months, and optimal M represents the optimal moneyness of the call (given the rebalancing interval and the level of transaction costs), defined as strike price divided by current stock price. All options have an initial time to maturity of six months. ϕ_0^* and ψ_0^* denote the optimal fractions of wealth invested in the stock and the call, respectively. θ_0^{B1}, θ_0^{B2}, and θ_0^N represent the optimal exposures to stock diffusion risk, volatility diffusion risk, and jump risk at time $t = 0$. $\mathcal{R}_{disc \to cont,w}$ (expressed in percent) is defined in Equation (8) and measures the utility gain from continuous instead of discrete rebalancing. $\mathcal{R}_{disc,w/o \to w}$ (expressed in percent) is defined in Equation (9) and measures the utility improvement due to the availability of the two call options. $\mathcal{R}_{total}$ is the sum of $\mathcal{R}_{disc \to cont,w}$ and $R_{disc,w/o \to w}$.

become smaller, which corresponds to an increase in the stock price diffusion exposure and a reduction in the volatility and jump exposure.

With two options, the optimal $\mathcal{R}_{disc,w/o \to w}$ goes down to 1.83% compared to 1.99% without margin requirements and thus remains practically the same as with one option. Interestingly, the optimal moneyness of the second call increases from 89% to 113%. This is due to the fact that for a stock investment we require only a margin of 50% while for the in-the-money call the whole purchase price is deducted from the margin account. This, in turn, leads to an almost complete loss of the potential utility gains generated by the second option. In total, margin requirements do not exhibit a very strong influence on the investor's utility gain.

Parameter Uncertainty

In this section we want to analyze how sensitive the utility gains from derivatives are to the mis-measurement of the model parameters. We assume that the investor can infer the parameters under the risk-neutral measure Q perfectly from market data, since there are many option prices at hand to calibrate the model. We further assume that there is no uncertainty about the general model for option prices, i.e., the SVCJ model represents the true *type* of data-generating process. The investor is only uncertain about the risk premia η_1, η_2, and $\lambda^P - \lambda^Q$.

The idea behind our approach is an investor who has a Bayesian view on the world. He does not believe that the true parameters can be perfectly identified from the data, but bases his decisions on some distribution for the uncertain parameter values. Specifically, he assumes that risk premia are normally distributed with means and variances derived from empirical studies. The standard deviations used in our analysis are 0.92 for η_1, 1.10 for η_2, and 0.28 for λ^P. Furthermore, we vary the assumed mean $\hat{\mu}$ of the respective risk premium to capture that the investor's beliefs can even be away from the true value on average. The investor then derives optimal portfolio strategies conditional on his beliefs, which we evaluate on the 'true' parameter set, to quantify the impact of parameter mis-measurement of certain parameters.

For each scenario, we draw from a normal distribution of the risk premium with the given mean (which is either equal to the true one or shifted by a certain number of standard deviations) and standard deviation, and generate one stock price and volatility path. Then we repeat this procedure 500,000 times to obtain once again 500,000 stock price paths. We then solve the optimization problem as before and analyze how parameter uncertainty affects the investor's optimal allocations and the utility gain.

The results are shown in Exhibit 9. As long as the correct mean is assumed, parameter uncertainty basically causes zero utility loss, so the results are not shown in detail. Even an estimation error for the mean of half a standard deviation causes only moderate utility losses. The largest utility losses are observed when the mean for η_1 is assumed to be equal to the true value plus half a standard deviation, where $\mathcal{R}_{disc,w/o\to w}$ decreases from 1.786% to 1.698%. As one may expect the losses are larger when the mean is over- or underestimated by a full standard deviation. For η_1, $\mathcal{R}_{disc,w/o\to w}$ then falls to 1.447% and 1.497%, respectively. When η_2 is mis-estimated, the utility losses are much smaller although the standard deviation is larger and the parameters are of similar size. $\mathcal{R}_{disc,w/o\to w}$ falls to just 1.735% and 1.772%, respectively. This is due to the fact that under discrete trading the optimal position in volatility risk does not increase linearly in the absolute value of η_2. It is restricted by the stability of the factor exposures and therefore does not change much if the risk premium is changed. All in all the optimal positions are

EXHIBIT 9

Optimal Initial Asset Allocation Under Parameter Uncertainty

Parameter	Mean	M	ϕ_0^*	ψ_0^*	θ_0^{B1}	θ_0^{B2}	θ_0^N	$\mathcal{R}_{disc,w/o\to w}$
None		1.08	1.508	−0.074	0.466	−0.228	−0.329	1.786
None		1.04	1.559	−0.094	0.447	−0.189	−0.324	1.762
None		1.00	1.606	−0.118	0.450	−0.149	−0.317	1.689
None		0.96	1.655	−0.145	0.469	−0.113	−0.307	1.585
η_1	$\hat{\mu} = \mu + 0.5$std	1.08	1.526	−0.066	0.603	−0.202	−0.339	1.698
η_1	$\hat{\mu} = \mu + 1.0$std	1.08	1.537	−0.057	0.742	−0.174	−0.348	1.447
η_1	$\hat{\mu} = \mu - 0.5$std	1.08	1.497	−0.083	0.330	−0.256	−0.320	1.717
η_1	$\hat{\mu} = \mu - 1.0$std	1.08	1.479	−0.092	0.196	−0.281	−0.311	1.497
η_2	$\hat{\mu} = \mu + 0.5$std	1.08	1.490	−0.071	0.502	−0.217	−0.327	1.770
η_2	$\hat{\mu} = \mu + 1.0$std	1.08	1.458	−0.066	0.537	−0.202	−0.322	1.735
η_2	$\hat{\mu} = \mu - 0.5$std	1.08	1.540	−0.079	0.430	−0.243	−0.334	1.786
η_2	$\hat{\mu} = \mu - 1.0$std	1.08	1.560	−0.083	0.395	−0.255	−0.336	1.772
λ^P	$\hat{\mu} = \mu + 0.5$std	1.08	1.486	−0.073	0.460	−0.225	−0.324	1.771
λ^P	$\hat{\mu} = \mu + 1.0$std	1.04	1.517	−0.092	0.435	−0.184	−0.315	1.745
λ^P	$\hat{\mu} = \mu + 2.0$std	1.08	1.389	−0.068	0.443	−0.207	−0.304	1.711
λ^P	$\hat{\mu} = \mu - 0.5$std	1.08	1.532	−0.076	0.470	−0.233	−0.334	1.786
λ^P	$\hat{\mu} = \mu - 1.0$std	1.08	1.553	−0.077	0.476	−0.236	−0.339	1.774
λ^P	$\hat{\mu} = \mu - 2.0$std	1.08	1.577	−0.078	0.485	−0.239	−0.344	1.567

Note: The optimal asset allocations and the associated indirect utility values are found by backward induction searching over 500,000 stock price paths. Each path was simulated with a new draw from the distribution of the risk premia with the indicated mean and the standard deviations given in the text. μ is the true value of the risk premium parameter as given in Exhibit 1, while $\hat{\mu}$ is the mean of the distribution the simulation is based on. std stands for standard deviation. M represents the moneyness of the call, defined as strike price divided by current stock price. ϕ_0^* and ψ_0^* denote the optimal fractions of wealth invested in the stock and the call, respectively. θ_0^{B1}, θ_0^{B2}, and θ_0^N represent the optimal exposures to stock diffusion risk, volatility diffusion risk, and jump risk at time $t = 0$. $\mathcal{R}_{disc,w/o\to w}$ (expressed in percent) denotes the annualized additional return due to the availability of the option and is defined in Equation (9).

not very sensitive to a potential mis-measurement of the risk premia.

CONCLUSION

An investor will generally benefit from having access to derivatives as soon as he uses the optimal strategy in discrete time instead of a naive discretization strategy. Optimal trading strategies in discrete time can generate up to two-thirds of the maximal gain which would be obtainable from an investment in derivatives in the ideal case of continuous rebalancing. And although two derivatives are needed in the continuous-time model to complete the market, the investor can realize almost the whole utility gain with one derivative—given that he chooses the optimal strike of the option or uses the variance contract.

This result implies that the group of investors benefiting from derivatives is rather large. It also justifies why a retail investor might want to have access to derivatives. Since variance contracts yield better results than options, investors would benefit from the introduction of these contracts where they do not exist yet.

In general, the investor chooses a more conservative portfolio if he can only trade at discrete points in time. He foregoes part of the risk premia, in particular on volatility risk, to avoid extreme positions which would make the portfolio exposure more unstable over time. The best choice for the investor is to short long-term calls or the variance contract.

A short position in long-term OTM calls together with a stock investment are packaged as retail derivatives called "discount certificates" in Germany. According to our results these contracts are a good and simple way for small investors to benefit from derivatives. Hedge funds, which in general have greater resources and face lower transaction costs, can realize large utility gains by following more sophisticated strategies and by rebalancing their positions more often.

APPENDIX

Margin Requirements

The margin requirements used in this study are adapted from Interactivebrokers.com [2006]. In case of differences between day and overnight requirements, overnight requirements are used. We use relative margin requirements, i.e., we impose restrictions on the portfolio weights. Let ϕ and ψ denote the relative investments in the stock and a specific call option, respectively. Furthermore, we assume that margin requirements are only imposed at rebalancing times. As a consequence, they are independent of the stock price and current wealth, so that we can set $S_t = W_t = 1$ to simplify the exposition. Monitoring the margin in continuous time, as is the case in practice, would introduce path-dependence into the problem and would make it computationally basically intractable.

For long stock positions, the required margin $MR(S)$ is 50% of the stock price, which in our case equals the relative weight ϕ of the stock investment, i.e., $MR(S) = 0.5\phi$. Short stock positions practically never turned out to be optimal in our analysis, so there is no need to specify margin requirements for them.

For a long position in a call, the margin requirement $MR(LC)$ is given by the value of the call position, i.e., analogous to the case of the stock $MR(LC) = \psi$.

The margin requirement for a short position depends on whether it is naked or covered. With a portfolio weight of $\psi < 0$ for the calls, the number of options per unit of wealth is $\frac{-\psi}{O}$, where O stands for the price of one call option. The number of covered and naked calls is then given by min $\{\frac{-\psi}{O}, \phi\}$ and $\frac{-\psi}{O} - \min\{\frac{-\psi}{O}, \phi\}$, respectively. Note that the second expression can be rewritten as max $\{\frac{-\psi}{O}, \phi\}$.

For the covered call, the margin requirement is equal to the option price if the option is in the money and zero otherwise:

$$MR(CC) = \begin{cases} \min\left\{\dfrac{-\psi}{O}, \phi\right\} \cdot O & \text{ITM-call} \\ 0 & \text{OTM-call, ATM-call} \end{cases}$$

Margin requirements for short naked calls are more complicated. They are given by 100% of the option price plus the maximum of 15% of the underlying market value minus the out-of-the-money amount and 10% of the underlying market value:

$$MR\ (NC) = \max\left\{\frac{-\psi}{O} - \phi, 0\right\} \times [O + \max\ \{0.15 - \max(\text{Strike} - 1.0, 0), 0.1\}]$$

The overall value of the portfolio then has to be larger than the sum of the margin requirements:

$$MR(S) + MR(NC) + MR(CC) + MR(LC) \leq 1.0.$$

In case of two options, the margin requirements for the second option are calculated in an analogous way.

ENDNOTES

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